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Original Article

Approximate left ϕ -biprojectivity of θ -Lau product algebras

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ABSTRACT: We continue [8] and we discuss approximately left ϕ -biprojectivity for θ -Lau product algebras. We give some Banach algebras among the category of θ -Lau product algebras which are not approximately left ϕ -biprojective. In fact, some class of matrix algebras under the notion of approximate left ϕ -biprojectivity is also discussed here.

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1. Introduction and Preliminaries

Helemskii studied the structure of Banach algebras by homological theory. There are two important notions in the homological theory, namely biflatness and biprojectivity. A Banach algebra A is called biprojective if there exists a bounded A-bimodule morphism $\rho : A \to A \widehat{\otimes} A$ such that $\pi_A \circ \rho(a) = a$, for all $a \in A$. Here $A \widehat{\otimes} A$ denotes the projective tensor product of A with A and $\pi_A : A \widehat{\otimes} A \to A$ is the product morphism which is given by $\pi_A(a \otimes b) = ab$ for all $a, b \in A$. For more information about homological Banach algebra's history see [6].

Zhang gave an approximate version of biprojectivity for Banach algebras. In fact A is approximately biprojective if there exists a net of A-bimodule morphism $\rho_{\alpha} : A \to A \widehat{\otimes} A$ such that $\pi_A \circ \rho_{\alpha}(a) \to a$ for all $a \in A$. He studied nilpotent ideals of Banach algebra using this notion, see [9].

Motivated by Zhang and Helemskii, Sahami and Pourabbas defined a notion of Banach homology with respect to a non-zero multiplicative linear functional. In fact for a non-zero multiplicative linear functional ϕ on A, the Banach algebras A is called approximate left ϕ -biprojective if there exists a net of bounded linear map $\rho_{\alpha} : A \longrightarrow A \widehat{\otimes} A$ such that

 $\rho_{\alpha}(ab) - a \cdot \rho_{\alpha}(b) \to 0, \quad \rho_{\alpha}(ab) - \phi(b)\rho_{\alpha}(a) \to 0 \quad and \quad \phi \circ \pi_{A} \circ \rho_{\alpha}(a) - \phi(a) \to 0,$

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for all $a, b \in A$. They studied approximately left ϕ -biprojectivity of group algebras, Segal algebras and measure algebras over a locally compact group.

In this paper, We continue [8] and we discuss approximately left ϕ -biprojectivity for θ -Lau product algebras. The relations with its subalgebras also studied here. We give some Banach algebras among the category of θ -Lau product algebras which are not approximately left ϕ -biprojective.

We remind some definitions and notations which we need in this paper. For an arbitrary Banach algebra A, the character space is denoted by $\sigma(A)$ consists of all non-zero multiplicative linear functionals on A and any element of $\sigma(A)$ is called a character. The θ -Lau product was first introduced by Lau [4] for F-algebras. Monfared [5] introduced and investigated θ -Lau product space $A \times_{\theta} B$, for Banach algebras in general. Indeed for two Banach algebras A and B such that $\sigma(B) \neq \emptyset$ and θ be a non-zero character on B, the Cartesian product $A \times B$ by following multiplication and norm

$$(a,b)(a',b') = (aa' + \theta(b')a + \theta(b)a',bb'),$$

$$||(a,b)|| = ||a||_A + ||b||_B,$$

is a Banach algebra, for all $a, a' \in A$ and $b, b' \in B$. The Cartesian product $A \times B$ with the above properties called the θ -Lau product of A and B which is denoted by $A \times_{\theta} B$. From [5] we identify $A \times \{0\}$ with A, and $\{0\} \times B$ with B. Thus, it is clear that A is a closed two-sided ideal while B is a closed subalgebra of $A \times_{\theta} B$, and $(A \times_{\theta} B)/A$ is isometrically isomorphic to B. If $\theta = 0$, then we obtain the usual direct product of A and B. Since direct products often exhibit different properties, we have excluded the possibility that $\theta = 0$. Moreover, if $B = \mathbb{C}$, the complex numbers, and θ is the identity map on \mathbb{C} , then $A \times_{\theta} B$ is the unitization A^{\sharp} of A. Note that, by [5, Proposition 2.4], the character space $\sigma(A \times_{\theta} B)$ of $A \times_{\theta} B$ is equal to

$$\{(\phi,\theta): \phi \in \sigma(A)\} \bigcup \{(0,\psi): \psi \in \sigma(B)\}.$$

Also, the dual space $(A \times_{\theta} B)^*$ of $A \times_{\theta} B$ is identified with $A^* \times B^*$ such that for each $(a, b) \in A \times_{\theta} B$, $\phi \in \sigma(A)$ and $\psi \in \sigma(B)$ we have

$$\langle (\phi, \psi), (a, b) \rangle = \phi(a) + \psi(b).$$

Now, suppose that A^{**} , B^{**} and $(A \times_{\theta} B)^{**}$ are equipped with their first Arens products. Then $(A \times_{\theta} B)^{**}$ is isometrically isomorphic with $A^{**} \times_{\theta} B^{**}$. Also, for all $(m, n), (p, q) \in (A \times_{\theta} B)^{**}$ the first Arens product is defined by

$$(m,n)\Box(p,q) = (m\Box p + n(\theta)p + q(\theta)m, n\Box q);$$

see [5, Proposition 2.12]. Note that every $\phi \in \sigma(A)$ has a unique extension to a character on A^{**} is given by ϕ where $\tilde{\phi}(m) = m(\phi)$, for all $m \in A^{**}$.

Note that A and B are closed two-sided ideal and closed subalgebra of $L := A \times_{\theta} B$, respectively. So, we can write a = (a, 0) and b = (0, b) for all $a \in A$ and $b \in B$. Therefore, $L = A \times_{\theta} B$ is a Banach A-bimodule and also is a Banach B-bimodule.

We recall that if X is a Banach A-bimodule, then with the following actions X^* is also a Banach A-bimodule:

$$a \cdot f(x) = f(x \cdot a), \quad f \cdot a(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*)$$

The projective tensor product of A with A is denoted by $A \widehat{\otimes} A$. The Banach algebra $A \widehat{\otimes} A$ is a Banach A-bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

Let $\phi \in \sigma(A)$. Then ϕ has a unique extension on A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$. Clearly this extension remains to be a character on A^{**} .

2. Approximate left ϕ -biprojectivity

Here $p_A : L \longrightarrow A$ and $p_B : L \longrightarrow B$ denote the usual projections defined by $p_A(a, b) = a$ and $p_B(a, b) = b$ for all $a \in A$ and $b \in B$. Let $q_A : A \longrightarrow L$ and $q_B : B \longrightarrow L$ be injections given by $q_A(a) = (a, 0)$ and $q_B(b) = (0, b)$. Thus for q_A and p_B we define

 $q_A {\otimes} q_A : A \widehat{\otimes} A \longrightarrow L \widehat{\otimes} L$

and

$$p_B \otimes p_B : L \widehat{\otimes} L \longrightarrow B \widehat{\otimes} B$$

with

$$(q_A \otimes q_A)(a \otimes c) = (a, 0) \otimes (c, 0)$$

and

$$(p_B \otimes p_B)((a,b) \otimes (c,d)) = b \otimes d,$$

for all $a, c \in A$ and $b, d \in B$, respectively. One can show that q_A and $q_A \otimes q_A$ are A-bimodule morphisms and also p_B, q_B and $p_B \otimes p_B$ are B-bimodule morphisms.

For a unital Banach algebra A with unit e. Set $r_A : L \longrightarrow A$ and $S_B : B \longrightarrow L$ with $r_A(a, b) = a + \theta(b)e$ and $S_B(b) = (-\theta(b)e, b)$, respectively, for every $a \in A, b \in B$. Now

$$r_A \otimes r_A : L \widehat{\otimes} L \longrightarrow A \widehat{\otimes} A$$

and

$$S_{\scriptscriptstyle B} {\otimes} S_{\scriptscriptstyle B} : B \widehat{\otimes} B \longrightarrow L \widehat{\otimes} L$$

follows that

$$(r_A \otimes r_A)((a, b) \otimes (c, d)) = (a + \theta(b)e) \otimes (c + \theta(d)e)$$

and

$$(S_{\scriptscriptstyle B} \otimes S_{\scriptscriptstyle B})(b \otimes d) = (-\theta(b)e, b) \otimes (-\theta(d)e, d)$$

respectively. Clearly $r_{\scriptscriptstyle A}$ and $r_{\scriptscriptstyle A}\otimes r_{\scriptscriptstyle A}$ are A-bimodule morphism and $S_{\scriptscriptstyle B}$ is a B-bimodule morphism.

Proposition 2.1. Suppose that A and B are Banach algebras. Let A has a unit e. Also let $\phi \in \sigma(A)$ and $\theta \in \sigma(B)$. If L is approximately left (ϕ, θ) -biprojective. Then A is approximately left ϕ -biprojective.

Proof. Let L be left (ϕ, θ) -biprojective. Then there exists a net of bounded linear maps $\rho_{\alpha} : L \longrightarrow L \widehat{\otimes} L$ such that

$$\rho_{\alpha}(xy) - x \cdot \rho_{\alpha}(y) \to 0 \quad \rho_{\alpha}(xy) - \phi(y)\rho_{\alpha}(x) \to 0, \quad \phi \circ \pi_{L} \circ \rho_{\alpha}(x) - \phi(x) \to 0, \quad (x, y \in L).$$

It is easy to see that

$$r_{\scriptscriptstyle A} \circ \pi_{\scriptscriptstyle L} = \pi_A \circ (r_{\scriptscriptstyle A} \otimes r_{\scriptscriptstyle A}), \qquad \quad \phi \circ r_{\scriptscriptstyle A} = (\phi, \theta)$$

Now define $\eta_{\alpha}: A \longrightarrow A \widehat{\otimes} A$ by $\eta_{\alpha} = (r_A \otimes r_A) \circ \rho_{\alpha} \circ q_A$. Consider

$$\begin{split} \eta_{\alpha}(a_{1}a_{2}) - a_{1}\eta_{\alpha}(a_{2}) &= (r_{A} \otimes r_{A}) \circ \rho_{\alpha} \circ q_{A}(a_{1}a_{2}) - a_{1} \cdot (r_{A} \otimes r_{A}) \circ \rho_{\alpha} \circ q_{A}(a_{2}) \\ &= (r_{A} \otimes r_{A}) \circ \rho_{\alpha} \circ q_{A}(a_{1}a_{2}) - (r_{A} \otimes r_{A}) \circ \rho_{\alpha}(a_{1} \cdot q_{A}(a_{2})) \to 0 \end{split}$$

and

$$\begin{split} \eta_{\alpha}(a_{1}a_{2}) - \phi(a_{2})\eta_{\alpha}(a_{1}) &= (r_{A} \otimes r_{A}) \circ \rho_{\alpha} \circ q_{A}(a_{1}a_{2}) - \phi(a_{2})(r_{A} \otimes r_{A}) \circ \rho_{\alpha} \circ q_{A}(a_{1}) \\ &= (r_{A} \otimes r_{A}) \circ \rho_{\alpha}(q_{A}(a_{1}) \cdot a_{2}) - \phi(a_{2})(r_{A} \otimes r_{A}) \circ \rho_{\alpha} \circ q_{A}(a_{1}) \\ &= (r_{A} \otimes r_{A}) \circ \rho_{\alpha}(q_{A}(a_{1}) \cdot a_{2}) - (r_{A} \otimes r_{A})(\phi(a_{2})\rho_{\alpha}(q_{A}(a_{1}))) \to 0 \end{split}$$

for every a_1 and a_2 in A. Also we have

$$\begin{split} \phi \circ \pi_A \circ \eta_\alpha(a) - \phi(a) &= \phi \circ \pi_A \circ (r_A \otimes r_A) \circ \rho_\alpha \circ q_A(a) - \phi(a) \\ &= (\phi \circ r_A \circ \pi_L \circ \rho_\alpha)(a, 0) - \phi(a) \\ &= ((\phi, \theta) \circ \pi_L \circ \rho_\alpha)(a, 0) - \phi(a) \to 0 \end{split}$$

for all $a \in A$. So A is approximately left ϕ -biprojective.

Proposition 2.2. Suppose that A and B are Banach algebras and $\psi \in \sigma(B)$. If L is approximately left $(0, \psi)$ -biprojective, then B is approximately left ψ -biprojective. Converse holds whenever A is unital.

Proof. Since L is approximately left $(0, \psi)$ -biprojective, there exists a net of bounded linear maps $\rho_L^{\alpha} : L \longrightarrow L \widehat{\otimes} L$ such that $(0, \psi) \circ \pi_L \circ \rho_L^{\alpha} - (0, \psi) \to 0$ and

$$\rho_L^{\alpha}(l_1 l_2) - l_1 \cdot \rho_L^{\alpha}(l_2) \to 0, \quad \rho_L^{\alpha}(l_1 l_2) - (0, \psi)(l_2) \cdot \rho_L^{\alpha}(l_1) \to 0, \qquad (l_1, l_2 \in L).$$

Set $\rho_B^{\alpha}: B \longrightarrow B \widehat{\otimes} B$ which is given by $\rho_B^{\alpha} = (p_B \otimes p_B) \circ \rho_L^{\alpha} \circ q_B$. It is easy to see that

$$\pi_B \circ (p_B \otimes p_B) = p_B \circ \pi_L, \qquad \psi \circ p_B = (0, \psi).$$

Now consider

$$\begin{split} \rho_B^{\alpha}(b_1b_2) - \psi(b_2)\rho_B^{\alpha}(a_1) &= (p_B \otimes p_B) \circ \rho_L^{\alpha} \circ q_B(b_1b_2) - \psi(b_2)(p_B \otimes p_B) \circ \rho_L^{\alpha} \circ q_B(a_1) \\ &= (p_B \otimes p_B) \circ \rho_L^{\alpha}(q_B(b_1) \cdot b_2) - \psi(b_2)(p_B \otimes p_B) \circ \rho_L^{\alpha}(q_B(b_1)) \to 0 \end{split}$$

and

$$\begin{split} \rho_B^{\alpha}(b_1b_2) - b_1 \cdot \rho_B^{\alpha}(b_2) &= (p_B \otimes p_B) \circ \rho_L^{\alpha} \circ q_B(b_1b_2) - b_1 \cdot (p_B \otimes p_B) \circ \rho_L^{\alpha} \circ q_B(b_2) \\ &= (p_B \otimes p_B) \circ \rho_L^{\alpha} \circ q_B(b_1b_2) - (p_B \otimes p_B) \circ \rho_L^{\alpha}(b_1 \cdot q_B(b_2)) \to 0 \end{split}$$

for every b_1 and b_2 in B. Also we have

$$\begin{pmatrix} \psi \circ \pi_B \circ \rho_B^{\alpha} \end{pmatrix}(b) - \psi(b) = \left(\psi \circ \pi_B \circ (p_B \otimes p_B) \rho_L^{\alpha} \circ q_B \right)(b) - \psi(b)$$
$$= \left(\psi \circ p_B \circ \pi_L \circ \rho_L^{\alpha} \right)(0,b) - \psi(b)$$
$$= \left((0,\psi) \circ \pi_L \circ \rho_L^{\alpha} \right)(0,b) - \psi(b) \to 0,$$

for each $b \in B$.

For converse, suppose that B is approximately left ψ -biprojective. Then there exists a net of bounded linear maps $\rho_B : B \longrightarrow B \widehat{\otimes} B$ such that

$$\rho_B^{\alpha}(ab) - a \cdot \rho_B^{\alpha}(b) \to 0, \rho_B^{\alpha}(ab) - \psi(b)\rho_B^{\alpha}(a) \to 0$$

and

$$\psi \circ \pi_B \circ \rho_B^{\alpha}(b) - \psi(b) \to 0$$

for each $a, b \in B$. Define $\rho_L^{\alpha} : L \longrightarrow L \widehat{\otimes} L$ by

$$\rho_L^{\alpha}(a,b) := (S_{\scriptscriptstyle B} \otimes S_{\scriptscriptstyle B}) \circ \rho_B^{\alpha}(b),$$

for all $a \in A$ and $b \in B$. It is easy to see that

$$\pi_L \circ (S_{\scriptscriptstyle B} \otimes S_{\scriptscriptstyle B}) = S_{\scriptscriptstyle B} \circ \pi_B, \qquad (0,\psi) \circ S_{\scriptscriptstyle B} = \psi, \qquad ((S_{\scriptscriptstyle B} \otimes S_{\scriptscriptstyle B}) \circ \lambda_B(b)) \cdot x = 0,$$

for all $b \in B$ and $x \in A$. By these facts we can show that ρ_L^{α} is a net of bounded linear maps such that

$$\rho_L^{\alpha}(l_1 l_2) - (0, \psi)(l_2)\rho_L^{\alpha}(l_1) \to 0, \quad \rho_L^{\alpha}(l_1 l_2) - l_1 \cdot \rho_L^{\alpha}(l_2) \to 0$$

for all $l_1, l_2 \in L$. Also we have

$$(0,\psi)\circ\pi_L\circ\rho_L^{\alpha}(l)-(0,\psi)(l)\to 0$$

for each $l \in L$. It follows that L is approximately left $(0, \psi)$ -biprojective.

Remark 2.3. We show that approximately left (ϕ, θ) -biprojectivity of L implies that B is approximately left θ biprojective. To see this, we know that there exists a net of bounded linear maps $\rho_L^{\alpha} : L \longrightarrow L \widehat{\otimes} L$ such that

$$\rho_L^{\alpha}(ab) - a \cdot \rho_L^{\alpha}(b) \to 0, \quad \rho_L^{\alpha}(ab) - (\phi, \theta)(b)\rho_L^{\alpha}(a) \to 0$$

and

$$(\phi, \theta) \circ \pi_L \circ \rho_L(a) - (\phi, \theta)(a) \to 0, \qquad (a, b \in L).$$

Note that, we have

$$p_{_B} \circ \pi_L = \pi_B \circ (p_{_B} \otimes p_{_B}), \qquad \qquad r_A \circ \pi_L = \pi_A \circ (r_A \otimes r_A), \qquad \qquad \theta \circ p_{_B} = (0, \theta).$$

 $\textit{Define } \rho_B^\alpha: B \longrightarrow B \widehat{\otimes} B \textit{ by } \rho_B^\alpha := (p_{\scriptscriptstyle B} \otimes p_{\scriptscriptstyle B}) \circ \rho_L^\alpha \circ q_{\scriptscriptstyle B}. \textit{ So by using }$

$$((\phi, 0) \circ \pi_L \circ \rho_L^{\alpha})(0, b) \to 0$$

we have

$$\begin{pmatrix} \theta \circ \pi_B \circ \rho_B^{\alpha} \end{pmatrix}(b) - \theta(b) = \langle (\phi, \theta), (0, b) \rangle - ((\phi, 0) \circ \pi_L \circ \rho_L^{\alpha})(0, b) - \theta(b) \\ = ((\phi, 0) \circ \pi_L \circ \rho_L^{\alpha})(0, b) \to 0,$$

for every $b \in B$. Also we have

$$\rho_B^{\alpha}(b_1b_2) - b_1 \cdot \rho_B^{\alpha}(b_2) \to 0, \quad \rho_B^{\alpha}(b_1b_2) - \theta(b_2)\rho_B(b_1) \to 0, \quad (b_1, b_2 \in B).$$

It follows that B is approximately left θ -biprojective.

3. Applications and examples

Suppose that A is a Banach algebra and $\phi \in \sigma(A)$. We remind that a Banach algebra A is approximately left ϕ -amenable if there exists a net (m_{α}) in A such that $am_{\alpha} - \phi(a)m_{\alpha} \to 0$ and $\phi(m_{\alpha}) \to 1$ for all $a \in A$, see [1]. A Banach algebra A is called approximately left character amenable, if A is approximately left ϕ -amenable for all $\phi \in \sigma(A)$ and A posses a left approximate identity see [1].

Example 3.1. We give a Law product Banach algebra which is not approximately left ϕ -biprojective.

To see this, suppose that $C^1[0,1]$ is the set of all differentiable functions which its derivation is continuous. With the point-wise multiplication and the sup-norm, $C^1[0,1]$ becomes a Banach algebra. It is well-known that $\sigma(C^1[0,1]) = \{\phi_t : t \in [0,1]\}$, where $\phi_t(f) = f(t)$ for all $t \in [0,1]$. We assume conversely that $C^1[0,1] \times_{\theta} C^1[0,1]$ is approximately left (ϕ_t, θ) -biprojective or approximately left $(0, \phi_t)$ -biprojective, where $\phi_t(f) = f(t)$ for each $t \in [0,1]$. It is easy to see that function 1 is an identity for $C^1[0,1]$. Using Proposition 2.1 and Proposition 2.2 follows that $C^1[0,1] \approx C^1[0,1]$ is approximately left ϕ_t -biprojective. So there exists a net of bounded linear map $\rho_{C^1[0,1]}^{\alpha} : C^1[0,1] \longrightarrow C^1[0,1] \hat{\otimes} C^1[0,1]$

and

$$\tilde{\phi}_t \circ \pi_{C^1[0,1]} \circ \rho^{\alpha}_{C^1[0,1]}(f) - \phi_t(f) \to 0$$

 $\rho^{\alpha}_{C^1[0,1]}(fg) - f \cdot \rho^{\alpha}_{C^1[0,1]}(g) \to 0, \quad \rho^{\alpha}_{C^1[0,1]}(fg) - \phi_t(g)\rho^{\alpha}_{C^1[0,1]}(f) \to 0$

for all $f, g \in C[0, 1]$. Define $m_{\alpha} = \pi_{C_{[0,1]}} \circ \rho_{C^1[0,1]}^{\alpha}(1) \in A$. Then

$$f \cdot m_{\alpha} - \phi_t(f)m_{\alpha} = f \cdot \pi_{C_{[0,1]}} \circ \rho_{C^1[0,1]}^{\alpha}(1) - \phi_t(f)\pi_{C_{[0,1]}} \circ \rho_{C^1[0,1]}^{\alpha}(1)$$

= $\pi_{C_{[0,1]}}(f \cdot \rho_{C^1[0,1]}^{\alpha}(1)) - \pi_{C_{[0,1]}}(\phi_t(f)\rho_{C^1[0,1]}^{\alpha}(1)) \to 0$

and

$$\phi_t(m_\alpha) - 1 = \phi_t \circ \pi_{C_{[0,1]}} \circ \rho_{C^1[0,1]}^\alpha(1) - 1 \to \phi(1) - 1 = 0$$

for all $f \in C^1[0,1]$. It follows that $C^1[0,1]$ is approximately left ϕ_t -amenable which is impossible by similar arguments as in [3, Example 2.5].

The Banach algebra A is called approximately left character biprojective if A is approximately left ϕ -biprojective for each $\phi \in \sigma(A)$, respectively, see [8].

Proposition 3.1. Suppose that G is a locally compact group and also M(G) is the measure algebra with respect to G. Let $\theta \in \sigma(M(G))$. Then $M(G) \times_{\theta} M(G)$ is approximately left character biprojective if and only if G is discrete and amenable.

Proof. Suppose that $M(G) \times_{\theta} M(G)$ is approximately left character biprojective. Since M(G) has an identity, Proposition 2.1 implies that M(G) is approximately left ϕ -biprojective for all $\phi \in \sigma(M(G))$. Following the arguments of previous Example, gives that M(G) is approximately character amenable. Now by [1, Theorem 7.2], G is discrete and amenable.

For converse, suppose that G is discrete and amenable. Then we have $M(G) = \ell^1(G)$. Thus by Johnson Theorem $\ell^1(G)$ is amenable. So [2, Corollary 2.1] finishes the proof.

Example 3.2. Let $A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$ be a matrix algebra. With matrix operation and ℓ^1 -norm A becomes a Banach algebra. Define $\phi : A \longrightarrow \mathbb{C}$ by

$$\phi(\left(\begin{array}{cc}a&b\\0&c\end{array}\right))=c.$$

It is easy to see that is a character on A. We claim that $A \times_{\theta} A$ is neither approximately (ϕ, θ) -biprojective nor is approximately left $(0, \phi)$ -biprojective, where $\theta \in \sigma(A)$. Suppose conversely that $A \times_{\theta} A$ is approximately left (ϕ, θ) -biprojective or approximately left $(0, \phi)$ -biprojective. Since A is unital, by Proposition 2.1 and Proposition 2.2 A is approximately left ϕ -biprojective. The existence of unit for A gives that A is approximately left ϕ -amenable. Define

$$J := \left\{ \left(\begin{array}{cc} 0 & b \\ 0 & d \end{array} \right) : \ b, d \in \mathbb{C} \right\}$$

One can see that J is a closed ideal of A and $\phi_{|_J} \neq 0$. Since A is left ϕ -amenable, by [3, Lemma 3.1] we have that J is $\phi_{|_J}$ -amenable. Now [7, Proposition 5.1] follows that, there exists a net (u_{α}) in J such that $ju_{\alpha} - \phi(j)u_{\alpha} \longrightarrow 0$ and $\phi(u_{\alpha}) \rightarrow 1$ for all $j \in J$. Set $j = \begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix}$ and $u_{\alpha} = \begin{pmatrix} 0 & w_{\alpha} \\ 0 & v_{\alpha} \end{pmatrix}$, for some $j_1, j_2, w_{\alpha}, v_{\alpha} \in \mathbb{C}$. Thus,

$$ju_{\alpha} - \phi(j)u_{\alpha} = \begin{pmatrix} 0 & j_1w_{\alpha} \\ 0 & j_2v_{\alpha} \end{pmatrix} - \begin{pmatrix} 0 & j_2w_{\alpha} \\ 0 & j_2v_{\alpha} \end{pmatrix} \longrightarrow 0.$$

It gives that $j_1v_{\alpha} - j_2w_{\alpha} \longrightarrow 0$. If we put $j_1 = 1$ and $j_2 = 0$, then we have $v_{\alpha} \rightarrow 0$ which contradicts with $\phi(u_{\alpha}) = v_{\alpha} \rightarrow 1$.

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