AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 6(2) (2025) 109-120
https://doi.org/10.22060/AJMC.2024.22702.1185

Original Article

Analysis of the vacuum solution of the five-dimensional Einstein field equations with negative cosmological constant via variational symmetries

Fatemeh Ahangari*[a]

[a] *Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, P.O.Box 1993893973, Iran*

ABSTRACT: The Kaluza-Klein theory can be reckoned as a classical unified field theory of two of the significant forces of nature gravitation and electromagnetism. This formulation geometrically demonstrates the effects of a gravitational and an electromagnetic field by investigating a five-dimensional space with a metric constructed via the spacetime metric and the four-potential of the electromagnetic field. For the purpose of exploring the influences of dimensionality on the distinct physical variables, inquiring into stationary Kaluza-Klein rotating fluids is of particular significance. In this research, an extensive investigation of the variational symmetries for a specific vacuum solution of the (4+1)-dimensional Einstein field equations with negative cosmological constant is presented. For this purpose, first of all, the variational symmetries of our analyzed model are completely determined and the construction of the Lie algebra of the resulted symmetries is accurately analyzed. It is represented that the Lie algebra of local symmetries interrelated to the system of geodesic equations is non-solvable and not semi-simple and the algebraic organization of the derived quotient Lie algebra is accurately evaluated. Mainly, the adjoint representation group is effectively utilized intended for establishing an optimal system of group invariant solutions; which unequivocally yields a conjugate relation in the set of all one-dimensional symmetry subalgebras. Accordingly, the associated set of invariant solutions can be regarded as the slightest list from that the alternative invariant solutions of one-dimensional subalgebras are thoroughly determined unambiguously by virtue of transformations. Literally, all the corresponding local conservation laws of the resulted variational symmetries are totally calculated. Indeed, the symmetries of the metric of our analyzed space-time lead to the constants of motion for the point particles.

Review History:

Received:24 September 2023
Revised:21 December 2023
Accepted:02 January 2024
Available Online:01 March 2025

Keywords:

Variational symmetries
Einstein field equations
Kaluza-Klein theory
Adjoint representation
Conservation laws

MSC (2020):

53Z05; 34A26; 58E10

1. Introduction

The Origin of Kaluza-Klein theory dates back to shortly after the first publication of general relativity. Generically, the Kaluza-Klein theory refers to fundamental generalizations of pure or an improved interpretation of four-dimensional general relativity to a $(4 + D)$-dimensional spacetime which is explicitly fulfilled via the hypothesis that there exist D extra dimensions. Hence, the so-called spontaneous compactification is proposed, determined

**Corresponding author.*
E-mail addresses: f.ahangari@alzahra.ac.ir, fa.ahangari@gmail.com

by ordinary gravity plus gauge theories in their impressive low energy sectors. Reckoning on the special significance of quest of fundamentals interaction unification in physics, a great deal of research has been devoted to relativistic Kaluza-Klein theory in five and more dimensional space-time in recent years [1, 10].

There is no doubt that the start of contemporary investigations of singularities in the context of general relativity had its fundamental origination in Kurt Gödel's original publication [4]. In this research, he has unambiguously obtained an exact solution for Einstein's field equations in which matter admits the configuration of a pressure free fluid i.e., dust solution. This model is quite homogenous and non-isotropic and might be considered structurally as a non-expanding and singularity free rotating cosmological solution and it displays a particular rotational symmetry which leads to presence of close time-like curves. Significantly, Gödel's exact rotating fluid filled cosmological model can be undeniably contemplated as the exclusive universe which is inherently perfect fluid filled and is invariant under a five-dimensional group of isometries multiply transitive on spacetime; it is also rotationally symmetric in local coordinates.

Subsequent to Gödel's relativistic model describing a rotating dust universe [4], the scrutiny of rotating fluids in the framework of general relativity acquired remarkable attention (notice [10] and the related references). Various relativistic cylindrically symmetric, non-statics and inhomogeneous Kaluza-Klein fluid models have been proposed by Patel and Dadhich which admit dimensional reduction. Their analyzed models have basically the five-dimensional Kasnerian vacuum solution which scrutinizes one dimension reduction to the radiation Friedman-Robertson-Walker flat model [10]. In 1996 a set of one-parameter solutions for a fluid conceding the equation of state $p = (2/3)\rho$, rotating about a regular axis was proposed exhaustively by Davison [3]. The noteworthy issue is that stationary Kaluza-Klein perfect fluid models in standard Einstein theory are not explicitly attainable in literature. Hence, with the purpose of exploring the consequences of dimensionality on the distinct physical variables, acquiring and scrutinizing this category of solutions is extensively substantial. In [11], R. Tikekar and L. K. Patel have explicitly developed the Kaluza-Klein field equations for cylindrically symmetric rotating distributions regarding perfect fluid. They have outlined a series of physically feasible solutions that is definitely opined to be the first suchlike Kaluza-Klein solutions and intrinsically it incorporates the Davidson's solution's Kaluza-Klein counterpart.

In this section, considering [11], a summary explanation regarding Kaluza-Klein field equations for stationary cylindrically symmetric fluid models in the context of standard Einstein theory is presented. Generically, a typical stationary cylindrically symmetric five-dimensional spacetime is described by virtue of this metric:

$$ds^2 = D^2(dt + Hd\phi)^2 - A^2dr^2 - B^2dz^2 - r^2C^2d\phi^2 - E^2d\psi^2, \tag{1}$$

In above formula, r, z and ϕ represent cylindrical polar coordinates, t is the time coordinate, ψ pointedly denotes the coordinate corresponding to the extra spatial dimension and A, B, C, D and H are functions depending exclusively on radial coordinate r.

If the metric (1) is to designate the spacetime of a stationary perfect fluid rotating about the regular axis $r = 0$, the metric coefficients will be absolutely connected to the dynamical parameters by virtue of the Einstein field equations which are in the pentad notation utilizing the system of units providing $c = G = 1$, approved in the following construction:

$$\mathbf{R}_{(ab)} = -8\pi\Big[(\rho + p)v_{(a)}v_{(b)} - \frac{1}{3}(\rho - p)g_{(ab)}\Big],$$

In above formula, ρ, p reprent the matter density and the fluid pressure, respectively. Also, v_a explicitly indicated components in the pentad frame of the unit time-like flow vector v^i of the fluid, which satisfies $v^iv_i = 1$. Davidson [3], thoroughly presented a cosmological solution of the relativistic system of field equations for a perfect fluid in rigid rotation about a regular axis. The mentioned model, indicates the possibility that the system of Kaluza-Klein field equations can be solved unambiguously via presuming the succeeding expression for the metric coefficients A, B, C, D, E and H,

$$\begin{aligned} A &= (1 + k^2r^2)^a, \qquad & B &= (1 + k^2r^2)^b, \qquad & C &= (1 + k^2r^2)^c,\\ D &= (1 + k^2r^2)^d, \qquad & E &= (1 + k^2r^2)^e, \qquad & H &= \alpha r^2. \end{aligned}$$

where a, b, c, d, e and k are constants and α is regarded as the constant of integration.

In [11] explicit particular cases for physical relevance that act in accordance with definite specific options of the free parameters, are thoroughly studied. In this paper, we will inclusively investigate the problem of symmetries and conservation laws for this specific solution which is reported in [11].

In the particular case, when $a = -1/2$, $b = e = c = -d = 1/4$, $\alpha^2 = k^2$, the Kaluza-Klein equations are all fulfilled and the spacetime of this type of solutions possess the subsequent metric structure:

$$ds^2 = (1 + k^2r^2)^{-1/2}(dt + kr^2d\phi)^2 - (1 + k^2r^2)^{-1}dr^2 - (1 + k^2r^2)^{1/2}\big(dz^2 + r^2d\phi^2 + d\psi^2\big). \tag{2}$$

which describes a five-dimensional spacetime of a stationary fluid which is structurally cylindrically symmetric whose density and pressure are absolutely constant and are connected by virtue of the following equation of state: $\rho + p = 0$. By inserting $\Lambda = -(3/2)k^2$, the metric above represents a vacuum solution of the five-dimensional Einstein field equations: $\mathbf{R}_{ij} = \Lambda g_{ij}$, where Λ represents the cosmological constant.
For further complete details refer to [11].
As discussed above, exploring the influences of dimensionality on the distinctive physical parameters, preponderantly reveals the significance of inquiring into stationary Kaluza-Klein rotating fluids. Consequently, in the current research, we have extensively focused on detailed exploration of the issue of symmetries and conservation laws for the privileged model (2) which explicitly describes the vacuum solution of the (4+1)-dimensional Einstein equations with negative cosmological constant. The organization of the current research is as follows:
In section two, we will exhibit a thoroughgoing analysis of the Lie algebra of variational symmetries for our analyzed cosmological solution (2). Firstly, by appraising the Lagrangian which is characterized straightforwardly through the metric, we will calculate the associated geodesic equations as the Euler Lagrange equations. Subsequently, we determine the point generators of the one-parameter Lie groups of transformations that leave invariant the action integral associated to the Lagrangian (variational symmetries). Furthermore, we will briefly analyze the structure of the Lie algebra of the resulted variational symmetries from the algebraic standpoint. For the motive of thorough comprehending the invariant solutions, a mean of characterizing which subgroups yield distinct solutions is indispensable. Hence, in section three, we create an optimal system of group invariant solutions by establishing the adjoint representation group, which implies a conjugate relation in the set of all one-dimensional resulted symmetry subalgebras. This procedure literally leads to a slightest listicle of invariant solutions from which the further related invariant solutions of one dimensional subalgebras can be deduced by virtue of transformations. Ultimately, in section three of the current paper, reckon with the fact that the symmetries of our five-dimensional analyzed cosmological solution (2) yield the constants of motion for the point particles, all the associated local conservation laws are accurately calculated. Finally, some significant consequences are presented.

2. Lie Algebra of variational symmetries

Generically, the Euler-Lagrange equations corresponding to the first order Lagrangians are second order differential equations. For our principal objectives in this paper, the Lagrangian is determined straightforwardly through the metric which unambiguously leads to the geodesic equations. We presume the following Lagrangian for minimizing the arc length that is for convenience expressed by the square of the arc length and accordingly provides the geodesic equations as the associated Euler Lagrange equations is concluded as follows:

$$L[x^\mu, \frac{dx^\mu}{ds}] = g_{\mu\nu}(x^\kappa)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}.$$

Hence, a collection of second order ordinary differential equations:

$$\frac{d^2x^\mu}{ds^2} = g\Big(s, x^\mu, \frac{dx^\mu}{ds}\Big).$$

In the following, the vector field X which is determined totally on a real parameter fiber bundle over a typical manifold [2, 5]:

$$\mathbf{X} = \xi(s, x^\mu)\frac{\partial}{\partial s} + \eta^\nu(s, x^\mu)\frac{\partial}{\partial x^\nu},$$

where $\mu, \nu = 1, \ldots, 5$. Accordingly, the first order prolongation of this vector field is identically characterized on the real parameter fiber bundle over the tangent bundle of the manifold and is explicitly demonstrated as follows:

$$\mathbf{X}^{[1]} = \mathbf{X} + \big(\eta^\nu_{,s} + \eta^\nu_{,\mu}\dot{x}^\mu - \xi_{,s}\dot{x}^\nu - \xi_{,\mu}\dot{x}^\mu\dot{x}^\nu\big)\frac{\partial}{\partial \dot{x}^\nu}.$$

Then $\mathbf{X}$ is denoted by a variational symmetry (or Noether point symmetry) of the analyzed Lagrangian whenever there exists a gauge function, $A = A(s, x^\mu)$, in such a manner that the following important identity holds:

$$\mathbf{X}^{[1]}L + (D_s\xi)L = D_sA. \tag{3}$$

where $D_s = \frac{\partial}{\partial s} + \dot{x}^\mu\frac{\partial}{\partial x^\mu}$, which is analogously characterized on the real parameter fiber bundle over the corresponding tangent bundle to the manifold. For extra details refer to [2, 5].

In this section, firstly by concentrating on the Lagrangian which is derived straightforwardly through the metric (2), we will calculate the geodesic equations as the corresponding Euler Lagrange equations. Subsequently, we totally determine the point generators associated to the one parameter Lie groups of transformations which leave invariant the action integral of our analyzed Lagrangian i.e., variational symmetries.
The Lagrangian for the metric (2) is

$$L=\frac{\dot{t}^2}{\sqrt{1+k^2r^2}}-\frac{\dot{r}^2}{1+k^2r^2}-\sqrt{1+k^2r^2}\dot{z}^2+\Big(\frac{k^2r^4}{\sqrt{1+k^2r^2}}-r^2\sqrt{1+k^2r^2}\Big)\dot{\phi}^2+\frac{2kr^2}{\sqrt{1+k^2r^2}}\dot{t}\dot{\phi}-\sqrt{1+k^2r^2}\dot{\psi}^2. \tag{4}$$

The interrelated simplified Euler-Lagrange equations are the geodesic equations entirely determined as follows:

$$\mathbf{E}:\begin{cases}\mathbf{E_1}: \ddot{t}+\dfrac{k^2r(4k^2r^2+7)}{(k^2r^2+1)(4k^2r^2+1)}\ \dot{t}\dot{r}=0,\\[2mm] \mathbf{E_2}: \ddot{r}-\dfrac{k^2r}{2\sqrt{1+k^2r^2}}\ \dot{t}^2+\dfrac{2kr(k^2r^2+2)}{\sqrt{1+k^2r^2}}\ \dot{t}\dot{\phi}-\dfrac{k^2r}{1+k^2r^2}\ \dot{r}^2-\dfrac{1}{2}\sqrt{1+k^2r^2}\ k^2r\ \dot{z}^2\\[2mm] \qquad\qquad -\dfrac{r(k^2r^2+2)}{2\sqrt{1+k^2r^2}}\ \dot{\phi}^2-\dfrac{1}{2}\sqrt{1+k^2r^2}\ k^2r\ \dot{\psi}^2=0,\\[2mm] \mathbf{E_3}: \ddot{z}+\dfrac{k^2r}{1+k^2r^2}\ \dot{r}\dot{z}=0,\\[2mm] \mathbf{E_4}: \ddot{\phi}-\dfrac{4k}{r(1+4k^2r^2)}\ \dot{t}\dot{r}+\dfrac{k^2r^2+2}{r(1+k^2r^2)}\ \dot{r}\dot{\phi}=0,\\[2mm] \mathbf{E_5}: \ddot{\psi}+\dfrac{k^2r}{1+k^2r^2}\ \dot{r}\dot{\psi}=0.\end{cases} \tag{5}$$

By applying (4) in (3), the corresponding determining PDEs for seven unknown functions ξ, η^μ and A, are thoroughly characterized. Moreover, each of these is a function of six parameters, i.e. s, t, r, z, ϕ and ψ. By solving the resulted equations for the metric (2), it is inferred that:

Theorem 2.1. *The Lie group of variational symmetries associated to solution (2) has a Lie algebra which is precisely generated by the vector fields* $\mathbf{X}=\xi\frac{\partial}{\partial s}+\eta^1\frac{\partial}{\partial t}+\eta^2\frac{\partial}{\partial r}+\eta^3\frac{\partial}{\partial z}+\eta^4\frac{\partial}{\partial \phi}+\eta^5\frac{\partial}{\partial \psi}$, *where*

$$\xi(s,t,r,z,\phi,\psi)=c_1,\qquad \eta^1(s,t,r,z,\phi,\psi)=c_5z+c_6\psi+c_9,\qquad \eta^2(s,t,r,z,\phi,\psi)=0,$$

$$\eta^3(s,t,r,z,\phi,\psi)=c_5t+c_3\psi+c_4,\qquad \eta^4(s,t,r,z,\phi,\psi)=c_5kz+c_6k\psi+c_8,$$

$$\eta^5(s,t,r,z,\phi,\psi)=-c_3z+c_6t+c_7,\qquad A(s,t,r,z,\phi,\psi)=c_2.$$

and $c_i,\ i=1,\dots,9$ *are optional constants.*

Hence, we explicitly find out the eight dimensional Lie algebra of variational symmetries with this basis:

Corollary 2.2. *Infinitesimal generators of every one parameter Lie group of variational symmetries corresponding to (5) are unequivocally demonstrated by:*

$$\mathbf{X_1}=\frac{\partial}{\partial s},\qquad \mathbf{X_2}=\frac{\partial}{\partial t},\qquad \mathbf{X_3}=\frac{\partial}{\partial z},$$

$$\mathbf{X_4}=\frac{\partial}{\partial \phi},\qquad \mathbf{X_5}=\frac{\partial}{\partial \psi},\qquad \mathbf{X_6}=-\psi\frac{\partial}{\partial z}+z\frac{\partial}{\partial \psi},$$

$$\mathbf{X_7}=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z}+kz\frac{\partial}{\partial \phi},\qquad \mathbf{X_8}=\psi\frac{\partial}{\partial t}+k\psi\frac{\partial}{\partial \phi}+t\frac{\partial}{\partial \psi},\qquad A(s,t,r,z,\phi,\psi)=c\qquad (constant).$$

The commutator table of variational symmetries of the system of geodesic equations (5) is presented in Table 1, where the entry in the i^{th} *row and* j^{th} *column is expressed by* $[X_i,X_j]=X_iX_j-X_jX_i,\ \ i,j=1,\dots,8.$

Table 1: Commutation relations satisfied by infinitesimal generators of $\mathfrak{g}^{\mathrm{III}}$

$[\,,]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$	$\mathbf{X}_7$	$\mathbf{X}_8$
$\mathbf{X}_1$	0	0	0	0	0	0	0	0
$\mathbf{X}_2$	0	0	0	0	0	0	$\mathbf{X}_3$	$\mathbf{X}_5$
$\mathbf{X}_3$	0	0	0	0	0	$-\mathbf{X}_5$	$\mathbf{X}_2 + k\mathbf{X}_4$	0
$\mathbf{X}_4$	0	0	0	0	0	0	0	0
$\mathbf{X}_5$	0	0	0	0	0	$\mathbf{X}_3$	0	$\mathbf{X}_2 + k\mathbf{X}_4$
$\mathbf{X}_6$	0	0	$\mathbf{X}_5$	0	$-\mathbf{X}_3$	0	$\mathbf{X}_8$	$-\mathbf{X}_7$
$\mathbf{X}_7$	0	$-\mathbf{X}_3$	$-\mathbf{X}_2 - k\mathbf{X}_4$	0	0	$-\mathbf{X}_8$	0	$-\mathbf{X}_6$
$\mathbf{X}_8$	0	$-\mathbf{X}_5$	0	0	$-\mathbf{X}_2 - k\mathbf{X}_4$	$\mathbf{X}_7$	$\mathbf{X}_6$	0

Let $\mathfrak{g}^{\mathrm{III}}$ represent the Lie algebra of local symmetries related to the system of geodesic equations (5). In the following, a concise investigation concerning the algebraic structure of $\mathfrak{g}^{\mathrm{III}}$ is pointed out. The Lie algebra $\mathfrak{g}^{\mathrm{III}}$ is non-solvable, because if $\mathfrak{g}^{\mathrm{III}(1)} =< X_i, [X_i, X_j] >= [\mathfrak{g}^{\mathrm{III}}, \mathfrak{g}^{\mathrm{III}}]$, be the derived subalgebra of $\mathfrak{g}^{\mathrm{III}}$, then we have:

$$\mathfrak{g}^{\mathrm{III}(1)} = [\mathfrak{g}^{\mathrm{III}}, \mathfrak{g}^{\mathrm{III}}] =< X_3, X_2 + kX_4, X_5, X_6, X_7, X_8 >,$$

$$\mathfrak{g}^{\mathrm{III}(2)} = [\mathfrak{g}^{\mathrm{III}(1)}, \mathfrak{g}^{\mathrm{III}(1)}] = \mathfrak{g}^{\mathrm{III}(1)}.$$

Hence, we have this chain of ideals $\mathfrak{g}^{\mathrm{III}} \supset \mathfrak{g}^{\mathrm{III}(1)} = \mathfrak{g}^{\mathrm{III}(2)} \neq 0$, which definitely confirms the non-solvability of $\mathfrak{g}^{\mathrm{III}}$. Besides, $\mathfrak{g}^{\mathrm{III}}$ is not semisimple, since its killing form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

is degenerate. $\mathfrak{g}^{\mathrm{III}}$ has a Levi decomposition of the form $\mathfrak{g}^{\mathrm{III}} = \mathfrak{r} \ltimes \mathfrak{h}$ where $\mathfrak{r} =< X_1, X_2, X_3, X_4, X_5 >$ is the radical namely the largest solvable ideal of $\mathfrak{g}^{\mathrm{III}}$ and $\mathfrak{h} =< X_6, X_7, X_8 >$ is a semisimple and non-solvable subalgebra of $\mathfrak{g}^{\mathrm{III}}$. Hereupon, according to [7] the quotient algebra created from $\mathfrak{g}$ can be identified in a manner that

$$\mathfrak{g}_1^{\mathrm{III}} = \mathfrak{g}^{\mathrm{III}}/\mathfrak{r} = \left\{ X + \mathfrak{r} \mid X \in \mathfrak{g}^{\mathrm{III}} \right\}$$

the members of $\mathfrak{g}_1^{\mathrm{III}}$ are denoted by Y_i and the commutator table of the resulted quotient Lie algebra is represented in Table 2, where the entry in the i^{th} row and j^{th} column is determined as $[\mathcal{Z}_i, \mathcal{Z}_j] = \mathcal{Z}_i\mathcal{Z}_j - \mathcal{Z}_j\mathcal{Z}_i,\ i, j = 1, 2, 3.$

Table 2: Commutation table of $\mathfrak{g}_1^{\mathrm{III}}$

$[\ ,\ ]$	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$
$\mathcal{Z}_1$	0	$\mathcal{Z}_3$	$-\mathcal{Z}_2$
$\mathcal{Z}_2$	$-\mathcal{Z}_3$	0	$-\mathcal{Z}_1$
$\mathcal{Z}_3$	$\mathcal{Z}_2$	$\mathcal{Z}_1$	0

The quotient algebra $\mathfrak{g}_1^{\mathrm{III}}$ is not only semisimple but also a non-solvable Lie algebra. It is semsimple, since its killing form $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is non-degenerate. In addition, $\mathfrak{g}_1^{\mathrm{III}}$ is non-solvable, since whenever ${\mathfrak{g}_1}^{\mathrm{III}(1)} =< \mathcal{Z}_i, [\mathcal{Z}_i, \mathcal{Z}_j] >= [\mathfrak{g}_1^{\mathrm{III}}, \mathfrak{g}_1^{\mathrm{III}}]$ be the derived subalgebra of $\mathfrak{g}_1^{\mathrm{III}}$, we have $\mathfrak{g}_1^{\mathrm{III}} = {\mathfrak{g}_1}^{\mathrm{III}(1)} = [\mathfrak{g}_1^{\mathrm{III}}, \mathfrak{g}_1^{\mathrm{III}}] =< \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3 >$ so, we have $\mathfrak{g}_1^{\mathrm{III}} = {\mathfrak{g}_1}^{\mathrm{III}(1)} \neq 0$, which precisely illustrates the non-solvability of $\mathfrak{g}_1^{\mathrm{III}}$.

3. Preliminary group classification of variational symmetries for the geodesic equations

It is worth mentioning that to every one parameter subgroups of the full symmetry group of an arbitrary system of differential equations, a family of solutions distinguished by invariant solutions can be explicitly associated. Moreover, any arbitrary linear combination of infinitesimal generators is as well an infinitesimal generator; so, corresponding to a differential equation infinitely many symmetry subgroups might be designated. Accordingly, for the purpose of entire recognition of the invariant solutions, a systematic mechanism of characterizing which subgroups yield distinct types of solutions is undeniably crucial. In the specific case of one dimensional subalgebras in fact the mentioned classification is elementally analogous to classifying the orbits of the adjoint representation. Whenever only one representative is picked out from each family of equivalent subalgebras, an optimal set of subalgebras is unambiguously established. As a consequence, by utilizing these transformations, the slightest collection of invariant solutions from which all the other invariant solutions associated to one dimensional subalgebras are straightforwardly resulted, is exhaustively formulated [8, 9].

Each X_i, $i = 1, \dots, 8$, of the basis symmetries generates an adjoint representation or interior automorphism $\mathrm{Ad}(\exp(\varepsilon X_i))$ expressed by means of the Lie series:

$$\mathrm{Ad}(\exp(\varepsilon.X_i).X_j) = X_j - \varepsilon.[X_i, X_j] + \frac{\varepsilon^2}{2}.[X_i, [X_i, X_j]] - \dots$$

where $[X_i, X_j]$ is the commutator for the Lie algebra, ε is a variable, and $i, j = 1, \dots, 8$.
We can look forward to simplify an optional element,

$$\mathbf{X} = a_1\mathbf{X_1} + a_2\mathbf{X_2} + \dots + a_8\mathbf{X_8} \tag{6}$$

of the Lie algebra of variational symmetries associated to the geodesic Lagrangian (4) which was designated by $\mathfrak{g}^{\mathrm{III}}$. Take notice of the elements of $\mathfrak{g}^{\mathrm{III}}$ can be characterized by vectors $a = (a_1, \dots, a_8) \in \mathbb{R}^8$ since each of them can be explained in the form (6) for arbitrary constants $a_1, \dots, a_8$. Consequently, the adjoint action is indeed regarded as a group of linear transformations of the vectors $(a_1, \dots, a_8)$.
Subsequently, we can express this impressive theorem:

Theorem 3.1. *An optimal system of one-dimensional Lie subalgebras of variational symmetries corresponding to the geodesic Lagrangian (4) is established by virtue of the following operators:*

$$\begin{aligned}
\mathbf{(1)} : a_1\mathbf{X_1} + a_2\mathbf{X_2} + \mathbf{X_3} + a_4\mathbf{X_4} + a_8\mathbf{X_8} &= a_1\frac{\partial}{\partial s} + \Big(a_2 + a_8\psi\Big)\frac{\partial}{\partial t} + \frac{\partial}{\partial z} \\
&\quad + \Big(a_4 + a_8\psi k\Big)\frac{\partial}{\partial \varphi} + a_8 t\frac{\partial}{\partial \psi}, \qquad A = c \\
\mathbf{(2)} : a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + \mathbf{X_5} + a_7\mathbf{X_7} &= a_1\frac{\partial}{\partial s} + \Big(a_2 + a_7 z\Big)\frac{\partial}{\partial t} + a_7 t\frac{\partial}{\partial z} \\
&\quad + \Big(a_4 + a_7 z k\Big)\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \psi}, \qquad A = c \\
\mathbf{(3)} : a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + a_6\mathbf{X_6} + \mathbf{X_7} &= a_1\frac{\partial}{\partial s} + \Big(a_2 + z\Big)\frac{\partial}{\partial t} + \Big(a_6\psi + t\Big)\frac{\partial}{\partial z} \\
&\quad + \Big(a_4 + zk\Big)\frac{\partial}{\partial \varphi} - a_6 z\frac{\partial}{\partial \psi}, \qquad A = c \\
\mathbf{(4)} : a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + a_6\mathbf{X_6} + a_8\mathbf{X_8} &= a_1\frac{\partial}{\partial s} + \Big(a_2 + a_8\psi\Big)\frac{\partial}{\partial t} + a_6\psi\frac{\partial}{\partial z} \\
&\quad + \Big(a_4 + a_8\psi k\Big)\frac{\partial}{\partial \varphi} + \Big(-a_6 z + a_8 t\Big)\frac{\partial}{\partial \psi}, \qquad A = c
\end{aligned}$$

where a_i are real arbitrary constants.

Proof. $F_i^\varepsilon : \mathfrak{g}^{\mathrm{III}} \to \mathfrak{g}^{\mathrm{III}}$ characterized by $\mathbf{X} \mapsto \mathrm{Ad}(\exp(\varepsilon_i\mathbf{X_i}).\mathbf{X})$ is a linear map, for $i = 1, \dots, 8$. The matrix M_i^ε of F_i^ε, with respect to the basis $\{\mathbf{X_1}, \dots, \mathbf{X_8}\}$ is

$$M_1^\varepsilon = I_8, \quad M_2^\varepsilon = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&-\varepsilon&0&0&0&1&0\\ 0&0&0&0&-\varepsilon&0&0&1 \end{pmatrix}, \quad M_3^\varepsilon = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&\varepsilon&1&0&0\\ 0&-\varepsilon&0&-\varepsilon k&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$M_4^{\varepsilon} = I_8, \quad M_5^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\varepsilon & 0 & -\varepsilon k & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_6^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\varepsilon & 0 & -\sin\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\varepsilon & 0 & \cos\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix}$$

$$M_7^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\big(e^{\varepsilon}+e^{-\varepsilon}\big) & \frac{1}{2}\big(e^{\varepsilon}-e^{-\varepsilon}\big) & \frac{1}{2}k\big(e^{\varepsilon}+e^{-\varepsilon}-2\big) & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\big(e^{\varepsilon}-e^{-\varepsilon}\big) & \frac{1}{2}\big(e^{\varepsilon}+e^{-\varepsilon}\big) & \frac{1}{2}k\big(e^{\varepsilon}-e^{-\varepsilon}\big) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\big(e^{\varepsilon}+e^{-\varepsilon}\big) & 0 & \frac{1}{2}\big(e^{\varepsilon}-e^{-\varepsilon}\big) \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\big(e^{\varepsilon}-e^{-\varepsilon}\big) & 0 & \frac{1}{2}\big(e^{\varepsilon}+e^{-\varepsilon}\big) \end{pmatrix}$$

$$M_8^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\big(e^{\varepsilon}+e^{-\varepsilon}\big) & 0 & \frac{1}{2}k\big(e^{\varepsilon}+e^{-\varepsilon}-2\big) & \frac{1}{2}\big(e^{\varepsilon}-e^{-\varepsilon}\big) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\big(e^{\varepsilon}-e^{-\varepsilon}\big) & 0 & \frac{1}{2}k\big(e^{\varepsilon}-e^{-\varepsilon}\big) & \frac{1}{2}\big(e^{\varepsilon}+e^{-\varepsilon}\big) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\big(e^{\varepsilon}+e^{-\varepsilon}\big) & \frac{1}{2}\big(e^{-\varepsilon}-e^{\varepsilon}\big) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\big(e^{-\varepsilon}-e^{\varepsilon}\big) & \frac{1}{2}\big(e^{\varepsilon}+e^{-\varepsilon}\big) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For the purpose of classifying the one-dimensional Lie subalgebras of variational symmetries corresponding to the geodesic Lagrangian (4), the following cases are outlined in a manner that in each case, by virtue of individually performing a finite number of the adjoint representations M_i^{ε} $(i = 1, \dots, 8)$ on $\mathbf{X}$, by appropriate choice of parameters ε_i in any step, it is moderately attemped to make the coefficients of $\mathbf{X}$ vanish and to obtain the simplest form of $\mathbf{X}$.

Let $\mathbf{X} = \sum_{i=1}^{8} a_i \mathbf{X_i}$, then

$$\begin{aligned}
F_8^{\varepsilon_8} \circ F_7^{\varepsilon_7} \circ \cdots \circ F_1^{\varepsilon_1} \,:\, \mathbf{X} \longmapsto\ & a_1 \mathbf{X_1} + \Bigg[a_2 \Big(\tfrac{1}{2}e^{\varepsilon_7} + \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8} + \tfrac{1}{2}e^{-\varepsilon_8}\Big) + a_3 \Big(\tfrac{1}{2}e^{\varepsilon_7} - \tfrac{1}{2}e^{-\varepsilon_7}\Big) \\
&+ a_4 \Bigg(\tfrac{k}{2}\Big(\tfrac{1}{2}e^{\varepsilon_7} + \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(e^{\varepsilon_8} + e^{-\varepsilon_8} - 2\Big) + \tfrac{k}{2}\Big(e^{\varepsilon_7} + e^{-\varepsilon_7} - 2\Big)\Bigg) + a_5 \Big(\tfrac{1}{2}e^{\varepsilon_7} + \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8} - \tfrac{1}{2}e^{-\varepsilon_8}\Big) \Bigg] \mathbf{X_2} \\
&+ \Bigg[a_2 \Bigg(\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7} - \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8} + \tfrac{1}{2}e^{-\varepsilon_8}\Big) - \sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8} - \tfrac{1}{2}e^{-\varepsilon_8}\Big) \Bigg) + a_3 \cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7} + \tfrac{1}{2}e^{-\varepsilon_7}\Big) \\
&+ a_4 \Bigg(\tfrac{k}{2}\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7} - \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(e^{\varepsilon_8} + e^{-\varepsilon_8} - 2\Big) + \tfrac{k}{2}\cos(\varepsilon_6)\Big(e^{\varepsilon_7} - e^{-\varepsilon_7}\Big) - \tfrac{k}{2}\sin(\varepsilon_6)\Big(e^{\varepsilon_8} - e^{-\varepsilon_8}\Big) \Bigg) \\
&+ a_5 \Bigg(\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7} - \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8} - \tfrac{1}{2}e^{-\varepsilon_8}\Big) - \sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8} + \tfrac{1}{2}e^{-\varepsilon_8}\Big) \Bigg) \Bigg] \mathbf{X_3} + a_4 \mathbf{X_4} \\
&+ \Bigg[a_2 \Bigg(\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7} - \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8} + \tfrac{1}{2}e^{-\varepsilon_8}\Big) + \cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8} - \tfrac{1}{2}e^{-\varepsilon_8}\Big) \Bigg) + a_3 \sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7} + \tfrac{1}{2}e^{-\varepsilon_7}\Big) \\
&+ a_4 \Bigg(\tfrac{k}{2}\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7} - \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(e^{\varepsilon_8} + e^{-\varepsilon_8} - 2\Big) + \tfrac{k}{2}\sin(\varepsilon_6)\Big(e^{\varepsilon_7} - e^{-\varepsilon_7}\Big) + \tfrac{k}{2}\cos(\varepsilon_6)\Big(e^{\varepsilon_8} - e^{-\varepsilon_8}\Big) \Bigg) \\
&+ a_5 \Bigg(\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7} - \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8} - \tfrac{1}{2}e^{-\varepsilon_8}\Big) + \cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8} + \tfrac{1}{2}e^{-\varepsilon_8}\Big) \Bigg) \Bigg] \mathbf{X_5} \\
&+ \Bigg[a_2 \Bigg(\Big(-\varepsilon_5 \cos(\varepsilon_6) + \varepsilon_3 \sin(\varepsilon_6) \Big)\Big(\tfrac{1}{2}e^{\varepsilon_7} - \tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8} + \tfrac{1}{2}e^{-\varepsilon_8}\Big) + \Big(\varepsilon_5 \sin(\varepsilon_6) + \varepsilon_3 \cos(\varepsilon_6)\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8} - \tfrac{1}{2}e^{-\varepsilon_8}\Big) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ a_3\Big(-\varepsilon_5\cos(\varepsilon_6)+\varepsilon_3\sin(\varepsilon_6)\Big)\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big) + a_4\Bigg(\tfrac{k}{2}\Big(-\varepsilon_5\cos(\varepsilon_6)+\varepsilon_3\sin(\varepsilon_6)\Big)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\\
&\times\Big(e^{\varepsilon_8}+e^{-\varepsilon_8}-2\Big)+\tfrac{k}{2}\Big(-\varepsilon_5\cos(\varepsilon_6)+\varepsilon_3\sin(\varepsilon_6)\Big)\Big(e^{\varepsilon_7}-e^{-\varepsilon_7}\Big)+\tfrac{k}{2}\Big(\varepsilon_5\sin(\varepsilon_6)+\varepsilon_3\cos(\varepsilon_6)\Big)\Big(e^{\varepsilon_8}-e^{-\varepsilon_8}\Big)\Bigg)\\
&+ a_5\Bigg(\Big(-\varepsilon_5\cos(\varepsilon_6)+\varepsilon_3\sin(\varepsilon_6)\Big)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8}-\tfrac{1}{2}e^{-\varepsilon_8}\Big)+\Big(\varepsilon_5\sin(\varepsilon_6)+\varepsilon_3\cos(\varepsilon_6)\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)\Bigg)\\
&+ a_6\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)+a_7\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{-\varepsilon_8}-\tfrac{1}{2}e^{\varepsilon_8}\Big)+a_8\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg]\mathbf{X_6}\\
+&\Bigg[a_1\cos(\varepsilon_6)+a_2\Bigg(\Bigg(-\varepsilon_3\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)-\varepsilon_2\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg)\Big(\tfrac{1}{2}e^{\varepsilon_8}-\tfrac{1}{2}e^{-\varepsilon_8}\Big)\\
&+\varepsilon_2\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8}-\tfrac{1}{2}e^{-\varepsilon_8}\Big)\Bigg)+a_3\Bigg(-\varepsilon_3\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)-\varepsilon_2\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg)\\
&+a_4\Bigg(\tfrac{k}{2}\varepsilon_2\sin(\varepsilon_6)\Big(e^{\varepsilon_8}-e^{-\varepsilon_8}\Big)-\varepsilon_3 k-\tfrac{k}{2}\varepsilon_2\cos(\varepsilon_6)\Big(e^{\varepsilon_7}-e^{-\varepsilon_7}\Big)-\tfrac{k}{2}\varepsilon_3\cos(\varepsilon_6)\Big(e^{\varepsilon_7}+e^{-\varepsilon_7}-2\Big)\\
&+\tfrac{k}{2}\Big(e^{\varepsilon_8}+e^{-\varepsilon_8}-2\Big)\Bigg(-\varepsilon_3\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)-\varepsilon_2\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg)\Bigg)+a_5\Bigg(\varepsilon_2\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)\\
&+\Big(\tfrac{1}{2}e^{\varepsilon_8}-\tfrac{1}{2}e^{-\varepsilon_8}\Big)\Bigg(-\varepsilon_3\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)-\varepsilon_2\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg)\Bigg)+a_6\Bigg(\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{-\varepsilon_8}-\tfrac{1}{2}e^{\varepsilon_8}\Big)\\
&-\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)\Bigg)+a_7\Bigg(\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)-\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\\
&\times\Big(\tfrac{1}{2}e^{-\varepsilon_8}-\tfrac{1}{2}e^{\varepsilon_8}\Big)\Bigg)-a_8\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg]\mathbf{X_7}\\
+&\Bigg[a_1\sin(\varepsilon_6)+a_2\Bigg(-\varepsilon_2\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8}-\tfrac{1}{2}e^{-\varepsilon_8}\Big)-\Bigg(\varepsilon_5\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)+\varepsilon_2\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)\Bigg)\\
&+a_3\Bigg(-\varepsilon_5\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)-\varepsilon_2\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg)\\
&+a_4\Bigg(-\tfrac{k}{2}\varepsilon_2\cos(\varepsilon_6)\Big(e^{\varepsilon_8}-e^{-\varepsilon_8}\Big)-\varepsilon_5 k-\tfrac{k}{2}\varepsilon_2\sin(\varepsilon_6)\Big(e^{\varepsilon_7}-e^{-\varepsilon_7}\Big)-\tfrac{k}{2}\varepsilon_5\Big(e^{\varepsilon_7}+e^{-\varepsilon_7}-2\Big)\\
&+\tfrac{k}{2}\Big(e^{\varepsilon_8}+e^{-\varepsilon_8}-2\Big)\Bigg(-\varepsilon_5\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)-\varepsilon_2\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg)\Bigg)\\
&+a_5\Bigg(-\varepsilon_2\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)+\Big(\tfrac{1}{2}e^{\varepsilon_8}-\tfrac{1}{2}e^{-\varepsilon_8}\Big)\Bigg(-\varepsilon_5\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)-\varepsilon_2\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg)\Bigg)\\
&+a_6\Bigg(\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{-\varepsilon_8}-\tfrac{1}{2}e^{\varepsilon_8}\Big)+\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)\Bigg)+a_7\Bigg(\sin(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_8}+\tfrac{1}{2}e^{-\varepsilon_8}\Big)\\
&+\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}-\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Big(\tfrac{1}{2}e^{-\varepsilon_8}-\tfrac{1}{2}e^{\varepsilon_8}\Big)\Bigg)+a_8\cos(\varepsilon_6)\Big(\tfrac{1}{2}e^{\varepsilon_7}+\tfrac{1}{2}e^{-\varepsilon_7}\Big)\Bigg]\mathbf{X_8}
\end{aligned}
$$

At this stage, we can simplify $\mathbf{X}$ in the following manner:

If $a_3 \neq 0$ we can make the coefficients of X_5, X_6 and X_7 vanish by $F_6^{\varepsilon_6}$, $F_5^{\varepsilon_5}$ and $F_2^{\varepsilon_2}$. By inserting $\varepsilon_6 = -\arctan\big(\frac{a_5}{a_3}\big)$, $\varepsilon_5 = \frac{a_6}{a_3}$ and $\varepsilon_2 = \frac{a_7}{a_3}$, respectively. Scaling $\mathbf{X}$ if necessary, we can suppose that $a_3 = 1$. So, $\mathbf{X}$ is reduced to the case (1).

If $a_3 = 0$ and $a_5 \neq 0$ we can make the coefficients of X_8, and X_6 vanish by $F_2^{\varepsilon_2}$ and $F_3^{\varepsilon_3}$. By setting $\varepsilon_2 = \arctan\big(\frac{a_8}{a_5}\big)$ and $\varepsilon_3 = -\frac{a_6}{a_5}$, respectively. Scaling $\mathbf{X}$ if necessary, we can presume that $a_5 = 1$. So, $\mathbf{X}$ is simplified to the case (2).

If $a_3 = 0$, $a_5 = 0$ and $a_7 \neq 0$ we can make the coefficient of X_8, vanish by $F_6^{\varepsilon_6}$. By considering $\varepsilon_6 = -\arctan\big(\frac{a_8}{a_7}\big)$. Scaling $\mathbf{X}$ if necessary, we can assume that $a_7 = 1$. So, $\mathbf{X}$ is simplified to the case (3).

If $a_3 = 0$, $a_5 = 0$ and $a_7 = 0$ then $\mathbf{X}$ is simplified to the case (4). □

4. Computation of the conservation laws

The foremost prominence of Noether symmetries is apparent from the reputable Noether's theorem. This theorem crucially depends on the accessibility of a Lagrangian and the associated variational symmetries which leave the action integral invariant. In accordance with this theorem, there is a strategy which connects the constants of the motion of a given Lagrangian system to its corresponding symmetry transformations [2, 6].
In this section, first of all, we will determine all the conserved flows corresponding to the variational symmetries $\mathbf{X_1}, \ldots, \mathbf{X_8}$ obtained in corollary 2.2. Each of these conserved quantities provides a conservation law for the system of geodesic equations (5).
For instance, for the variational symmetry $\mathbf{X_7} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} + kz\frac{\partial}{\partial \phi}$, we obtain this conserved vector:

$$
\begin{aligned}
T^7 &= \xi L + (\Omega^1 - \dot{t}\xi)\frac{\partial L}{\partial \dot{t}} + (\Omega^2 - \dot{r}\xi)\frac{\partial L}{\partial \dot{r}} + (\Omega^3 - \dot{z}\xi)\frac{\partial L}{\partial \dot{z}} + (\Omega^4 - \dot{\varphi}\xi)\frac{\partial L}{\partial \dot{\phi}} + (\Omega^5 - \dot{\psi}\xi)\frac{\partial L}{\partial \dot{\psi}} - A \\
&= z\Big(\frac{2\dot{t}}{\sqrt{1+k^2r^2}} + \frac{2kr^2\dot{\phi}}{\sqrt{1+k^2r^2}}\Big) - 2t\sqrt{1+k^2r^2}\ \dot{z} + kz\Big(2(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2})\dot{\phi} + \frac{2kr^2\ \dot{t}}{\sqrt{1+k^2r^2}}\Big) - c.
\end{aligned}
$$

Analogously, we have calculated the conserved vectors of the other computed variational symmetries. The conclusions are thoroughly demonstrated in Table 3.

Table 3: Conservation laws of (5) resulted from the Noether's theorem

	Noether Symmetry	Conserved Vectors
1	$\mathbf{X_1} = \partial s$	$T^1 = -\frac{\dot{t}^2}{\sqrt{1+k^2r^2}} + \frac{\dot{r}^2}{1+k^2r^2} + \sqrt{1+k^2r^2}\ \dot{z}^2 - \frac{2kr^2}{\sqrt{1+k^2r^2}}\ \dot{t}\dot{\phi} - (\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2})\dot{\phi}^2 + \sqrt{1+k^2r^2}\ \dot{\psi}^2 - c$
2	$\mathbf{X_2} = \partial t$	$T^2 = \frac{2\dot{t}}{\sqrt{1+k^2r^2}} + \frac{2kr^2\ \dot{\phi}}{\sqrt{1+k^2r^2}} - c$
3	$\mathbf{X_3} = \partial z$	$T^3 = -2\sqrt{1+k^2r^2}\ \dot{z} - c$
4	$\mathbf{X_4} = \partial \varphi$	$T^4 = 2\Big(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2}\Big)\dot{\phi} + \frac{2kr^2\ \dot{t}}{\sqrt{1+k^2r^2}} - c$
5	$\mathbf{X_5} = \partial \psi$	$T^5 = -2\sqrt{1+k^2r^2}\ \dot{\psi} - c$
6	$\mathbf{X_6} = \psi\partial z - z\partial\psi$	$T^6 = -2\psi\sqrt{1+k^2r^2}\ \dot{z} + 2z\sqrt{1+k^2r^2}\ \dot{\psi} - c$
7	$\mathbf{X_7} = z\partial t + t\partial z + kz\partial\phi$	$T^7 = z\Big(\frac{2\dot{t}}{\sqrt{1+k^2r^2}} + \frac{2kr^2\dot{\phi}}{\sqrt{1+k^2r^2}}\Big) - 2t\sqrt{1+k^2r^2}\ \dot{z} + kz\Big(2(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2})\dot{\phi} + \frac{2kr^2\ \dot{t}}{\sqrt{1+k^2r^2}}\Big) - c$
8	$\mathbf{X_8} = \psi\partial t + k\psi\partial\phi + t\partial\psi$	$T^8 = \psi\Big(\frac{2\dot{t}}{\sqrt{1+k^2r^2}} + \frac{2kr^2\dot{\phi}}{\sqrt{1+k^2r^2}}\Big) - 2t\sqrt{1+k^2r^2}\ \dot{\psi} + k\psi\Big(2(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2})\dot{\phi} + \frac{2kr^2\ \dot{t}}{\sqrt{1+k^2r^2}}\Big) - c$

Subsequently, the conserved flows of those infinitesimal generators resulting via establishing an optimal system of one-dimensional subalgebras of the Lie algebra of variational symmetries as illustrated in theorem 3.1 are totally computed.

(1) For the symmetry operator $a_1\mathbf{X_1} + a_2\mathbf{X_2} + \mathbf{X_3} + a_4\mathbf{X_4} + a_8\mathbf{X_8} = a_1\frac{\partial}{\partial s} + (a_2 + a_8\psi)\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + (a_4 + a_8\psi k)\frac{\partial}{\partial \varphi} + a_8 t\frac{\partial}{\partial \psi}$, $A = c$ by applying Noether's theorem the following conservation law is resulting:

$$D_s\Bigg(\frac{a_1}{\sqrt{k^2r^2+1}}\ \dot{t}^2-\frac{a_1}{k^2r^2+1}\ \dot{r}^2-a_1\sqrt{k^2r^2+1}\ \dot{z}^2+a_1\Bigg(\frac{k^2r^4}{\sqrt{k^2r^2+1}}-r^2\sqrt{k^2r^2+1}\Bigg)\ \dot{\varphi}^2+\frac{2a_1r^2k}{\sqrt{k^2r^2+1}}\ \dot{t}\ \dot{\varphi}$$
$$-a_1\sqrt{k^2r^2+1}\ \dot{\psi}^2+(-a_1\dot{t}+a_8\psi+a_2)\Bigg(\frac{2}{\sqrt{k^2r^2+1}}\ \dot{t}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\ \dot{\varphi}\Bigg)-2(-a_1z+1)\sqrt{k^2r^2+1}\ \dot{z}$$
$$+\frac{2a_1}{k^2r^2+1}\ \dot{r}^2+\Big(a_8k\psi-a_1\dot{\varphi}+a_4\Big)\Bigg(\frac{2r^4k^2}{\sqrt{k^2r^2+1}}\ \dot{\varphi}-2r^2\sqrt{k^2r^2+1}\ \dot{\varphi}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\ \dot{t}\Bigg)$$
$$-2(-a_1\dot{\psi}+a_8t)\sqrt{k^2r^2+1}\ \dot{\psi}-c\Bigg)=0.$$

(2) For the symmetry operator $a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_5}+a_7\mathbf{X_7}=a_1\frac{\partial}{\partial s}+(a_2+a_7z)\frac{\partial}{\partial t}+a_7t\frac{\partial}{\partial z}+(a_4+a_7zk)\frac{\partial}{\partial \varphi}+\frac{\partial}{\partial \psi}$, $A=c$ by applying Noether's theorem the following conservation law is resulting:

$$D_s\Bigg(\frac{a_1}{\sqrt{k^2r^2+1}}\ \dot{t}^2-\frac{a_1}{k^2r^2+1}\ \dot{r}^2-a_1\sqrt{k^2r^2+1}\ \dot{z}^2+a_1\Bigg(\frac{k^2r^4}{\sqrt{k^2r^2+1}}-r^2\sqrt{k^2r^2+1}\Bigg)\ \dot{\varphi}^2+\frac{2a_1r^2k}{\sqrt{k^2r^2+1}}\ \dot{t}\ \dot{\varphi}$$
$$-a_1\sqrt{k^2r^2+1}\ \dot{\psi}^2+(-a_1\dot{t}+a_7z+a_2)\Bigg(\frac{2}{\sqrt{k^2r^2+1}}\ \dot{t}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\ \dot{\varphi}\Bigg)-2(-a_1z+a_7t)\sqrt{k^2r^2+1}\ \dot{z}$$
$$+\frac{2a_1}{k^2r^2+1}\ \dot{r}^2+\Big(a_7kz-a_1\dot{\varphi}+a_4\Big)\Bigg(\frac{2r^4k^2}{\sqrt{k^2r^2+1}}\ \dot{\varphi}-2r^2\sqrt{k^2r^2+1}\ \dot{\varphi}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\ \dot{t}\Bigg)$$
$$-2(-a_1\dot{\psi}+1)\sqrt{k^2r^2+1}\ \dot{\psi}-c\Bigg)=0.$$

(3) For the symmetry operator $a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+a_6\mathbf{X_6}+\mathbf{X_7}=a_1\frac{\partial}{\partial s}+(a_2+z)\frac{\partial}{\partial t}+(a_6\psi+t)\frac{\partial}{\partial z}+(a_4+zk)\frac{\partial}{\partial \varphi}-a_6z\frac{\partial}{\partial \psi}$, $A=c$ by applying Noether's theorem the following conservation law is resulting:

$$D_s\Bigg(\frac{a_1}{\sqrt{k^2r^2+1}}\ \dot{t}^2-\frac{a_1}{k^2r^2+1}\ \dot{r}^2-a_1\sqrt{k^2r^2+1}\ \dot{z}^2+a_1\Bigg(\frac{k^2r^4}{\sqrt{k^2r^2+1}}-r^2\sqrt{k^2r^2+1}\Bigg)\dot{\varphi}^2+\frac{2a_1r^2k}{\sqrt{k^2r^2+1}}\ \dot{t}\ \dot{\varphi}$$
$$-a_1\sqrt{k^2r^2+1}\ \dot{\psi}^2+(-a_1\dot{t}+a_2+z)\Bigg(\frac{2}{\sqrt{k^2r^2+1}}\ \dot{t}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\ \dot{\varphi}\Bigg)-2(-a_1\dot{z}+a_6\psi+t)\sqrt{k^2r^2+1}\ \dot{z}$$
$$+\frac{2a_1}{k^2r^2+1}\ \dot{r}^2+(-a_1\dot{\varphi}+kz+a_4)\Bigg(\frac{2r^4k^2}{\sqrt{k^2r^2+1}}\ \dot{\varphi}-2r^2\sqrt{k^2r^2+1}\ \dot{\varphi}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\ \dot{t}\Bigg)$$
$$+2\Big(a_1\dot{\psi}+a_6z\Big)\sqrt{k^2r^2+1}\ \dot{\psi}-c\Bigg)=0.$$

(4) For the symmetry operator $a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+a_6\mathbf{X_6}+a_8\mathbf{X_8}=a_1\frac{\partial}{\partial s}+\Big(a_2+a_8\psi\Big)\frac{\partial}{\partial t}+a_6\psi\frac{\partial}{\partial z}+\Big(a_4+a_8\psi k\Big)\frac{\partial}{\partial \varphi}+\Big(-a_6z+a_8t\Big)\frac{\partial}{\partial \psi}$, $A=c$ by applying Noether's theorem the following conservation law is resulting:

$$D_s\Bigg(\frac{a_1}{\sqrt{k^2r^2+1}}\ \dot{t}^2-\frac{a_1}{k^2r^2+1}\ \dot{r}^2-a_1\sqrt{k^2r^2+1}\ \dot{z}^2+a_1\Bigg(\frac{k^2r^4}{\sqrt{k^2r^2+1}}-r^2\sqrt{k^2r^2+1}\Bigg)\ \dot{\varphi}^2+\frac{2a_1r^2k}{\sqrt{k^2r^2+1}}\ \dot{t}\ \dot{\varphi}$$
$$-a_1\sqrt{k^2r^2+1}\ \dot{\psi}^2+\Big(-a_1\dot{t}+a_2+a_8\psi\Big)\Bigg(\frac{2}{\sqrt{k^2r^2+1}}\ \dot{t}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\ \dot{\varphi}\Bigg)-2\Big(-a_1z+a_6\psi\Big)\sqrt{k^2r^2+1}\ \dot{z}$$
$$+\frac{2a_1}{k^2r^2+1}\ \dot{r}^2+\Big(a_8k\psi-a_1\dot{\varphi}+a_4\Big)\Bigg(\frac{2r^4k^2}{\sqrt{k^2r^2+1}}\ \dot{\varphi}-2r^2\sqrt{k^2r^2+1}\ \dot{\varphi}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\ \dot{t}\Bigg)$$
$$-2\Big(-a_1\dot{\psi}-a_6z+a_8t\Big)\sqrt{k^2r^2+1}\ \dot{\psi}-c\Bigg)=0.$$

$$D_s\Bigg(\frac{a_1}{\sqrt{k^2r^2+1}}\,\dot{t}^2-\frac{a_1}{k^2r^2+1}\,\dot{r}^2-a_1\sqrt{k^2r^2+1}\,\dot{z}^2+a_1\Big(\frac{k^2r^4}{\sqrt{k^2r^2+1}}-r^2\sqrt{k^2r^2+1}\Big)\,\dot{\varphi}^2+\frac{2a_1r^2k}{\sqrt{k^2r^2+1}}\,\dot{t}\,\dot{\varphi}$$
$$-a_1\sqrt{k^2r^2+1}\,\dot{\psi}^2+\Big(-a_1\dot{t}+a_2+a_8\psi\Big)\Big(\frac{2}{\sqrt{k^2r^2+1}}\,\dot{t}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\,\dot{\varphi}\Big)-2\Big(-a_1z+a_6\psi\Big)\sqrt{k^2r^2+1}\,\dot{z}$$
$$+\frac{2a_1}{k^2r^2+1}\,\dot{r}^2+\Big(a_8k\psi-a_1\dot{\varphi}+a_4\Big)\Big(\frac{2r^4k^2}{\sqrt{k^2r^2+1}}\,\dot{\varphi}-2r^2\sqrt{k^2r^2+1}\,\dot{\varphi}+\frac{2r^2k}{\sqrt{k^2r^2+1}}\,\dot{t}\Big)$$
$$-2\Big(-a_1\dot{\psi}-a_6z+a_8t\Big)\sqrt{k^2r^2+1}\,\dot{\psi}-c\Bigg)=0.$$

Conclusion

The Kaluza-Klein theory can be fundamentally considered as a precursor to string theory and is undoubtedly the first example of a unification theory by adding an extra dimension. Literally, Kaluza-Klein theories describe a physically viable plan for the unification of gravity with other interactions which is originally based on the hypothesis that the gauge symmetries are intrinsically geometrical. Predominantly, after Gödel's relativistic model proposal for rotating dust universe, inquiring into rotating fluids in the framework of general relativity has acquired remarkable contemplation. In this research, a thoroughgoing investigation of variational symmetries and conservation laws for a specific vacuum solution of the (4+1)-dimensional Einstein field equations with negative cosmological constant is presented. For this purpose, firstly by exploring the Lagrangian which is straightforwardly characterized from the metric, the corresponding geodesic equations as the Euler Lagrange equations are totally computed. Subsequently, we have obtained the point generators of the one parameter Lie groups of transformations which leave the action integral of the Lagrangian invariant, viz., variational symmetries. It is demonstrated that the Lie algebra of local symmetries of the system of geodesic equations is non-solvable and not semi-simple and a brief discussion regarding the algebraic structure of the derived quotient Lie algebra is proposed. Principally, a thorough classification of symmetry subalgebras for our analyzed system of geodesic equations is constructed via the adjoint representation group. As a main consequence, the optimal system of group invariant solutions, can be predominantly reckoned as the minimal list from which all the other invariant solutions of one-dimensional subalgebras are characterized simply by virtue of transformations. Eventually, considering the point that the symmetries of the metric of our analyzed five-dimensional space-time lead to the constants of motion for the point particles, all the corresponding local conservation laws of the computed variational symmetries are totally calculated.

References

[1] I. Barukčić, *N-th index D-dimensional Einstein gravitational field equations*, BoD–Books on Demand, Norderstedt, Germany, 2020.

[2] G. W. Bluman, A. F. Cheviakov, and S. C. Anco, *Applications of symmetry methods to partial differential equations*, vol. 168 of Applied Mathematical Sciences, Springer, New York, 2010.

[3] W. Davidson, *A Petrov type I cylindrically symmetric solution for perfect fluid in steady rigid body rotation*, Classical Quantum Gravity, 13 (1996), pp. 283–287.

[4] K. Gödel, *An example of a new type of cosmological solutions of Einstein's field equations of gravitation*, Rev. Modern Physics, 21 (1949), pp. 447–450.

[5] N. H. Ibragimov, *Elementary Lie group analysis and ordinary differential equations*, vol. 4 of Wiley Series in Mathematical Methods in Practice, John Wiley & Sons, Ltd., Chichester, 1999.

[6] A. H. Kara and F. M. Mahomed, *A basis of conservation laws for partial differential equations*, J. Nonlinear Math. Phys., 9 (2002), pp. 60–72. Special issue in honour of P. G. L. Leach on the occasion of his 60th birthday.

[7] A. Kushner, V. Lychagin, and V. Rubtsov, *Contact Geometry and Nonlinear Differential Equations*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2006.

[8] P. J. Olver, *Applications of Lie groups to differential equations*, vol. 107 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1986.

[9] L. V. Ovsiannikov, *Group analysis of differential equations*, Academic Press, Inc., New York-London, 1982. Translated from the Russian by Y. Chapovsky, Translation edited by William F. Ames.

[10] A. Prakash, *Studies on gravitational field equations and important results of relativistic cosmology*, Lulu Publication, NC, United States, 2019.

[11] R. Tikekar and L. K. Patel, *Rotating cylindrically symmetric Kaluza-Klein fluid model*, Pramana J. Phys., 55 (2000), pp. 361–368.

Please cite this article using:

Fatemeh Ahangari, Analysis of the vacuum solution of the five-dimensional Einstein field equations with negative cosmological constant via variational symmetries, AUT J. Math. Comput., 6(2) (2025) 109-120
https://doi.org/10.22060/AJMC.2024.22702.1185