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On The Simulation of Partial Differential Equations Using the Hybrid of Fourier Transform and Homotopy Perturbation Method

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ABSTRACT

In the present work, a hybrid of Fourier transform and homotopy perturbation method is developed for solving the non-homogeneous partial differential equations with variable coefficients. The Fourier transform is employed with combination of homotopy perturbation method (HPM), the so called Fourier transform homotopy perturbation method (FTHPM) to solve the partial differential equations. The closed form solutions obtained from the series solution of recursive sequence forms are obtained. We show that the solutions to the non-homogeneous partial differential equations are valid for the entire range of problem domain. However the validity of the solutions using the previous semi-analytical methods in the entire range of problem domain fails to exist. This is the deficiency of the previous HPMs caused by unsatisfied boundary conditions that is overcome by the new method, the Fourier transform homotopy perturbation method. Moreover, it is shown that solutions approach very rapidly to the exact solutions of the partial differential equations. The effectiveness of the new method for three non-homogenous differential equations with variable coefficients is shown schematically. The very rapid approach to the exact solutions is also shown schematically.

KEYWORDS

Fourier transformation, homotopy perturbation method, Non-homogeneous partial differential equation

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1. INTRODUCTION

Most of the governing differential equations do not have analytical solutions in genuine events in nature and Physics. Besides, regarding the non-linear feature and also variable coefficients of afore-mentioned equations, the analytical solutions are not sought. Therefore, the researchers are driven to approximate solutions, obtaining from semi analytical methods such as the homotopy perturbation method (HPM) [1-23], variational iteration method (VIM) and Adomian decomposition method (ADM) [24-32]. By application of standard homotopy and perturbation methods, the HPM [1-23] is developed and modified by some researchers, namely [5-20], to acquire accuracy, convergence rapidity and decrease in computational work. The modified homotopy perturbation method has got its way in solution of many actual problems in real situations, although the validity of the solutions is limited to small range of space and time domains in the problems. This means that, the unsatisfied boundary conditions in the solution of HPM and other semi-analytical methods is not effective in the last results [14-15]. Consequently, to overcome this inherent insufficiency in the solutions of semi-analytical methods, an enormous computational endeavour has been made. Madani et al. [15] recently developed a hybrid of Laplace transform and homotopy perturbation method (LHPM) for solving of one dimensional non-homogeneous partial differential equation. The principle achievements of their work, was to get solutions for the differential equation with wide range of validity in the problem domain [15]. They proved, that the results of LHPM method, is more accurate in the range of the problem domain comparing HPM method [15].

However, most recently Nourazar et al [33] proposed a new idea of combination of the Fourier transform and the homotopy perturbation method, FTHPM, to solving the partial linear and non-linear differential equations. They showed that using the FTHPM is capable of solving the partial differential equations with greater accuracy than the previous homotopy perturbation method.

The chief intention of this current work is to get better validity of solution in small range of problem domain by the so called new proposal, FTHPM, given by Nourazar et al [33]. In this method the deficiency in the semi-analytical methods such as HPM occurring by boundary conditions which are earlier satisfied in one dimension [14-15], are improved. By combining the Fourier transform and HPM method, the new modified HPM is established. In the new modified HPM, all the conditions over the entire range of time and space problem domains are satisfied. By mathematic relations it is proved that as the number of partial sum of the infinite series of approximate solution are increased, the rapid approach to the closed form solution is obtainable. This solution is valid for the whole range of the problem domain. Three different time dependent non-homogeneous partial differential equations, by using the new modified HPM (FTHPM), are analyzed and solved in this work. The exact solutions of the problems as the closed form solutions are acquired. Besides, the ways of quick approaches of the solution to closed form solution are shown in the case studies schematically.

2. HOMOTOPY PERTURBATION METHOD (HPM)

The homotopy perturbation method (HPM) is originally initiated by He [1-13]. This is a combination of the classical perturbation technique and homotopy technique. The basic idea of the HPM for solving nonlinear differential equations is as follow; consider the following differential equation:

$$E(u) = 0, \tag{1}$$

where E is any differential operator. We construct a homotopy as follow:

$$H(u,p) = F(u) + p\big(E(u) - F(u)\big). \tag{2}$$

Where $F(u)$ is a functional operator with the known solution v_0. It is clear that when p is equal to zero then $H(u,0) = F(u) = 0$, and when p is equal to 1, then $H(u,1) = E(u) = 0$. It is worth noting that as the embedding parameter p increases monotonically from zero to unity the zero order solution $v_0$continuously deforms into the original problem $E(u) = 0$. The embedding parameter, $p \in [0,1]$, is considered as an expanding parameter [1-14]. In the homotopy perturbation method the embedding parameter p is used to get series expansion for solution as:

$$u = \sum_{i=0}^{\infty} p^i\, v_i = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots \tag{3}$$

when $p \to 1$, then Eq. (3) becomes the approximate solution to Eq. (1) as:

$$u = v_0 + v_1 + v_2 + v_3 + \cdots \tag{4}$$

The series (4) is a convergent series and the rate of convergence depends on the nature of Eq. (1) [1-14]. It is also assumed that Eq. (4) has a unique solution and by comparing the like powers of p the solution of various orders is obtained.

3. BASIC IDEA OF FTHPM

The general forms of one-dimensional non homogeneous partial differential equations with variable coefficients are considered for illustrating the basic idea of the present method.

$$\mu(t)\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\varphi(x,t), \tag{5}$$

and

$$\mu(t)\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\varphi(x,t), \tag{6}$$

The initial condition is as follows:

$$u(x,0)=f(x), \tag{7}$$

And the boundary conditions are:

$$u(0,t)=g_0(t), \qquad u_x(0,t)=g_1(t) \tag{8}$$

Here, it is to be noted that the FTHPM is a technique that may be used to solve the problems with the initial-initial conditions only.

Applying Fourier transform to Eq. (5) and Eq. (7) we obtain as:

$$\mu(t)\frac{d\hat{u}}{dt}+\hat{u}_x(0,t)+i\omega\hat{u}(0,t)+\omega^2\hat{u}-\hat{\varphi}(\omega,t)=0 \tag{9}$$
$$\hat{u}(\omega,0)=\hat{f}(\omega),$$

In Eq. (9) $\hat{u}$ is the Fourier transform of u. We develop a homotopy as follow:

$$H(\hat{u},p)=\mu(t)\frac{d\hat{u}}{dt}-\hat{\varphi}_1(\omega,t)+p\big(\hat{u}_x(0,t)+i\omega\hat{u}(0,t)+\omega^2\hat{u}-\hat{\varphi}_2(\omega,t)\big), \tag{10}$$

where,

$$\hat{\varphi}(\omega,t)=\hat{\varphi}_1(\omega,t)+\hat{\varphi}_2(\omega,t). \tag{11}$$

when p is equal to the unity, Eq. (10) converts back to the main differential equation and when p is equal to zero the zero order solution is obtained. According to the concept of the HPM the solution of Eq. (10) can be expressed in series solution as:

$$\hat{u}(\omega,t)=\sum_{i=0}^{\infty}p^i\hat{v}_i(\omega,t)=\hat{v}_0(\omega,t)+p^1\hat{v}_1(\omega,t)+p^2\hat{v}_2(\omega,t)+\cdots \tag{12}$$

The inverse Fourier transform is taken from both sides of Eq. (12) as:

$$u(x,t)=\sum_{i=0}^{\infty}p^i v_i(x,t)=v_0(x,t)+p^1v_1(x,t)+p^2v_2(x,t)+\cdots \tag{13}$$

When $p\rightarrow 1$ the approximate solution of Eq. (5) can be readily obtained as:

$$u(x,t)=\lim_{p\to 1}\sum_{i=0}^{\infty}p^i v_i(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+\cdots \tag{14}$$

4. CASE STUDY

We solve three one-dimensional transient and non-homogeneous partial differential equations with variable coefficients to demonstrate the effectiveness and the strength of the present method, FTHPM, in the entire range of problem domain.

Example 1. Consider the following non-homogeneous partial differential equation as the first case study problem:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+e^{-x}(cos(t)-sin(t)),$$
$$u(x,0)=x, \tag{15}$$
$$u(0,t)=sin(t)\ ,\quad u_x(0,t)=1-sin(t).$$

Eq. (15) is the Cauchy reaction-diffusion equation that has a wide range of applications in physical phenomenon of real life. The application of the Cauchy reaction-diffusion equations may be categorized as the spatial effects of ecology that can be modeled by the Cauchy reaction-diffusion equation. Different types of ecological phenomena such as the minimal patch size necessary to sustain a population, wave fronts propagation of biological invasions, and the formation of spatial patterns in the distributions of populations are supported and analyzed by Cauchy reaction-diffusion model. The Cauchy reaction-diffusion equations are also used in the modeling of chemical reactions in combustion phenomena. The interactions between the convection and dispersion generating the solitary waves, compactions are studied with the aid of the Cauchy problem of reaction-diffusion model.

By applying the Fourier transform to Eq. (15) we obtain as follow:

$$\frac{\partial \hat{u}}{\partial t} + (sin(t) - cos(t))\left(\frac{1}{i\omega + 1}\right) + (1 - sin(t)) + i\omega\, sin(t) + \omega^2 \hat{u} = 0, \qquad (16)$$

$$\hat{u}(\omega, 0) = -\frac{1}{\omega^2}$$

We construct a homotopy as:

$$H(u,p) = \frac{\partial \hat{u}}{\partial t} + cos(t)\left(\frac{1}{i\omega + 1}\right) + p\left(1 + \omega^2 \hat{u} + sin(t)\left(\frac{1}{i\omega + 1} + i\omega - 1\right)\right) = 0, \ \ p \in [0,1]. \qquad (17)$$

The solution of Eq. (17) can be written in power series of p as:

$$\hat{u} = \hat{u}(\omega, t) = \hat{v}_0(\omega, t) + p^1 \hat{v}_1(\omega, t) + p^2 \hat{v}_2(\omega, t) + \qquad (18)$$

Substituting Eq. (18) into Eq. (17) and setting the sum of terms of identical powers of p equal to zero, we obtain:

$$p^0: \ \frac{d\hat{v}_0}{dt} - cos(t)\left(\frac{1}{i\omega + 1}\right) = 0 \quad , \quad \hat{v}_0(\omega, 0) = \frac{-1}{\omega^2}$$

$$p^1: \ \frac{d\hat{v}_1}{dt} + \omega^2 \hat{v}_0 + 1 + sin(t)\left(\frac{1}{i\omega + 1} + i\omega - 1\right) = 0 \quad , \quad \hat{v}_1(\omega, 0) = 0 \qquad (19)$$

$$p^2: \ \frac{d\hat{v}_2}{dt} + \omega^2 \hat{v}_1 = 0 \quad , \quad \hat{v}_2(\omega, 0) = 0$$

$$p^3: \ \frac{d\hat{v}_3}{dt} + \omega^2 \hat{v}_2 = 0 \quad , \quad \hat{v}_3(\omega, 0) = 0$$

$$p^4: \ \frac{d\hat{v}_4}{dt} + \omega^2 \hat{v}_3 = 0 \quad , \quad \hat{v}_4(\omega, 0) = 0$$

.
.

$$p^i: \ \frac{d\hat{v}_i}{dt} + \omega^2 \hat{v}_{i-1} = 0 \quad , \quad \hat{v}_i(\omega, 0) = 0 \ , \ i = 2, 3 \ \ldots$$

Solving for the zero order term, $\hat{v}_0(\omega, t)$, we obtain:

$$\hat{v}_0(\omega, t) = \frac{i - \omega - i\omega^2\, sin(t)}{\omega^2(-i + \omega)} \qquad (20)$$

The inverse Fourier transform of Eq. (20) is obtained as:

$$S_0(x, t) = v_0(x, t) = x + e^{-x}\, sin(t). \qquad (21)$$

This is the exact solution of the problem, Eq. (15). It is worth noting that in Eq. (19), the inverse Fourier transforms of the higher orders than zero are zero. This exact solution is not possible to be achieved by the HPM and modified HPM, because the conditions cannot completely be satisfied in there. In other words, even the boundary conditions of the problem change totally; the solution using the HPM will not be affected at all. Moreover, the very rapid convergence of the results toward the exact solution is achieved using the FTHPM. This shows that the method is capable of obtaining the solution of non-homogeneous parabolic differential equations with very rapid convergence towards the exact solutions.

Example 2. The below differential equation is considered as the second case study problem:

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + u - e^{-x}(1 + 2t) = 0, \qquad (22)$$

$$u(x, 0) = x,$$

$$u(0, t) = t\,, \ \ u_x(0, t) = e^{-t} - t.$$

The Fourier transform of Eq. (22) is:

$$\frac{\partial \hat{u}}{\partial t} + \hat{u} - e^{-t} + t - i\omega t - \omega^2 \hat{u} - \left(\frac{1 + 2t}{1 + i\omega}\right) = 0, \qquad (23)$$

$$\hat{u}(\omega, 0) = \frac{-1}{\omega^2}$$

We construct a homotopy as:

$$H(\hat{u},p)=\frac{\partial \hat{u}}{\partial t}+\hat{u}+p\left(-e^{-t}+t-i\omega t-\omega^2\hat{u}-\left(\frac{1+2t}{1+i\omega}\right)\right)=0,\quad p\in[0,1]\,. \tag{24}$$

The solution of equation (24) can be written in power series of p as:

$$\hat{u}=\hat{u}(\omega,t)=\hat{v}_0(\omega,t)+p^1\hat{v}_1(\omega,t)+p^2\hat{v}_2(\omega,t)+ \tag{25}$$

By substituting of Eq. (25) into Eq. (24) and setting the sum of terms of identical powers in p equal to zero, we obtain:

$$\begin{aligned}
&p^0:\ \frac{d\hat{v}_0}{dt}+\hat{v}_0=0 \quad,\quad \hat{v}_0(\omega,0)=\frac{-1}{\omega^2}\\
&p^1:\ \frac{d\hat{v}_1}{dt}+\hat{v}_1-\omega^2\hat{v}_0+(-e^{-t}+t-i\omega t)-\left(\frac{1+2t}{1+i\omega}\right)=0 \quad,\quad \hat{v}_1(\omega,0)=0\\
&p^2:\ \frac{d\hat{v}_2}{dt}+\hat{v}_2-\omega^2\hat{v}_1=0 \quad,\quad \hat{v}_2(\omega,0)=0\\
&p^3:\ \frac{d\hat{v}_3}{dt}+\hat{v}_3-\omega^2\hat{v}_2=0 \quad,\quad \hat{v}_3(\omega,0)=0\\
&p^4:\ \frac{d\hat{v}_4}{dt}+\hat{v}_4-\omega^2\hat{v}_3=0 \quad,\quad \hat{v}_4(\omega,0)=0\\
&..\\
&p^i:\ \frac{d\hat{v}_i}{dt}+\hat{v}_i-\omega^2\hat{v}_{i-1}=0 \quad,\quad \hat{v}_i(\omega,0)=0\ ,\ i=2,3\ \ldots
\end{aligned} \tag{26}$$

By solving the recursive differential equations, Eq. (26), we obtain the followings:

$$\begin{aligned}
&\hat{v}_0(\omega,t)=\frac{-1}{\omega^2}e^{-t}\\
&\hat{v}_1(\omega,t)=-\frac{\omega^2e^{-t}+\omega^2t-\omega^2-t}{1+i\omega}\\
&\hat{v}_2(\omega,t)=\frac{e^{-t}(\omega^2-2\omega^4)}{1+i\omega}-\frac{\omega^2(\omega^2t+\omega^2te^t-2\omega^2e^t-te^t+e^t)e^{-t}}{1+i\omega}\\
&\hat{v}_3(\omega,t)=\frac{e^{-t}(-3\omega^6+2\omega^4)}{1+i\omega}-\frac{1}{2}\frac{\omega^4(-2t+4t\omega^2+\omega^2t^2+2e^t\omega^2t-6e^t\omega^2-2te^t+4e^t)e^{-t}}{1+i\omega}\\
&\hat{v}_4(\omega,t)=-\frac{1}{6}\frac{e^{-t}\omega^6(-18+24\omega^2+18\omega^2t-12t-3t^2+6\omega^2t^2+\omega^2t^3+6e^t\omega^2t-24e^t\omega^2-6te^t+18e^t)}{1+i\omega}
\end{aligned} \tag{27}$$

Then, we have

$$\begin{aligned}
&v_0(x,t)=xe^{-t}\\
&v_1(x,t)=e^{-x}(2t+e^{-t}-1)\\
&v_2(x,t)=e^{-x}(3-2t-3e^{-t}-\ te^{-t})\\
&v_3(x,t)=e^{-x}\left(\frac{t^2}{2}e^{-t}+3te^{-t}+5e^{-t}+2t-5\right)\\
&v_4(x,t)=e^{-x}\left(-\frac{1}{6}t^3e^{-t}-\frac{3}{2}t^2e^{-t}-5te^{-t}-7e^{-t}-2t+7\right)
\end{aligned} \tag{28}$$

.

By setting $p=1$ in Eq. (25) and taking the inverse Fourier transform, the solution of Eq. (22) can be written as $u(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+\cdots$

Therefore, in view of Eq. (28) the solution in series form is written as:

$$u(x,t)=xe^{-t}+e^{-x}(2t+e^{-t}-1)+e^{-x}(3-2t-3e^{-t}-\ te^{-t})+e^{-x}\left(\frac{t^2}{2}e^{-t}+3te^{-t}+5e^{-t}+2t-5\right)+e^{-x}\left(-\frac{1}{6}t^3e^{-t}-\frac{3}{2}t^2e^{-t}-5te^{-t}-7e^{-t}-2t+7\right)+ \tag{29}$$

The Taylor series expansion for $\mathrm{e}^{-\mathrm{t}}$ is a below:

$$e^{-t}=\sum_{n=0}^{\infty}\frac{(-1)^nt^n}{n!} \tag{30}$$

Substituting Eq. (30) into Eq. (29) and by the help of some algebraic manipulations the closed form of Eq. (29) is obtained as:

$$u(x,t)=xe^{-t}\ +te^{-x}\,. \tag{31}$$

Eq. (22) is the exact solution of the problem. In the continuation the trend of convergence towards the exact solution is sketched by the results of $S_0(x,t)=v_0(x,t)$ to $S_5(x,t)=\sum_{i=0}^{5}v_i(x,t)$ of the FTHPM solution of Eq. (22). As can be seen from figure (1) a trend of very rapid convergence (Figs. 1a-1f) towards the exact solution (Fig. 2) is clearly shown. Table 1 shows the percentage of relative errors of the results of $S_0(x,t)=v_0(x,t)$ to $S_5(x,t)=\sum_{i=0}^{5}v_i(x,t)$ of the FTHPM solution of Eq. (22). The trend of very rapid The inverse Fourier transform of Eq. (27) is convergence, the maximum relative error of less than 1%, of the solution towards the exact solution is clearly shown.

Example 3. Consider the following differential equation as the third case study problem:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-u=0,$$

$$u(x,0)=\sin(x)+1\,,\quad u_t(x,0)=0, \tag{32}$$

$$u(0,t)=\cosh(t)\,,\quad u_x(0,t)=1.$$

By applying the Fourier transform to Eq. (32) we obtain as follow:

$$\frac{\partial^2 \hat{u}}{\partial t^2}+i\omega\, cosh(t)+1+(\omega^2-1)\hat{u}=0,$$

$$\hat{u}(\omega,0)=\frac{i-(1+i\omega)\omega}{\omega(\omega^2-1)}\;,\qquad \hat{u}_t(\omega,0)=0\,. \tag{33}$$

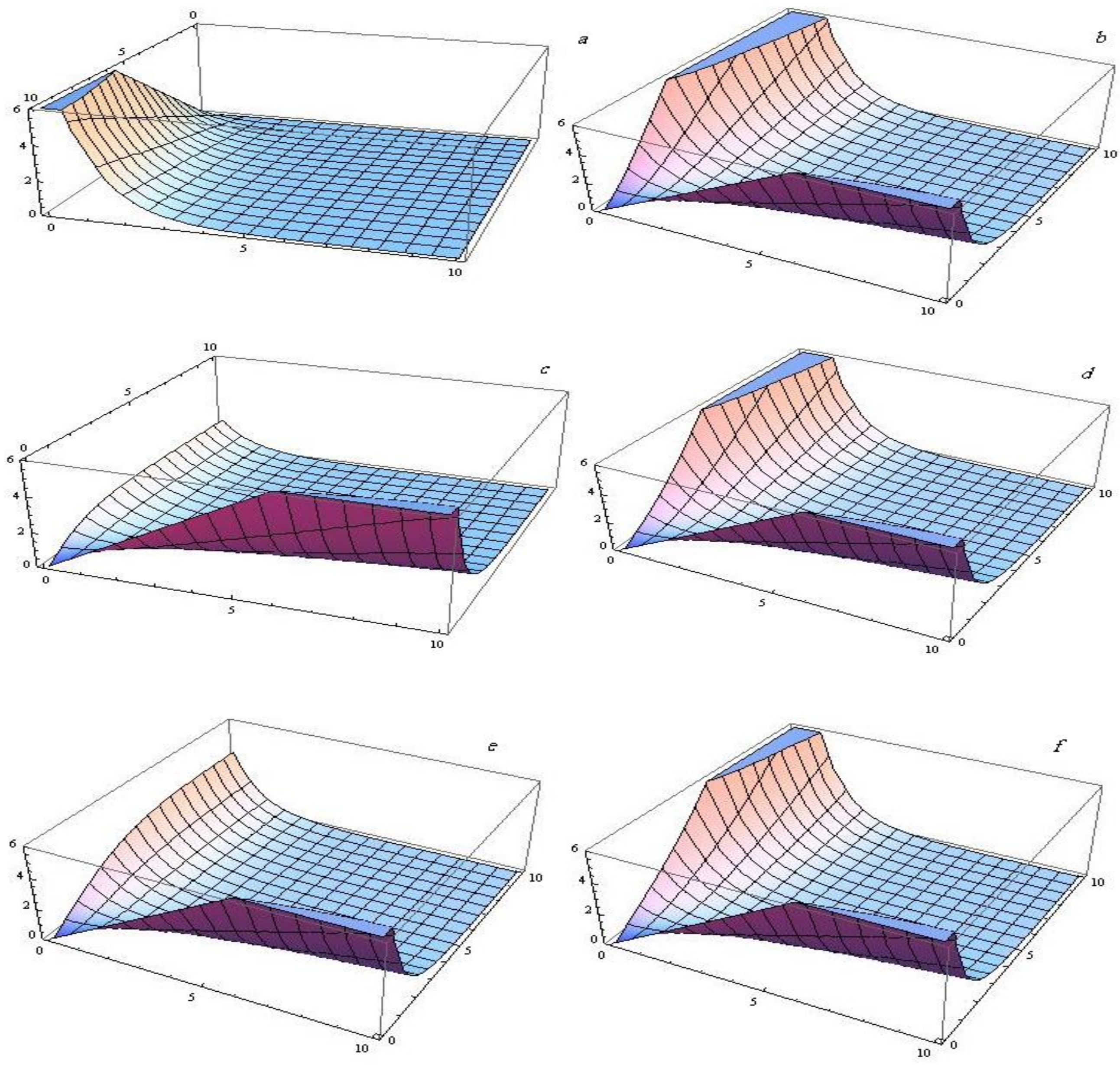

Fig. 1 . the sketch of (a-f): $S_0(x,t)=v_0(x,t)$ to $S_5(x,t)=\sum_{i=0}^{5}v_i(x,t)$ of the FTHPM solution of Eq. (22).

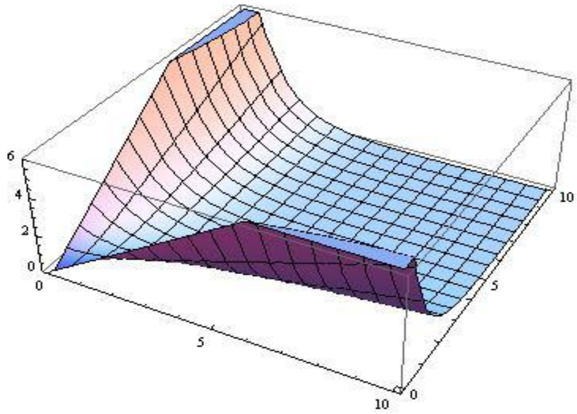

Fig. 2 . the sketch of the exact solution of Eq. (22).

TABLE 1. THE TABLE OF THE PERCENTAGE OF RELATIVE ERRORS OF THE RESULTS OF $\boldsymbol{S_0(x,t)} = \boldsymbol{v_0(x,t)}$ TO $\boldsymbol{S_5(x,t)} = \sum_{i=0}^{5} \boldsymbol{v_i(x,t)}$ OF THE FTHPM SOLUTION OF EQ. (22).

		percentage of relative error (%RE)			
		$x=1$	$x=5$	$x=10$	$x=15$
$t=0.1$	$S_0(x,t)$	*3.9069*	*0.01489*	*0.00005017*	*2.25e-7*
	$S_1(x,t)$	*0.1890*	*0.00072*	*2.43e-7*	*0*
	$S_2(x,t)$	*0.006195*	*0.000023*	*0*	*0*
	$S_3(x,t)$	*0.0001533*	*5.84e-7*	*0*	*0*
	$S_4(x,t)$	*3.046e-6*	*0*	*0*	*0*
	$S_5(x,t)$	*0*	*0*	*0*	*0*
$t=1$	$S_0(x,t)$	*50*	*0.3650*	*0.00123*	*5.54e-6*
	$S_1(x,t)$	*18.394*	*0.13427*	*0.000454*	*2.04e-6*
	$S_2(x,t)$	*5.1819*	*0.03782*	*0.0000128*	*5.74e-7*
	$S_3(x,t)$	*1.1668*	*0.0085*	*0.0000118*	*1.29e-7*
	$S_4(x,t)$	*0.21744*	*0.001587*	*5.36e-6*	*0*
	$S_5(x,t)$	*0.03444*	*0.000251*	*8.50e-7*	*0*
$t=2$	$S_0(x,t)$	*84.4638*	*1.9526*	*0.0067*	*0.0000301*
	$S_1(x,t)$	*47.9473*	*1.10843*	*0.0038*	*0.0000171*
	$S_2(x,t)$	*22.8619*	*0.52851*	*0.001816*	*8.157e-6*
	$S_3(x,t)$	*9.2073*	*0.21285*	*0.000731*	*3.285e-6*
	$S_4(x,t)$	*3.1733*	*0.0733*	*0.000252*	*1.13e-6*
	$S_5(x,t)$	*0.94971*	*0.0219*	*0.0000754*	*2.58e-7*

$$p^0: \ \frac{d^2\hat{v}_0}{dt^2} + i\omega\, cosh(t) = 0 \quad ,$$

$$\hat{v}_0(\omega,0) = \frac{i-(1+i\omega)\omega}{\omega(\omega^2-1)} \quad , \quad \hat{v}_{0t}(\omega,0) = 0$$

$$p^1: \ \frac{d^2\hat{v}_1}{dt^2} + (\omega^2-1)\hat{v}_0 + 1 = 0 \quad , \quad \hat{v}_1(\omega,0) = 0$$

$$p^2: \ \frac{d^2\hat{v}_2}{dt^2} + (\omega^2-1)\hat{v}_1 = 0 \quad , \quad \hat{v}_2(\omega,0) = 0 \qquad (36)$$

$$p^3: \ \frac{d^2\hat{v}_3}{dt^2} + (\omega^2-1)\hat{v}_2 = 0 \quad , \quad \hat{v}_3(\omega,0) = 0$$

$$p^4: \ \frac{d^2\hat{v}_4}{dt^2} + (\omega^2-1)\hat{v}_3 = 0 \quad , \quad \hat{v}_4(\omega,0) = 0$$

..

$$p^i: \ \frac{d^2\hat{v}_i}{dt^2} + (\omega^2-1)\hat{v}_{i-1} = 0 \quad , \quad \hat{v}_i(\omega,0) = 0 \ , \ i = 2,3 \ ...$$

We construct a homotopy as:

$$H(\hat{u},p)=\frac{\partial^2\hat{u}}{\partial t^2}+i\omega\, cosh(t)+p(1+(\omega^2-1)\hat{u})=0,\quad p\in[0,1]. \tag{34}$$

The solution of Eq. (34) can be written in power series of p as:

$$\hat{u}=\hat{u}(\omega,t)=\hat{v}_0(\omega,t)+p^1\hat{v}_1(\omega,t)+p^2\hat{v}_2(\omega,t)+ \tag{35}$$

Substituting Eq. (35) into Eq. (34) and setting the sum of terms of identical powers of p equal to zero, we get:

By solving the recursive differential equations, Eq. (36), we obtain the followings:

$$v_0(x,t)=1+sin(x)$$

$$v_1(x,t)=\frac{t^2}{2}$$

$$v_2(x,t)=\frac{t^4}{24} \tag{37}$$

$$v_3(x,t)=\frac{t^6}{720}$$

$$v_4(x,t)=\frac{t^8}{40320}$$

By setting $p=1$ in Eq. (35) and taking the inverse Fourier transform, the solution of Eq. (32) can be written as $u(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+\cdots$ Therefore, in view of Eq. (37) the solution in series form is written as:

$$u(x,t)=sin(x)+1+\frac{t^2}{2}+\frac{t^4}{24}+\frac{t^6}{720}+\frac{t^8}{40320}+\cdots \tag{38}$$

The Taylor series expansion of $\cosh(t)$ is the following:

$$cosh(t)=1+\frac{t^2}{2}+\frac{t^4}{24}+\frac{t^6}{720}+\frac{t^8}{40320}+\cdots=\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!} \tag{39}$$

Substituting Eq. (39) into Eq. (38), the closed form of Eq. (38) is given by:

$$u(x,t)=sin(x)+cosh(t) \tag{40}$$

This is the exact solution of the problem, Eq. (32). In the following we have shown the trend of convergence of the solutions of Eq. (32) using the FTHPM towards the exact solution by sketching the results of $S_0(x,t)=v_0(x,t)$ to $S_5(x,t)=\sum_{i=0}^{5}v_i(x,t)$. As can be seen from Figs (3a-3f) a very rapid convergence towards the exact solution (Fig. 3) is clearly shown. Table 2 shows the percentage of relative errors of the results of $S_0(x,t)=v_0(x,t)$ to $S_5(x,t)=\sum_{i=0}^{5}v_i(x,t)$ of the FTHPM solution of Eq. (32). The trend of very rapid convergence, the relative error of almost zero, of the solution towards the exact solution is clearly shown.

5. Conclusions

A new effective modification to the homotopy perturbation method, the Fourier transform homotopy perturbation method (FTHPM), is presented in this paper. The new modification to the HPM is the combination of the Fourier transform and homotopy perturbation method. The validity and effectiveness of the new method is shown by solving three non-homogenous differential equations with variable coefficients and the very rapid approach to the exact solutions is shown schematically. The very rapid approach towards the exact solutions of the new method, FTHPM, indicates that the amount of computational work is much less than those required for other previous semi-analytical methods. Moreover, the deficiency of the previous HPMs caused by unsatisfied boundary conditions is overcome by the new method where, the solution is shown to be valid in the entire range of problem domain. Therefore, it is concluded that the FTHPM can be considered as a powerful and efficient tool in obtaining the accurate solutions as well as other effective numerical methods.

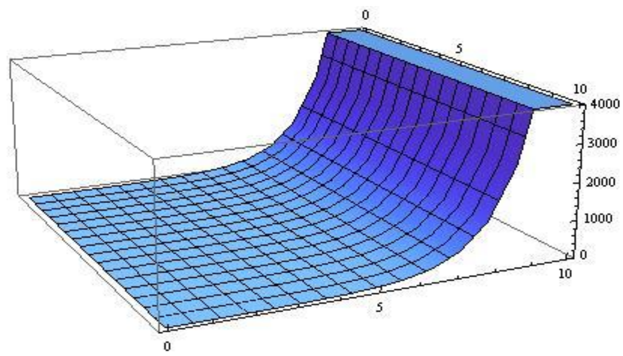

Fig. 3 . the sketch of the exact solution of Eq. (32)

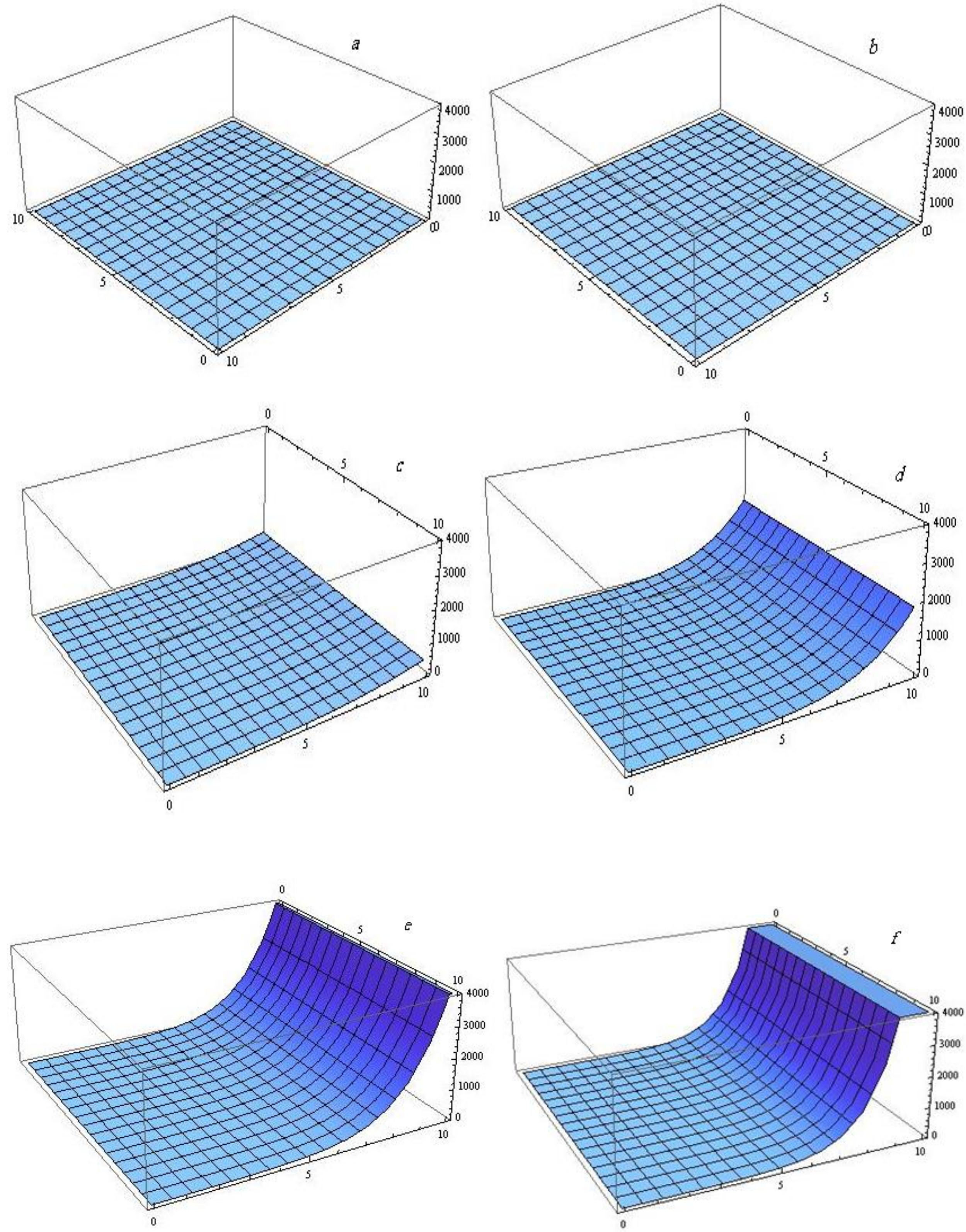

Fig. 4 . the sketch of (a-f): $S_0(x,t) = v_0(x,t)$ to $S_5(x,t) = \sum_{i=0}^{5} v_i(x,t)$ of the FTHPM solution of Eq. (32).

TABLE 2. THE TABLE OF THE PERCENTAGE OF RELATIVE ERRORS OF THE RESULTS OF $S_0(x,t)=v_0(x,t)$ TO $S_5(x,t)=\sum_{i=0}^{5} v_i(x,t)$ OF THE FTHPM SOLUTION OF EQ. (32).

		percentage of relative error (%RE)			
		$x=1$	$x=5$	$x=10$	$x=15$
$t=0.1$	$S_0(x,t)$	*0.271*	*10.8598*	*1.0855*	*0.3023*
	$S_1(x,t)$	*0.000022*	*0.0009045*	*0.0009041*	*0.000251*
	$S_2(x,t)$	*0*	*0*	*0*	*0*
	$S_3(x,t)$	*0*	*0*	*0*	*0*
	$S_4(x,t)$	*0*	*0*	*0*	*0*
	$S_5(x,t)$	*0*	*0*	*0*	*0*
$t=1$	$S_0(x,t)$	*22.775*	*92.9684*	*54.3592*	*24.7601*
	$S_1(x,t)$	*1.80666*	*7.3748*	*4.3121*	*1.9641*
	$S_2(x,t)$	*0.05930*	*0.2420*	*0.1415*	*0.0644*
	$S_3(x,t)$	*0.001052*	*0.00429*	*0.00251*	*0.00114*
	$S_4(x,t)$	*0.0000116*	*0.0000475*	*0.0000277*	*0.0000127*
	$S_5(x,t)$	*0*	*0*	*0*	*0*
$t=2$	$S_0(x,t)$	*59.999*	*98.5347*	*85.8311*	*62.5996*
	$S_1(x,t)$	*16.5563*	*27.1895*	*23.6841*	*17.2736*
	$S_2(x,t)$	*2.07506*	*3.40777*	*2.96842*	*2.16497*
	$S_3(x,t)$	*0.14424*	*0.23687*	*0.20633*	*0.150485*
	$S_4(x,t)$	*0.00631*	*0.010378*	*0.00904*	*0.006593*
	$S_5(x,t)$	*0.000189*	*0.000311*	*0.000271*	*0.000198*

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