AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 6(2) (2025) 151-162
https://doi.org/10.22060/AJMC.2024.22585.1178

Original Article

Dynamic monopolies in simple graphs

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ABSTRACT: This paper studies a repetitive polling game played on an n-vertex graph G. At first, each vertex is colored, Black or White. At each round, each vertex (simultaneously) recolors itself by the color of the majority of its closed neighborhood. The variants of the model differ in the choice of a particular tie-breaking rule. We assume the tie-breaking rule is Prefer-White and we study the relation between the notion of "dynamic monopoly" and "vertex cover" of G. In particular, we show that any vertex cover of G is a dynamic monopoly or reaches a $2-$periodic coloring. Moreover, we compute $\mathrm{dyn}(G)$ for some special classes of graphs including paths, cycles and links of some graphs.

Review History:

Received:27 July 2023
Revised:13 December 2023
Accepted:26 January 2024
Available Online:01 March 2025

Keywords:

Repetitive polling game
Dynamic monopoly
Vertex cover
Majority function

MSC (2020):

05C57; 05C69; 37E15

1. Introduction

In distributed computing on point-to-point networks, faulty processors can cause misbehavior in their neighbors. To restrict their influence and limit the overall detriment effected by faulty processors, the idea of majority-based voting models is often used. Voting models have been applied as a decision tool in some agreement protocols (Byzantine Agreement, Consensus, Interactive Consistency), distributed database inconsistency resolution protocols management system, key distribution in security and discrete time dynamical systems. However, in point-to-point systems, majority-voting may not be able to stop the diffusion of defective behavior of faulty elements (sometimes these elements are few but well-placed). The dynamics, with respect to faults, of a system using majority-based voting can be described as a repetitive polling game on graphs (see [8, 18]).

Let $G = (V, E)$ be a simple connected graph with the vertex set $V = \{v_1, \ldots, v_n\}$ modeling the topology of the system. Consider the following repetitive polling game on G. At round 0, each vertex v is colored the color $x^0(v)$, which may be Black (non-faulty) or White (faulty). At each round, each vertex (simultaneously) looks at the current colors of vertices in its (closed) neighborhood and adopts the more common color (namely, the one occurring at the majority of its neighbors) as its new color.

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Let us give more details. Denote the closed neighborhood of the vertex v by $N_G[v]$; that is, $N_G(v) \cup \{v\}$. The global state of the graph after round t is represented as a vector $X^t = (x^t(v_1), \ldots, x^t(v_n))$, where $x^t(v)$ is the color of vertex v after round t. For each vertex v, define $W_t(v) = |\{w \in N_G[v] \ ; \ x^t(w) = \text{ White }\}|$, then at round $r+1$ the vertex v will be White if

$$W_r(v) \geq \lceil \frac{\deg_G(v)+1}{2} \rceil \tag{1}$$

and it will be Black if (1) does not hold. Note that if at round t exactly half of the elements of $N_G[v]$ are colored White and half of it are colored Black, then we say there is a tie. Two well-known tie-breaking rules are "Prefer-White" and "Prefer-Current"; the first, in case of a tie, recolors the vertex White ($x^{t+1}(v) = White$), and the latter does not change the color ($x^{t+1}(v) = x^t(v)$). Our tie-breaking rule is "Prefer-White". An example of the resulting process is illustrated in Figure 1. Other types of the above polling game, such as the case that the tie-breaking rule is "Prefer-Current" could be found in [16, 17].

In this paper, we focus on cases where the computation converges into the all-White monochromatic configuration, this corresponds to the status when the whole system will finally have a faulty manner. Let $S \subseteq V$ and $X^0 = (x^0(v_1), \ldots, x^0(v_n))$ be an initial coloring in which $x^0(v)$ =White if and only if $v \in S$. Let $S_r = \{v \in V; \ v$ is colored White at round $r\}$. We say that S is a dynamic monopoly, abbreviated dynamo, if there exists r such that $S_r = V$. We show the smallest size of any dynamos with $\text{dyn}(G)$. The concept of dynamo was studied in [14, 17, 18, 20, 22]. The minimum size of a dynamo has been widely studied on different graph classes (see [9, 8, 14]), which are motivated by different literatures, such as statistical physics, viral marketing, and fault-local mending in distributed systems. Beside the mentioned results, reversible and irreversible dynamic monopolies were studied in [4, 5, 6, 7]. Related problems were studied from different perspectives and under different names in [1, 2, 3, 12, 15, 19, 21]. With a close look at all the aforementioned references, it can be said that the concept of dynamic monopoly is defined differently from one research to another which is due to the existence of different versions of the polling game. The definition of dynamic monopoly considered in this research is based on a version of the game called *Prefer-White, Self-Included.*

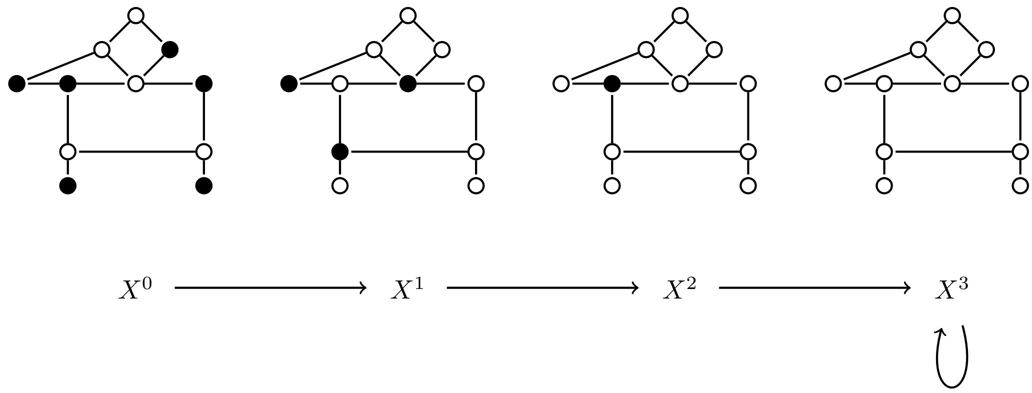

Figure 1: An example for a multi round repetitive polling game

The paper is organized as follows. In Section 2, we study the relation between the notion of "vertex cover" and the notion of "dynamic monopoly" for an arbitrary graph G. To seek this purpose, first, we examine this relation in some special classes of graphs including paths, cycles and stars (see Proposition 2.3) and, in particular, we compute $\text{dyn}(G)$ in each case (Corollary 2.4). Next, we study an arbitrary connected graph G and show that any vertex cover is a dynamic monopoly or reaches a 2−periodic coloring (see Theorem 2.6). In particular, any vertex cover of G is a dynamic monopoly provided that G does not have any even cycle (Corollary 2.7). Next, we find a bound for $\text{dyn}(G)$ when G is a tree (Corollary 2.7). In Section 3, we relate the problem of finding $\text{dyn}(G)$ to the problem of finding $\text{dyn}(G_1), \text{dyn}(G_2)$ when G is a link of G_1 and G_2 via a single edge $\{v, w\}$ ($v \in V(G_1), w \in V(G_2)$ (See Theorems 3.1, 3.2, 3.3). Finally, we assume that G is a link of two cycles or two paths, and in each case we compute $\text{dyn}(G)$ (see Subsections 3.1, 3.2 and 3.3).

2. Dynamic monopolies and vertex covers

Let $G = (V, E)$ be a graph on the vertex set $V = \{v_1, \ldots, v_n\}$. Following the notations of introduction, for any initial coloring $X^0 = (x^0(v_1), \ldots, x^0(v_n))$, let $S = \{v \ ; \ x^0(v) = White\}$ and corresponding to X^r, let $S_r = \{v \in V \ ; \ v$ is colored White at round $r\}$ for each $r \in \mathbb{N}$. By [11, Theorem 2.1] (see also [10]), $S_r = S_{r+2}$ for some $r \in \mathbb{N}$. We say X^r is a fixed coloring when $S_r = S_{r+1}$ and X^r is a 2-periodic coloring when $S_r = S_{r+2} \neq S_{r+1}$. So, S is a dynamic monopoly when there exists $r \in \mathbb{N}$ such that X^r is the fixed coloring and $S_r = V$. Also, the repetitive polling game on G is called a fixed coloring game if for each initial coloring X^0 there exists $r \in \mathbb{N}$ such that X^r is a fixed coloring. The next theorem shows the role of even cycles in characterizing fixed coloring games. It is a direct consequence of [13, Theorem 2.2], but for the convenience of the reader, we will state its proof briefly.

Theorem 2.1. *Let $X^0 = (x^0(v_1), \ldots, x^0(v_n))$ be a 2-periodic coloring for the repetitive polling game on $G = (V, E)$. Then G has a Black-White alternating even cycle.*

Proof. Since X^0 is a $2-$periodic coloring, $S = S_2 \neq S_1$. This shows that $(S_1 \setminus S) \cup (S \setminus S_1) \neq \emptyset$. For each $v \in V$ if $\deg(v) = 1$ and $v \in S_r$ for some $r \in \mathbb{N}$, then $v \in S_{r+1}$. Therefore, each vertex $v \in (S_1 \setminus S) \cup (S \setminus S_1)$ has degree more than 1 and $N_G(v)$ can be written as

$$N_G(v) = (N_G(v) \setminus (S \cup S_1)) \dot{\cup} (N_G(v) \cap S \cap S_1) \dot{\cup} (N_G(v) \cap (S \setminus S_1)) \dot{\cup} (N_G(v) \cap (S_1 \setminus S)) \tag{2}$$

According to the rule of the game, by comparing $\deg(v)$ with the cardinalities of the sets in Equation (2), one can see that

$$v \in S_1 \setminus S \Rightarrow |N_G(v) \cap (S \setminus S_1)| \geq 2 \tag{3}$$

and

$$v \in S \setminus S_1 \Rightarrow |N_G(v) \cap (S_1 \setminus S)| \geq 2. \tag{4}$$

Choose $v_{i_0} \in S_1 \setminus S$. By (3), we can choose distinct $v_{+i_1}, v_{-i_1} \in S \setminus S_1$. So, $v_{-i_1}, v_{i_0}, v_{+i_1}$ is a Black-White alternating simple path with 3 vertices. To find the desired cycle, we follow the following steps.

Step 1. As $v_{+i_1}, v_{-i_1} \in S \setminus S_1$, by (4), we can select $v_{+i_2} \in (N_G(v_{+i_1}) \cap (S_1 \setminus S)) \setminus \{v_{i_0}\}$ and $v_{-i_2} \in (N_G(v_{-i_1}) \cap (S_1 \setminus S)) \setminus \{v_{i_0}\}$. If $v_{+i_2} = v_{-i_2}$, then $v_{-i_2}, v_{-i_1}, v_{i_0}, v_{+i_1}, v_{+i_2}$ forms a Black-White alternating cycle of length 4. If $v_{+i_2} \neq v_{-i_2}$, we get a Black-White alternating simple path with 5 vertices.

Step 2. Let $j \geq 2$ and suppose we have not found the desired cycle by the end of $(j-1)$th step. This in particular shows that $v_{-i_j}, \ldots, v_{i_0}, \ldots, v_{+i_j}$ is a simple Black-White alternating path with $2j+1$ vertices. Let us assume $v_{-i_j}, v_{+i_j} \in S_1 \setminus S$ (The other situation can be discussed in the same way). By (3), we can choose $v_{+i_{j+1}} \in (N_G(v_{+i_j}) \cap (S \setminus S_1)) \setminus \{v_{+i_{j-1}}\}$ and $v_{-i_{j+1}} \in (N_G(v_{-i_j}) \cap (S \setminus S_1)) \setminus \{v_{-i_{j-1}}\}$. Suppose $v_{+i_{j+1}} = v_{-i_{j+1}}$ or $\{v_{+i_{j+1}}, v_{-i_{j+1}}\} \cap \{v_{-i_j}, \ldots, v_{i_0}, \ldots, v_{+i_j}\} \neq \emptyset$. Since $v_{-i_j}, \ldots, v_{i_0}, \ldots, v_{+i_j}$ is a simple Black-White alternating path, the walk $v_{-i_{j+1}}, v_{-i_j}, \ldots, v_{i_0}, \ldots, v_{+i_j}, v_{+i_{j+1}}$ contains a Black-White alternating cycle of even length. If $v_{+i_{j+1}} \neq v_{-i_{j+1}}$ and $\{v_{+i_{j+1}}, v_{-i_{j+1}}\} \cap \{v_{-i_j}, \ldots, v_{i_0}, \ldots, v_{+i_j}\} = \emptyset$, we get a Black-White alternating simple path with $2j+3$ vertices.

Because the number of graph vertices is finite, the above process will stop with a successful output. □

Let $G = (V, E)$ be a graph on the vertex set V and $T \subseteq V$. We say that T is a vertex cover of G if for each $\{i, j\} \in E$ we have $\{i, j\} \cap T \neq \emptyset$. If T is minimal with respect to inclusion, then T is called minimal vertex cover of G. Let

$$\mathcal{DM}(G) = \{S \subseteq V \ ; \ S \text{ is a dynamic monopoly of } G\}$$

and

$$\mathcal{VC}(G) = \{T \subseteq V \ ; \ T \text{ is a vertex cover of } G\}.$$

It would be interesting to know the relation between the notions of dynamic monopoly and vertex cover. In this section, we study this problem.

Lemma 2.2. *Let $G = (V, E)$ be a graph and S be a dynamic monopoly of it. If v and w are two vertices of degree at most 2 and $\{v, w\} \in E$, then $S \cap \{v, w\} \neq \emptyset$.*

Proof. Let X^0 and S be as above. Suppose that v and w are two vertices of degree at most 2 and $\{v, w\} \in E$. One can easily see that if $S \cap \{v, w\} = \emptyset$, then $S_r \cap \{v, w\} = \emptyset$ for each $r \in \mathbb{N}$. So, S is not a dynamic monopoly of G. □

Although the first lemma is simple, it plays a key role in finding $\operatorname{dyn}(G)$. The next proposition shows the relation between $\mathcal{DM}(G)$ and $\mathcal{VC}(G)$ when G is a cycle, a path or a star graph.

Proposition 2.3. *(1) Let C_n be a cycle with n vertices. If n is an odd number, then $\mathcal{VC}(C_n) = \mathcal{DM}(C_n)$ and if n is an even number, then the set of dynamic monopolies coincides with the set of non-minimal vertex covers of G.*
(2) Let P_n be a path with n vertices. Then $\mathcal{VC}(P_n) = \mathcal{DM}(P_n)$.
(3) Let $K_{n,1}$ be the star graph with $n+1$ vertices where $n > 1$. Then $\mathcal{VC}(K_{n,1}) \subsetneq \mathcal{DM}(K_{n,1})$.

Proof. (1) Consider a cycle graph $C_n = (V, E)$ where

$$V = \{v_1, v_2, \ldots, v_n\}, E = \{\{v_i, v_{i+1}\}; 1 \le i \le n-1\} \cup \{\{v_1, v_n\}\}$$

(for simplifying the notation, we assume that $v_{n+1} = v_1$ and $v_0 = v_n$). We claim that $S \subseteq V$ is a dynamo if and only if the following conditions hold:

(i) For each $v_i \in V - S$, $\{v_{i-1}, v_{i+1}\} \subseteq S$.
(ii) There exists $v_i \in S$ such that $\{v_{i+1}, v_{i-1}\} \cap S \neq \emptyset$.

To prove the claim, first suppose that $S \subseteq V$ is a dynamo and $v_i \in V - S$ for some $1 \le i \le n-1$. If $v_{i-1} \in V - S$ or $v_{i+1} \in V - S$, then by Lemma 2.2, S is not a dynamic monopoly of C_n. Now assume that for each $v_i \in S$ both v_{i-1} and v_{i+1} belong to $V - S$. So, one of the following situations happen:

(a) At each round all vertices of the graph change their color and in particular n is even. In this case, S is not a dynamo.
(b) There exists a round t such that all vertices of the graph are Black at each round $k \ge t$. Again, S is not a dynamo.

Conversely, suppose that two conditions (i) and (ii) are satisfied. We prove S is a dynamo. Let $X^0 = (x^0(v_1), \ldots, x^0(v_n))$ be an initial coloring that $x^0(v_i) = White$ if and only if $v_i \in S$. Let $S_0 = S$ and assume that S_r is defined as before. By induction on r, we prove that for each $r \ge 0$ the following statement holds.

$$\forall v_i \in V (v_i \notin S_r \Rightarrow \{v_{i-1}, v_{i+1}\} \subseteq S_r \text{ and } v_i \in S_{r+1}). \tag{5}$$

Let $i = 0$ and suppose $v_i \notin S_0$. Then by (i), $\{v_{i-1}, v_{i+1}\} \subseteq S_0$. As $\deg(v_i) = 2$, we conclude $v_i \in S_1$. So, (5) holds when $i = 0$. Now assume $r > 0$ and (5) holds for each $0 \le s < r$. If $v_i \notin S_r$, then by induction hypothesis $v_i \in S_{r-1}$ which implies that $\{v_{i-1}, v_{i+1}\} \cap S_{r-1} = \emptyset$. Since $v_{i-1} \notin S_{r-1}$ and $v_{i+1} \notin S_{r-1}$, by induction hypothesis, $\{v_{i-1}, v_{i+1}\} \subseteq S_r$. Using $\deg(v_i) = 2$, we see $v_i \in S_{r+1}$. Hence (5) holds for $r + 1$.

Next, by (ii) and without loss of generality, we assume that $\{v_1, v_2\} \subseteq S = S_0$. It is clear that $\{v_1, v_2\} \subseteq S_r$ for each $r \in \mathbb{N}$. We claim that $v_i \in S_r$ for each $i \ge 3$ and each $r \ge i - 2$. To prove the claim, we apply an inductive process. Let $i = 3$. If $v_3 \in S_0$, then it is clear that $v_3 \in S_r$ for each $r \in \mathbb{N}$. Assume that $v_3 \notin S_0$. By (5), $v_3 \in S_1$. As $v_2 \in S_r$ for each $r \in \mathbb{N}$ and $\deg(v_3) = 2$, we deduce that $v_3 \in S_r$ for each $r \in \mathbb{N}$. So the claim is proved for $i = 3$. Suppose that $i > 3$ and the claim has been proved for each $3 \le j < i$. This shows that $v_{i-1} \in S_r$ for each $r \ge i - 3$. As $\deg(v_i) = 2$, $v_i \in S_r$ for each $r \ge i - 2$ when $v_i \in S_{i-3}$. Assume that $v_i \notin S_{i-3}$. By (5), $v_i \in S_{i-2}$. So, the claim is proved by induction hypothesis and $\deg(v_i) = 2$. According to the above claim, we can say $V = S_{n-2}$. Hence, S is a dynamic monopoly.

On the other hand, $T \subseteq V$ is a vertex cover of C_n if and only if for every $v_i \in V - T$, $\{v_{i-1}, v_{i+1}\} \subset T$. Furthermore, if n is an odd number and $T \subseteq V$ is a vertex cover of C_n, then there exists $v_i \in T$ such that $\{v_{i+1}, v_{i-1}\} \cap T \neq \emptyset$. So (i) and (ii) hold and T is a dynamic monopoly. If n is an even number and $T \subseteq V$ is a vertex cover of C_n, then as previous case, (i) holds. But (ii) holds if and only if T is a non-minimal vertex cover of C_n. So, the conclusion follows.

(2) By a similar argument as case 1, the assertion follows.

(3) Consider a star graph $K_{n,1} = (V, E)$ where $V = \{v_0, v_1, \ldots, v_n\}, E = \{\{v_0, v_i\}; 1 \le i \le n\}$. One can check that

$$\mathcal{VC}(K_{n,1}) = \{\{v_0\} \cup T \ ; \ T \subseteq \{v_1, \ldots, v_n\}\} \cup \{\{v_1, \ldots, v_n\}\}.$$

One can easily check that every element of $\mathcal{VC}(K_{n,1})$ is a dynamic monopoly. On the other hand, each $A \subset V$ which has at least $\lceil \frac{n}{2} \rceil$ elements is a dynamic monopoly. □

The next corollary is an immediate consequence of Proposition 2.3.

Corollary 2.4. $\mathrm{dyn}(C_n) = \lceil \frac{n}{2} \rceil$, $\mathrm{dyn}(P_n) = \lfloor \frac{n}{2} \rfloor$, $\mathrm{dyn}(K_{n,1}) = 1$.

In general, there is no special relation between dynamic monopoly sets of G and its vertex covers. In the rest of this section, we intend to clarify this issue.

Lemma 2.5. *Consider the repetitive polling game on a simple graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$. For an arbitrary initial coloring $X^0 = (x^0(v_1), \ldots, x^0(v_n))$, if $V \setminus S \subseteq S_1$, then S is a dynamic monopoly or there exists $t \in \mathbb{N}$ such that X^t is a $2-$periodic coloring.*

Proof. Let $X^0 = (x^0(v_1), \ldots, x^0(v_n))$ be an arbitrary initial coloring in which $V \setminus S \subseteq S_1$. We claim that

$$\forall r \in \mathbb{N} \; ; \; V \setminus S_r \subseteq S_{r+1} \tag{6}$$

We prove the claim by induction on r. Let $r = 1$ and choose an arbitrary element v of $V \setminus S_1$. So $x^1(v) = Black$ and $x^0(v) = White$. This means that

$$|(V \setminus S) \cap N_G(v)| = |\{w \in N_G(v) \; ; \; x^0(w) = Black\}| > \lceil \frac{\deg_G(v)}{2} \rceil.$$

So,

$$|S_1 \cap N_G(v)| = |\{w \in N_G(v) \; ; \; x^1(w) = White\}| > \lceil \frac{\deg_G(v)}{2} \rceil.$$

This shows that $x^2(v) = White$ and so $v \in S_2$ and the assertion is proved for $r = 1$. Now let $r > 0$ and by induction hypothesis assume that $V \setminus S_r \subseteq S_{r+1}$. Choose an arbitrary element v of $V \setminus S_{r+1}$. So $x^{r+1}(v) = Black$ and $x^r(v) = White$. This means that

$$|(V \setminus S_r) \cap N_G(v)| = |\{w \in N_G(v) \; ; \; x^r(w) = Black\}| > \lceil \frac{\deg_G(v)}{2} \rceil.$$

So,

$$|S_{r+1} \cap N_G(v)| = |\{w \in N_G(v) \; ; \; x^{r+1}(w) = White\}| > \lceil \frac{\deg_G(v)}{2} \rceil.$$

This shows that $x^{r+2}(v) = White$ and so $v \in S_{r+2}$ and the claim is proved.

Now suppose that S is not a dynamic monopoly. So, for each positive integer r, we must have $S_r \subsetneq V$. On the other hand, by [11, Theorem 2.1] (see also [10]), there exists $t \in \mathbb{N}$ such that X^t is a fixed coloring or a $2-$periodic coloring. If X^t is a fixed coloring, then $S_t = S_{t+1}$. So, by Equation (6), $\emptyset \neq V \setminus S_{t+1} \subseteq S_{t+1}$ which is a contradiction. This proves that X^t is a $2-$periodic coloring. □

Now we are ready to present the relation between dynamic monopoly sets of G and its vertex covers.

Theorem 2.6. *Let S be an arbitrary vertex cover of a connected graph $G = (V, E)$ on the vertex set $V = \{v_1, \ldots, v_n\}$ and $X^0 = (x^0(v_1), \ldots, x^0(v_n))$ be the initial coloring in which $x^0(v) = White$ if and only if $v \in S$. Then S is a dynamic monopoly or there exists $t \in \mathbb{N}$ such that X^t is a $2-$periodic coloring.*

Proof. Since S is a vertex cover of G, for each $v \in V \setminus S$, $N_G(v) \subseteq S$ and so $v \in S_1$. Now the result follows by Lemma 2.5. □

A set T of vertices in a graph $G = (V, E)$ is an independent set if no pair of vertices of T is adjacent. The independence number of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set in G and the covering number of G, denoted by $\beta(G)$, is the minimum of the cardinalities of all vertex covers of G. It is well known that a minimal vertex cover corresponds to the complement of a maximal independent set and so, $\alpha(G) + \beta(G) = |V|$.

Corollary 2.7. *Consider the repetitive polling game on a simple connected graph $G = (V, E)$.*

(1) If the repetitive polling game on G is a fixed coloring game, then $\mathcal{VC}(G) \subseteq \mathcal{DM}(G)$ and so $\mathrm{dyn}(G) \leq \beta(G)$.

(2) If G doesn't have a cycle of even length, then $\mathrm{dyn}(G) \leq \beta(G)$.

Proof. (1) Since the game is a fixed coloring game, by theorem 2.6, any vertex cover of G is a dynamic monopoly. So the conclusion follows.

(2) Since G doesn't have any cycle of even length, by Theorem 2.1, the game is a fixed coloring game. So, the result follows by statement (1). □

Corollary 2.8. *Consider a repetitive polling game on a simple tree $T = (V, E)$ with $|V| > 2$.*

(1) $\mathrm{dyn}(T) \leq \lfloor \frac{|V(T)|}{2} \rfloor$.

(2) $\mathrm{dyn}(T) \leq |V| - |\{v \in V \; ; \; \deg(v) = 1\}|$.

Proof. (1) Since T is acyclic, by Theorem 2.1, the game on it is a fixed coloring game. On the other hand T is a bipartite graph and so its vertex set V can be partitioned into disjoint subsets S and S' such that $|S| \leq |S'|$ and each edge has an end point in S and an end point in S'. This shows that S is a vertex cover of T which has at most $\lfloor \frac{|V(T)|}{2} \rfloor$ elements. Now by Corollary 2.7, we have $\mathrm{dyn}(G) \leq \beta(G) \leq \lfloor \frac{|V(T)|}{2} \rfloor$.

(2) Note that $\{v \in V\ ;\ \deg(v) = 1\}$ is an independent set of T and so its complement is a vertex cover of T. Thus, $\mathrm{dyn}(T) \leq \beta(T) \leq |V| - |\{v \in V\ ;\ \deg(v) = 1\}|$. □

The following example shows that the bound $\mathrm{dyn}(G) \leq \beta(G)$ in Corollary 2.7 can be strict.

Example 2.1. Let $T = (V, E)$ where $V = \{v_1, v_2, \ldots, v_8\}$ and

$$E = \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_6, v_7\}, \{v_6, v_8\}\}.$$

Then one can check that $S = \{v_3, v_6\}$ is a dynamic monopoly of T of minimum cardinality and $S \cup \{v_4\}$ is a vertex cover of minimum cardinality. So, $\mathrm{dyn}(T) < \beta(T)$ (see Figure 2).

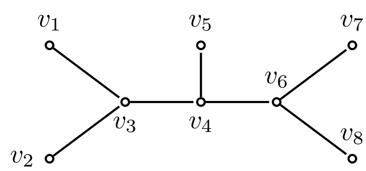

Figure 2: The graph of Example 2.1

Example 2.2. Let $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_{12}\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_1\}, \{v_6, v_7\}, \{v_7, v_8\}, \{v_8, v_9\}, \{v_9, v_{10}\}, \{v_{10}, v_{11}\}, \{v_{11}, v_{12}\}, \{v_{12}, v_7\}\}$. One can see that $S = \{v_2, v_4, v_6, v_7, v_9, v_{11}\}$ and $T = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}\}$ are two vertex covers of minimum size for G. The initial coloring corresponding to T is a $2-$periodic coloring while S is a dynamic monopoly of minimum size ($\mathrm{dyn}(G) = \beta(G)$, see Figure 3).

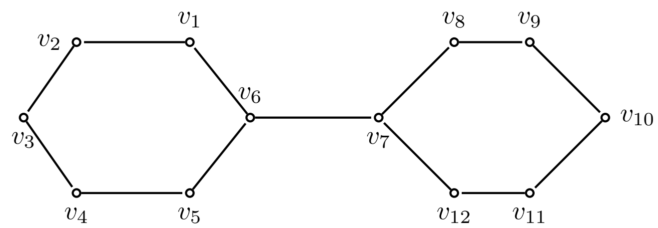

Figure 3: The graph of Example 2.2

In the following, we give another application of Lemma 2.5.

Corollary 2.9. *Consider the repetitive polling game on a simple graph $G = (V, E)$. Let $S \subseteq V$ be such that $|N_G(v) \cap S| > \lfloor \frac{\deg_G(v)}{2} \rfloor$ for each $v \in V \setminus S$. If the repetitive polling game on G is a fixed coloring game, then S is a dynamic monopoly.*

Proof. Let X^0 be an initial coloring in which $x^0(v) = White$ if and only if $v \in S$. Then one can easily see that $V \setminus S \subseteq S_1$. So the conclusion follows by Lemma 2.5. □

3. Dynamic monopolies in link of graphs

Consider the repetitive polling game on a simple graph $G = (V, E)$, where G is the link of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with a single edge $\{x, y\}$ where $x \in V(G_1)$ and $y \in V(G_2)$. Note that in the general case, both the inequalities $\mathrm{dyn}(G) < \mathrm{dyn}(G_1) + \mathrm{dyn}(G_2)$ and $\mathrm{dyn}(G) > \mathrm{dyn}(G_1) + \mathrm{dyn}(G_2)$ may occur (see Examples 3.1 and 3.3). Moreover, if S^i is a dynamic monopoly of G_i for $i = 1, 2$, then $S^1 \cup S^2$ is not necessarily a dynamic monopoly for G (See Example 3.2).

Example 3.1. Let G be a link of two copies of C_4. Then one can check that $\mathrm{dyn}(G) = 4$ while $\mathrm{dyn}(C_4) = 3$ (see Figure 4).

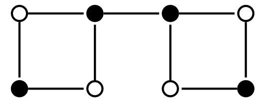

Figure 4: The link of two copies of C_4 for Example 3.1

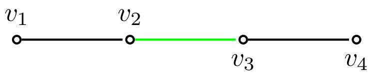

Figure 5: The link of two paths of Example 3.2

Example 3.2. Consider the graph P_4 on the vertex set $\{v_1, v_2, v_3, v_4\}$ as a link of two copies of P_2 on the vertex sets $\{v_1, v_2\}$ and $\{v_3, v_4\}$ respectively. Then $S^1 = \{v_1\}$ and $S^2 = \{v_4\}$ are dynamic monopolies of these two copies of P_2 while $S^1 \cup S^2$ is not a dynamic monopoly for P_4 (see Figure 5).

Example 3.3. Let G be the link of $K_{3,1}$ and $K_{4,1}$ with a single edge that connects two vertices of degree 1. Let v be the internal vertex of $K_{3,1}$ and w be the internal vertex of $K_{4,1}$. Then $S^1 = \{v\}$ and $S^2 = \{w\}$ are dynamic monopolies of $K_{3,1}$ and $K_{4,1}$ respectively but $\{v, w\}$ is not a dynamic monopoly of G and we have $\text{dyn}(G) = 3 > \text{dyn}(K_{3,1}) + \text{dyn}(K_{4,1})$ (see Figure 6).

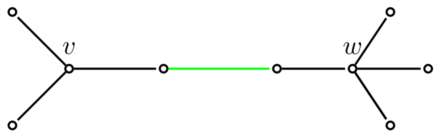

Figure 6: The link of $K_{3,1}$ and $K_{4,1}$ of Example 3.3

Theorem 3.1. *Let G be a link of an arbitrary graph G_1 and an isolated vertex y. Then* $\text{dyn}(G) \leq \text{dyn}(G_1) + 1$.

Proof. Suppose that S is a dynamic monopoly of minimum size for G_1. One can easily see that $S \cup \{y\}$ is a dynamic monopoly of G. □

The following example shows that the bound in Theorem 3.1 is sharp.

Example 3.4. Let G be the link of $K_{3,1}$ and an isolated vertex y with a single edge that connects a vertex of degree 1 to y. Then $\text{dyn}(G_1) = 1$ and $\text{dyn}(G) = 2$. (see Figure 7)

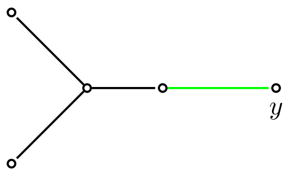

Figure 7: The link of $K_{3,1}$ and an isolated vertex of Example 3.4

Theorem 3.2. *Let $G = (V, E)$ be a link of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with a single edge $\{x, y\}$ where $x \in V_1$, $y \in V_2$, and $\deg_{G_1}(x)$ and $\deg_{G_2}(y)$ are even numbers. Suppose that S^1 and S^2 are dynamic monopolies of G_1 and G_2, respectively. Then $S^1 \cup S^2$ is a dynamic monopoly of G. In particular,*

$$\text{dyn}(G) \leq \text{dyn}(G_1) + \text{dyn}(G_2)$$

.

Proof. Let X^0 be the initial coloring of G_1 which $x^0(v)$ =White if and only if $v \in S^1$ and Y^0 be the initial coloring of G_2 which $y^0(w)$ =White if and only if $w \in S^2$. Keeping in mind that

$$S_t^1 = \{v \in V_1;\ v \text{ is colored White at round } t \text{ when the game is played on } G_1\}$$

and

$$S_t^2 = \{w \in V_2;\ w \text{ is colored White at round } t \text{ when the game is played on } G_2\},$$

by the fact that S^1 and S^2 are dynamic monopoly, there exists $r > 0$ such that $S_r^1 = V_1$ and $S_r^2 = V_2$.

Now let $S = S^1 \cup S^2$ and Z^0 be the initial coloring of G which $z^0(v)$ =White if and only if $v \in S$. Recall that for each $t \in \mathbb{N}$

$$S_t = \{v \in V;\ v \text{ is colored White at round } t \text{ when the game is played on } G\}.$$

We claim that $S_r = V$ and so S is a dynamic monopoly for G. To prove the claim, by induction on t, we show that for each positive integer t, $S_t^1 \cup S_t^2 \subseteq S_t$.

First, let $t = 1$ and $v \in S_1^1$. This means that $|N_{G_1}(v) \cap S^1| \geq \lceil \frac{\deg_{G_1}(v)+1}{2} \rceil$.

If $v \neq x$, then $N_G(v) = N_{G_1}(v)$. So by $S^1 \subset S$, $|N_G(v) \cap S| \geq \lceil \frac{\deg_G(v)+1}{2} \rceil$ and we conclude that $v \in S_1$.

If $v = x$, then $N_G(v) = N_{G_1}(v) \cup \{y\}$. Since $\deg_{G_1}(v)$ is even, $\lceil \frac{\deg_G(v)+1}{2} \rceil = \lceil \frac{\deg_{G_1}(v)+1}{2} \rceil$. So, again by $S^1 \subset S$, $|N_G(v) \cap S| \geq \lceil \frac{\deg_G(v)+1}{2} \rceil$ and $v \in S_1$.

By the same argument, we can show that if $w \in S_1^2$, then $w \in S_1$. So the assertion is proved for $t = 1$.

Now suppose that $t > 1$ and by induction hypothesis $S_{t-1}^1 \cup S_{t-1}^2 \subseteq S_{t-1}$. Let $v \in S_t^1$. Since

$$N_{G_1}(v) \cap S_{t-1}^1 \subseteq N_G(v) \cap S_{t-1} \text{ and } \lceil \frac{\deg_G(v)+1}{2} \rceil = \lceil \frac{\deg_{G_1}(v)+1}{2} \rceil,$$

we conclude that $v \in S_t$. By the same method, we can show that $S_t^2 \subset S_t$. So the conclusion follows. □

By a similar argument as used in the proof of Theorem 3.2, the next result can be proved.

Theorem 3.3. *Let $G = (V, E)$ be a link of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with a single edge $\{x, y\}$ where $\deg_{G_1}(x) \leq 2$ and $\deg_{G_2}(y) \leq 2$. Suppose that S^1 and S^2 are dynamic monopolies of G_1 and G_2, respectively. Then $S^1 \cup S^2 \cup \{x, y\}$ is a dynamic monopoly of G. In particular,* $\text{dyn}(G) \leq \text{dyn}(G_1) + \text{dyn}(G_2) + 2$.

In the rest of this section, we compute $\text{dyn}(G)$ when G is a link of some special graphs.

3.1. Dynamic monopoly for the link of cycle graphs

Consider the cycle $C_n = (V, E)$ where

$$V = \{v_1, v_2, \dots, v_n\} \text{ and } E = \{\{v_i, v_{i+1}\}; 1 \leq i \leq n-1\} \cup \{\{v_1, v_n\}\}.$$

Let $C_{n,1}$ be the link of C_n and an isolated vertex v_{n+1} with the single edge $\{v_n, v_{n+1}\}$. In the following, we compute $\text{dyn}(C_{n,1})$.

Theorem 3.4. $\text{dyn}(C_{n,1}) = \lceil \frac{n}{2} \rceil$.

Proof. In order to compute $\text{dyn}(C_{n,1})$, we find a dynamic monopoly of the minimum size.

Case 1. n is a natural odd number.

In this case, by Theorem 2.1, the repetitive polling game on $C_{n,1}$ is a fixed coloring game. Let $S = \{v_i; 1 \leq i \leq n$ and i is an odd number$\}$. Then S is a vertex cover of $C_{n,1}$ and by Theorem 2.6, it is a dynamic monopoly for $C_{n,1}$.

Now we show $\text{dyn}(G) = |S|$. Suppose that T is a dynamic monopoly of the minimum size for $C_{n,1}$ such that $|T| < |S|$. So, there exist two adjacent vertices $v, w \in V(C_{n,1}) \setminus T$. By Lemma 2.2, the only possible cases for v, w are $\{v, w\} \in \{\{v_{n-1}, v_n\}, \{v_1, v_n\}, \{v_n, v_{n+1}\}\}$. Suppose that $v_n, v_{n-1} \in V(C_{n,1}) \setminus T$. Then by Lemma 2.2, $v_{n-2} \in T$ and by a similar argument as the proof of Lemma 2.2, $\{v_1, v_{n+1}\} \subseteq T$. Now since for each $1 \leq i \leq n-2$ we have $|\{v_i, v_{i+1}\} \cap T| \geq 1$ we conclude that $|T| \geq \lceil \frac{n}{2} \rceil$ which is a contradiction. The cases $\{v, w\} = \{v_1, v_n\}$ or $\{v, w\} = \{v_n, v_{n+1}\}$ can be discussed similarly.

Case 2. n is a natural even number.

First, we show that the repetitive polling game on $C_{n,1}$ is a fixed coloring game. Suppose that $X^0 = (x^0(v_1), \ldots, x^0(v_n), x^0(v_{n+1}))$ is a 2−periodic coloring. Then by Theorem 2.1, $x^0(v_n) = x^2(v_n) \neq x^1(v_n)$. So, $x^1(v_{n+1}) = White$ or $x^2(v_{n+1}) = White$. Since $\deg_{C_{n,1}}(v_{n+1}) = 1$, for each positive integer $t \geq 2$ we have $x^t(v_{n+1}) = White$ which ensures that for each positive integer $t \geq 2$ we have $x^t(v_n) = White$. This is a contradiction and so $C_{n,1}$ doesn't have a 2−periodic coloring.

Let $S = \{v_i \ ; \ 1 \leq i \leq n \text{ and } i \text{ is an even number}\}$. The same argument as the case 1 shows that S is a dynamic monopoly of the minimum size for $C_{n,1}$. □

Consider the cycles $C_n = (V_1, E_1)$ and $C_m = (V_2, E_2)$ where

$$V_1 = \{v_1, v_2, \ldots, v_n\},\ E_1 = \{\{v_i, v_{i+1}\}; 1 \leq i \leq n-1\} \cup \{\{v_1, v_n\}\}$$

and

$$V_2 = \{v_{n+1}, v_{n+2}, \ldots, v_{n+m}\},\ E_2 = \{\{v_i, v_{i+1}\}; n+1 \leq i \leq n+m-1\} \cup \{\{v_{n+1}, v_{n+m}\}\}.$$

Let $C_{n,m}$ be the link of C_n and C_m with the single edge $\{v_n, v_{n+1}\}$. In the following, we compute $\text{dyn}(C_{n,m})$.

Theorem 3.5. $\text{dyn}(C_{n,m}) = \begin{cases} \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil, & n+m \text{ is an even;} \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 1, & \text{otherwise.} \end{cases}$

Proof. We follow the proof in three cases.

Case 1. Both n and m are odd.

In this case, by Theorem 2.1, the repetitive polling game on $C_{n,m}$ is a fixed coloring game. Let $S = \{v_{n+1}\} \cup \{v_i \ ; \ 1 \leq i \leq n+m \text{ and } i \text{ is an odd number}\}$. Then S is a vertex cover of $C_{n,m}$ and by Theorem 2.6, it is a dynamic monopoly for it.

Now suppose that T is a dynamic monopoly of the minimum size for $C_{n,m}$ such that $|T| < |S|$ and $X^0 = (x^0(v_1), \ldots, x^0(v_{n+m}))$ be an initial coloring in which $x^0(v_i) = White$ if and only if $v_i \in T$. Therefore, there exist two adjacent vertices $v, w \in V(C_{n,m})$ of degree 2 such that $x^0(v) = x^0(w) = Black$ or $x^1(v) = x^1(w) = Black$. So by Lemma 2.2, we get a contradiction. Hence, $\text{dyn}(C_{n,m}) = |S| = \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$.

Case 2. Both n and m are even.

Let $S = \{v_i \ ; \ 1 \leq i \leq n \text{ and } i \text{ is an even number}\} \bigcup \{v_i \ ; \ n+1 \leq i \leq n+m \text{ and } i \text{ is an odd number}\}$. We claim S is a dynamic monopoly of minimum size for $C_{n,m}$. Suppose $X^0 = (x^0(v_1), \ldots, x^0(v_{n+1}))$ be an initial coloring in which $x^0(v_i) = White$ if and only if $v_i \in S$. it is clear that for each $t \in \mathbb{N}$ we have $x^t(v_n) = x^t(v_{n+1}) = White$. So by Theorem 2.1, there exists $t \in \mathbb{N}$ such that X^t is a fixed coloring. Now by the fact that S is a vertex cover of $C_{n,m}$ and using Theorem 2.6, we conclude that S is a dynamic monopoly of $C_{n,m}$.

Now suppose that T is a dynamic monopoly of the minimum size for $C_{n,m}$ such that $|T| < |S|$. By the same argument as the case 1, we get a contradiction. So, $\text{dyn}(C_{n,m}) = |S| = \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$.

Case 3. One of n and m is even and the other is odd.

Suppose n is odd and m is even. Let $S = \{v_i; \ 1 \leq i \leq n+m \text{ and } i \text{ is an odd number}\}$. We claim S is a dynamic monopoly of minimum size for $C_{n,m}$. Suppose $X^0 = (x^0(v_1), \ldots, x^0(v_{n+1}))$ be an initial coloring in which $x^0(v_i) = White$ if and only if $v_i \in S$. It is easy to see that for each $t \in \mathbb{N}$ we have $x^t(v_{n+1}) = White$. So by Theorem 2.1, there exists $t \in \mathbb{N}$ such that X^t is a fixed coloring. Now, as the case 2, by the fact that S is a vertex cover of $C_{n,m}$ and theorem 2.6, we conclude that S is a dynamic monopoly of $C_{n,m}$.

Now suppose that T is a dynamic monopoly of the minimum size for $C_{n,m}$ such that $|T| < |S|$. By the same argument as the case 1, we get a contradiction. So $\text{dyn}(C_{n,m}) = |S| = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 1$. □

3.2. Dynamic monopoly for the link of path graphs

Consider the path $P_n = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{\{v_i, v_{i+1}\}; 1 \leq i \leq n-1\}$. Let $P^i_{n,1}$ be the link of P_n and an isolated vertex v_{n+1} with the single edge $\{v_i, v_{n+1}\}$ for $v_i \in V$. In the following, we compute $\text{dyn}(P^i_{n,1})$.

Lemma 3.6. $\text{dyn}(P^i_{n,1}) = \begin{cases} \lceil \frac{n}{2} \rceil & n \text{ is an odd number and } i \in \{1, n\}. \\ \lfloor \frac{n}{2} \rfloor & \text{otherwise.} \end{cases}$

Proof. Note that by Theorem 2.1, the repetitive polling game on $P^i_{n,1}$ is a fixed coloring game. We follow the proof in two cases.

Case 1. n is an odd number and $i \in \{1, n\}$.

In this case, it is enough to apply Corollary 2.4.

Case 2. n is an even number or $1 < i < n$.

In case $i = 1$, we suppose that

$$S = \{v_i;\ 1 \leq i \leq n \text{ and } i \text{ is an odd number}\}.$$

Otherwise, let

$$S = \{v_i;\ 1 \leq i \leq n \text{ and } i \text{ is an even number}\}.$$

In both conditions, we claim S is a dynamic monopoly of minimum size for $P^i_{n,1}$. Suppose $X^0 = (x^0(v_1), \dots, x^0(v_{n+1}))$ be an initial coloring in which $x^0(v_i) = White$ if and only if $v_i \in S$. It is easy to see that S is a vertex cover of $P^i_{n,1}$. So, by theorem 2.6, we conclude that S is a dynamic monopoly of $P^i_{n,1}$ which ensures that S is a dynamic monopoly.

Now suppose that T is a dynamic monopoly of the minimum size for $P^i_{n,1}$ such that $|T| < |S|$ and $X^0 = (x^0(v_1), \dots, x^0(v_{n+1}))$ be an initial coloring in which $x^0(v_i) = White$ if and only if $v_i \in T$. Therefore, there exist two adjacent vertices $v, w \in V(P^i_{n,1})$ of degree at least 2 such that $x^t(v) = x^t(w) = Black$ for each $t \geq 2$ which is a contradiction. □

Consider the paths $P_n = (V_1, E_1)$ and $P_m = (V_2, E_2)$ where

$$V_1 = \{v_1, v_2, \dots, v_n\},\ E_1 = \{\{v_i, v_{i+1}\}; 1 \leq i \leq n-1\},$$

and

$$V_2 = \{v_{n+1}, v_{n+2}, \dots, v_{n+m}\},\ E_2 = \{\{v_i, v_{i+1}\}; n+1 \leq i \leq n+m-1\}.$$

Let $P_{n,m}$ be the link of P_n and P_m with the single edge $\{v_n, v_{n+1}\}$. In the following, we compute $\text{dyn}(P_{n,m})$.

Lemma 3.7. $\text{dyn}(P_{n,m}) = \lfloor \frac{n+m}{2} \rfloor$.

Proof. By Corollary 2.4, the conclusion follows. □

Consider the paths $P_n = (V_1, E_1)$ and $P_m = (V_2, E_2)$, $n \geq 3,\ m \geq 2$, where

$$V_1 = \{v_1, v_2, \dots, v_n\}, E_1 = \{\{v_i, v_{i+1}\}; 1 \leq i \leq n-1\}$$

and

$$V_2 = \{v_{n+1}, v_{n+2}, \dots, v_{n+m}\},\ E_2 = \{\{v_i, v_{i+1}\}; n+1 \leq i \leq n+m-1\}.$$

Let $P_{i,n,m}$ be the link of P_n and P_m with the single edge $\{v_i, v_{n+1}\}$ such that $1 \leq i \leq n$. In the following, we compute $\text{dyn}(P_{i,n,m})$.

Lemma 3.8. $\text{dyn}(P_{i,n,m}) = \begin{cases} \lfloor \frac{n+m}{2} \rfloor & i \in \{1, n\} \\ \lfloor \frac{n+m-1}{2} \rfloor & i \in \{2, n-1\} \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor & 3 \leq i \leq n-2 \text{ and } (i \text{ is even or } m = 2) \\ \lceil \frac{n}{2} \rceil + \lfloor \frac{m}{2} \rfloor & \text{Otherwise.} \end{cases}$

Proof. Note that by Theorem 2.1, the repetitive polling game on $P_{i,n,m}$ is a fixed coloring game. We follow the proof in three cases.

Case 1. $i \in \{1, n\}$.

In this case, $P_{i,n,m} = P_{n,m}$. So, By Lemma 3.7, the result is obvious.

Case 2. $i \in \{2, n-1\}$.

In this case $P_{i,n,m} = P^i_{n+m-1,1}$. So, by Lemma 3.6, the conclusion follows.

Case 3. $3 \leq i \leq n-2$ and (i is even or $m = 2$).

Let $S = \{v_j;\ 1 \leq j \leq n \text{ and } j \text{ is an even number}\} \bigcup \{v_j\ ;\ n+2 \leq j \leq n+m \text{ and j} \equiv n+2 \mod 2\}$. One can see that S is a vertex cover of $P_{i,n,m}$, so by Theorem 2.6, S is a vertex cover of $P_{i,n,m}$ which ensures that S is a dynamic monopoly. By the same argument as Lemma 3.6, one can see that S is a dynamic monopoly of the minimum size for $P_{i,n,m}$. Thus, $\text{dyn}(P_{i,n,m}) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$.

Case 4. $3 \leq i \leq n-2$ and (i is odd and $m > 2$).

Let $S = \{v_j;\ 1 \leq j \leq n \text{ and } j \text{ is an odd number}\} \bigcup \{v_j;\ n+2 \leq j \leq n+m \text{ and j} \equiv n+2 \mod 2\}$. One can see that S is a vertex cover of $P_{i,n,m}$, so by Theorem 2.6, S is a dynamic monopoly. By the same argument as Lemma 3.6, one can see that S is a dynamic monopoly of the minimum size for $P_{i,n,m}$. Thus, $\text{dyn}(P_{i,n,m}) = \lceil \frac{n}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. □

Consider the paths $P_n = (V_1, E_1)$ and $P_m = (V_2, E_2)$, $n, m \geq 3$, where

$$V_1 = \{v_1, v_2, \ldots, v_n\},\ E_1 = \{\{v_i, v_{i+1}\}; 1 \leq i \leq n-1\},$$

and

$$V_2 = \{v_{n+1}, v_{n+2}, \ldots, v_{n+m}\},\ E_2 = \{\{v_i, v_{i+1}\}; n+1 \leq i \leq n+m-1\}.$$

Let $P_{i,j,n,m}$ be the link of P_n and P_m with the single edge $\{v_i, v_j\}$ such that $1 \leq i \leq n$ and $n+1 \leq j \leq n+m$. In the following, we compute $\text{dyn}(p_{i,j,n,m})$.

Theorem 3.9. $$\text{dyn}(P_{i,j,n,m}) = \begin{cases} \text{dyn}(P_{i,n,m}), & \textit{i=1,2,n-1,n and j=n+1,n+m.} \\ \text{dyn}(P_{j,m,n}), & \textit{i=1,n and j=n+1,n+2,n+m-1,n+m.} \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor, & \textit{otherwise.} \end{cases}$$

Proof. According to Lemma 3.8, it is enough to examine the third case. Choose

$$S = \{v_i;\ 1 \leq i \leq n \text{ and } \ i \equiv 2 \mod 2\} \bigcup \{v_i;\ n+1 \leq i \leq n+m \ \text{ and } \ i \equiv n+1 \mod 2\}.$$

Similar to the proofs of the previous Lemmas, the conclusion follows. □

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Please cite this article using:

Leila Musavizadeh Jazaeri, Leila Sharifan, Dynamic monopolies in simple graphs, AUT J. Math. Comput., 6(2) (2025) 151-162
https://doi.org/10.22060/AJMC.2024.22585.1178