Amirkabir University of Technology
(Tehran Polytechnic)
Vol. 46 - No. 1 - Spring 2014, pp. 57- 64

Amirkabir International Journal of Science & Research
(Modeling, Identification, Simulation & Control)
(AIJ-MISC)

$(n, 1, 1, \alpha)$-Center Problem

P. Kavand [1], A. Mohades [2*], and M. Eskandari [3]

1- PhD. Student of Computer Science, Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran.
2- Associate Professor, Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran,
3- Assistant Professor, Department of Mathematics, Alzahra University, Tehran, Iran.

ABSTRACT

Given a set S of n points in the plane and a constant α,$(n, 1, 1, \alpha)$-center problem is to find two closed disks which each covers the whole S, the diameter of the bigger one is minimized, and the distance of the two centers is at least α. Constrained $(n, 1, 1, \alpha)$-center problem is the $(n, 1, 1, \alpha)$-center problem in which the centers are forced to lie on a given line L. In this paper, we first introduce $(n, 1, 1, \alpha)$-center problem and its constrained version. Then, we present an $O(n \log n)$ algorithm for solving the $(n, 1, 1, \alpha)$-center problem. Finally, we propose a linear time algorithm for its constrained version.

KEYWORDS

Computational Geometry; K-Center Problem; Farthest Point Voronoi Diagram; Center Hull.

*Corresponding Author, Email: mohades@aut.ac.ir

1. INTRODUCTION

Facility location problems concern the choice of the location of one or multiple facilities, in a given geographical space and subject to some constraints, to optimally fulfill predetermined objectives. Facility location problems are usually categorized based on some characteristics such as: number of facilities (single/ multiple), objective function (single/ multiple), solution space (discrete/ network/ continuous), number of commodities (single/ multiple), capacity limitation (yes/no), shape of facility (point/ extensive), and demand (discrete/ continuous). However, it should be noted that these characteristics are not restricted to the above mentioned ones. In the literature, many concepts, tools and techniques of artificial intelligence such as fuzzy logic can be used for the implementation of numerous models in operations research. So, some of the works in the area of facility location have used fuzzy theory to fuzzify the parameters of the model or have dealt with the facility location problem as a fuzzy multiple attribute decision making problem [1].

An important problem in facility location is k-center problem. Let S be a set of n points in the plane, the Euclidean k-center problem in $\mathbb{R}^d$, is to find k congruent closed balls of minimum radius whose union covers S. The k-center problem has been known to be NP-complete when parameter k is a part of the input [2]. Agarwal and Procopiuc showed that k-center problem in $\mathbb{R}^d$ can be solved in $n^{O\left(k^{1-1/d}\right)}$ time [3].

The smallest enclosing circle problem in the Euclidean plane (Euclidean 1-center problem) is a special case of the planar k-center problem in the Euclidean space. In 1857 the 1-center problem was presented, for the first time, by Sylvester [4] and finally N. Megiddo presented a linear time algorithm for solving this problem in 1983 [5]. He also presented a linear time algorithm for finding the center of the smallest enclosing circle on a given line.

The other special case of the k-center problem is the 2-center problem which has many applications in transportation [6]. Hershberger and Suri [7] solved the decision version of 2-center problem in $O(n^2 \log n)$ time. Agarwal and Sharir [8] obtained an $O(n^2 log^3 n)$ time algorithm for solving the 2-center problem in the plane, using the parametric searching paradigm of Megiddo [9]. Eppstein presented a randomized algorithm whose expected running time is $O(n^2 log^2 n \log\log n)$ [10], and Jaromczyk and Kowaluk proposed a deterministic algorithm whose running time is $O(n^2 \log n)$ [11]. Sharir [12] gave a near-linear algorithm for the 2-center problem in the plane with $O(nlog^9 n)$ running time. Then, Eppstein [13] improved the previous bounds and gave a randomized algorithm for solving this problem with $O(nlog^2 n)$ expected time. Finally, Chan [14] presented a deterministic algorithm whose running time is $O(nlog^2 nlog^2 \log n)$.

A variant of the 2-center problem which is known as alpha-connected two-center problem was considered in [15]. Let S be a set of n points in the plane and α be a constant. The alpha-connected two-center problem is to find two congruent closed disks, whose radii is as small as possible, their union covers S, and the distance of the two centers is at most $2(1-\alpha)r$ in which r is the optimal radius. An $O(n^2 log^2 n)$ expected-time algorithm was presented for solving this problem in [15].

This paper considers the $(n, 1, 1, \alpha)$-center problem and its constrained version. Given a set S of n points in the plane and a constant α, the $(n, 1, 1, \alpha)$-center problem is to find two closed disks each of which covers the whole S, the diameter of the bigger one is minimized, and the distance of the two centers is at least α. The constrained $(n, 1, 1, \alpha)$-center problem is the $(n, 1, 1, \alpha)$-center problem in which the centers are forced to lie on a given line L. This paper presents an $O(n \log n)$ time algorithm for solving the $(n, 1, 1, \alpha)$-center problem, and a linear time algorithm for its constrained version using the farthest point Voronoi diagram.

An important application of this problem appears in "Urban Design". Assume that we want to locate some facility centers in an area. The goal is designing an efficient arrangement of such facility centers such that the maximum distance between the customers and their nearest facility center of each type is minimized. In addition, the distance between each two facility centers is an admissible distance. For example, it is not smaller than a predefined distance, due to the density of the facility centers that causes to have a crowded region. In this study we want to locate two facility centers of two different types in a city or along a street of a city (in the case of the constrained version) such that the maximum distance between the customers and each facility center is minimized. Meanwhile, the distance between these two facilities should not be less than the predefined distance α.

This paper is organized as follows. Section 2 contains some definitions and notations. Section 3 considers the $(n, 1, 1, \alpha)$-center decision problem. In Section 4 we present an $O(n \log n)$ algorithm for solving the $(n, 1, 1, \alpha)$-center problem. Section 5 gives an $O(n)$ algorithm for solving the constrained $(n, 1, 1, \alpha)$-center problem.

2. Definitions And Notations

Let $S = \{s_1, s_2, \dots, s_n\}$ be a set of n points and L be a line in the plane. The Euclidean distance between two points s_i and s_j in the plane is represented by $d(s_i, s_j)$. We also denote the Euclidean distance between the point s_i and the line L in the plane by $d(s_i, L)$.

Let p be a point in the plane and r be a positive real number, the circle and the closed disk each centered at p with radius r are represented by $C(p, r)$ and $D(p, r)$, respectively.

Let $S = \{s_1, s_2, \dots, s_n\}$ be a set of n demand points in the plane, α be a constant, and $P = \{a_1, a_2, \dots, a_p\}$ and $Q = \{b_1, b_2, \dots, b_q\}$ be two sets containing$p$ and q facility centers of two different types a and b, respectively. The (n, p, q, α)-center problem is to locate these $p + q$ faciliy centers in the plane such that:

$$min\, max \left(\max_{1 \le i \le n} \min_{1 \le j \le p} d(s_i, a_j) , \max_{1 \le i \le n} \min_{1 \le j \le q} d(s_i, b_j) \right) \qquad (1)$$

$$s.t. \qquad d(a_i, b_j) \ge \alpha \forall i, j; 1 \le i \le p, 1 \le j \le q.$$

The $(n, 1, 1, \alpha)$-center problem is a special case of the (n, p, q, α)-center problem. In other words, for a given set S of n points and a constant α, the $(n, 1, 1, \alpha)$-center problem is to find two closed disks each of which covers the whole S, the diameter of the bigger one is minimized, and the distance of the two centers is at least α.

3. $(\boldsymbol{n}, \mathbf{1}, \mathbf{1}, \boldsymbol{\alpha})$-Center Decision Problem

In this section, we first present the definition of a geometric object which is called center hull, also known as the circular hull [15, 16]. Then, we propose a trivial $O(n \log n)$ algorithm for solving the $(n, 1, 1, \alpha)$-center decision problem using the center hull.

Given a set S of n points in the plane, a constant α, and a positive real number r. The $(n, 1, 1, \alpha)$-center decision problem is to decide whether S can be covered by two congruent closed disks of radius r, such that each of them covers the whole S and the distance of the two centers is at least α.

Definition 3- 1- [16] The center hull of S with radius r, denoted by $CENH(S, r)$, is the locus of points which are the centers of circles of radius r that cover S, that is,

$$\text{CENH}(S, r) = \{p \mid C(p, r) \text{covers} S\}$$

In other words, the center hull of S with radius r, is the intersection of all the disks centered at the points of S with radius r, i.e., $CENH(S, r) = \bigcap_{s \in S} D(s, r)$ [15].

Lemma 3- 2- Let $S = \{s_1, s_2, \dots, s_n\}$ be a set of n points, $FPVD(S)$ be the farthest-point Voronoi diagram of S, and $V(s_i)$ be the Voronoi region associated with an element s_i. Within each Voronoi region, say $V(s_i)$, $\partial\text{CENH}(S, r)$ is a subarc of a circle of radius r centered at s_i [16].

Lemma 3- 3- $\partial\text{CENH}(S, r)$ is a circular list of subarcs of circles of radius $r$each centered at the associated farthest-neighbor in S [16].

Proposition 3- 4- The $(n, 1, 1, \alpha)$-center decision problem can be solved in $O(n \log n)$ time.

Proof. The solution of the decision problem is positive if and only if the diameter of the $CENH(S, r)$ is greater than or equal to α. Since the farthest point Voronoi diagram and the center hull can be constructed in $O(n \log n)$ time [16], and the diameter of the center hull can be computed in $O(n)$ time [17], the decision problem can be solved in $O(n \log n)$ time.

4. $(\boldsymbol{n}, \mathbf{1}, \mathbf{1}, \boldsymbol{\alpha})$-Center Problem

Let S be a set of n points in the plane and α be a constant. We wish to find two closed disks each of which covers the whole S, the diameter of the bigger one is minimized, and the distance of the two centers is at least α.

In this section, we propose an $O(n \log n)$ time algorithm for solving the $(n, 1, 1, \alpha)$-center problem.

Lemma 4- 1- If r and r' are the optimal radii of the $(n, 1, 1, \alpha)$-center problem, then $r = r'$.

Proof. Let $D(o, r)$ and $D(o',\ r')$ be the optimal disks of the $(n, 1, 1, \alpha)$-center problem. Assume without loss of generality that the line L passing through o and o' is horizontal and o' is on the right side of o. Let a and b be the upper and lower intersection points of $C(o, r)$ and $C(o',\ r')$, respectively, and ρ and ρ' be the subarcs of the circles $C(o, r)$ and $C(o', r')$ constructing the boundary of the intersection region of $D(o, r)$ and $D(o', r')$, respectively. Suppose that $r > r'$. For $r' > r$, we can do in a similar way. Since S has located in the intersection area of $D(o, r)$ and $D(o',\ r')$, by moving o on L towards o', $D(o_{new}, r_{new})$ in which o_{new} is the new position of o and $r_{new} = d(o_{new}, a)$, still covers S. To prove our claim, it is sufficient to show that any point t which lies on ρ is covered by $D(o_{new}, r_{new})$. Since $d(a, L) \ge d(t, L)$, the area of triangle Δoao_{new} is greater than or equal to the area of the triangle Δoto_{new}. By using the Heron's formula for calculating the area of a triangle, we can have $d(o_{new}, t) \le d(o_{new}, a)$; so the result is followed.

On the other hand, $r_{new} < r$: Suppose that the vertical line passing through a intersects L in a point z. We have:

$$r^2 = d^2(o,z) + d^2(a,z),$$
$$r_{new}^2 = d^2(o_{new},z) + d^2(a,z). \quad (2)$$

Since $d(o,z) > d(o_{new},z)$, $r > r_{new}$.Therefore, by moving o on Ltowardso', $D(o_{new}, r_{new})$ still covers S and $r_{new} < r$. Thus, if $d(o,o') > \alpha$, we can move o on L towards o' such that $d(o_{new}, o')$ is still more than or equal to α and also the solution becomes better which is a contradiction with the optimality, so $d(o,o') = \alpha$. In this case,o' can also be moved on L in a direction that its distance from o increases. Let q be a point of S farthest from o'_{new} (the new position of o'). It is obvious that $D(o'_{new}, r'_{new})$ covers S and $r'_{new} \geq r'$. Thus, according to what is said, o and o' can be moved on L such that $d(o_{new}, o'_{new}) \geq \alpha$, $r_{new} < r$ and $r'_{new} \leq r_{new}$. This means that r and r' cannot be the optimal radii which is a contradiction (See Fig. 1). □

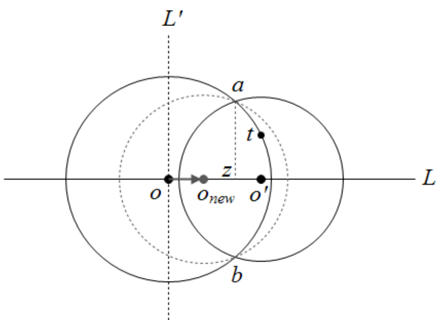

Fig. 1. The left disk is $D(o,r)$, the right disk is $D(o',\ r')$, and $r > r'$

Lemma 4- 2- If $D(o,r)$ and $D(o',\ r')$ are the optimal disks of the $(n,1,1,\alpha)$-center problem, then $d(o,o') = \alpha$.

Proof. Assume without loss of generality that the line L passing through o and o' is horizontal and o' lies on the right side of o. Let a be the upper intersection point of $C(o,r)$ and $C(o',\ r')$. It follows from Lemma 4-1 that $r = r'$. Suppose that $d(o,o') > \alpha$, hence o can be moved on L towards o' such that $d(o,o') \geq \alpha$. It can be proved in a similar way that is said in the previous Lemma that $D(o_{new}, r_{new})$, in which o_{new} is the new position of o and $r_{new} = d(o_{new}, a)$, covers S and $r_{new} < r$, so $r' = r > r_{new}$. It follows from Lemma 4-1 that r' and r_{new} can not be the optimal radii and they can be improved. Thus,o and o' cannot also be the optimal centers which is a contradiction with the first hypothesis (See Fig. 2). □

Lemma 4- 3- Let $D = aa'$ be the diameter of the $CENH(S,r)$. Only one of the following two cases can happen,

(i) Both a and a' are the vertices of the $CENH(S,r)$.

(ii) a is a vertex and a' is a point on an arc A' of the $CENH(S,r)$ which is the intersection point of A' and the line passing through a and c' (the associated center of A'), and also $d(c',a') \leq d(a,a')$.

An example is illustrated in Fig. 3 for the case (ii).

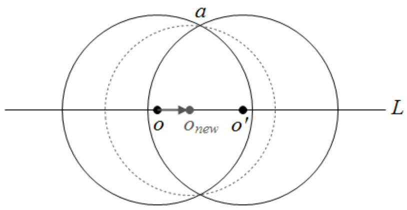

Fig. 2. The left disk is $D(o,r)$, the right disk is $D(o',\ r')$, $r = r'$, and $d(o,o') > \alpha$.

Proof. At least one of a and a' must be a vertex of the center hull, since otherwise we can move a or a' to be a vertex, and so the length of aa' will increase. For instance, in Fig. 4, by moving a and a' on A and A', respectively, with the same speed and in two different directions, clockwise and counterclockwise, $d(a,a')$ will increase. Thus, aa'cannot be the diameter of the $CENH(S,r)$. On the other hand, if a is a vertex and a' is a point on an arc $A'$of the center hull, a' must be the intersection point of A' and the line passing through a and c' (the associated center of A') and also $d(c',a') \leq d(a,a')$, since otherwise we can move a' on A' in a direction such that the length of aa' increases. For instance, in Fig. 5, by moving a' on A' the distance between a and a' will increase. As a result,aa' cannot be the diameter of the $CENH(S,r)$. Other cases are similar to Fig. 4 and Fig. 5.

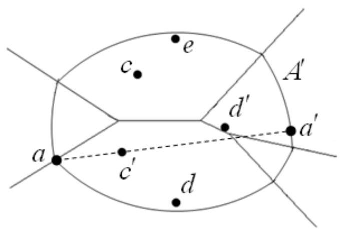

Fig. 3. The farthest point Voronoi diagram and the associated center hull of the point set $S = \{c, c', d, d', e\}$ with some radius r. $a'$is the intersection point of A' and the line passing through a and c' (the associated center of A'). $aa'$is a candidate for the diameter of the $CENH(S,r)$.

In Fig. 4 and Fig. 5, let o be the midpoint of the line segment aa', c and c' be the associated centers of the arcs A and A', respectively, L be the line passing through c and c', and b and b' be the intersection points of L and the arcs A and A', respectively. □

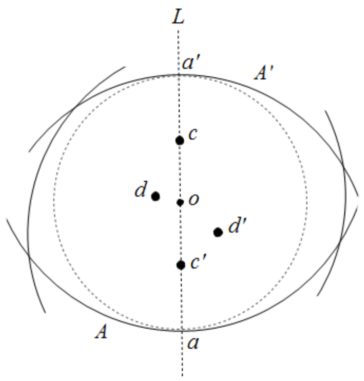

Fig. 4. The center hull of the point set $S = \{c, c', d, d'\}$ with a radius r. $a = b$and $a' = b'$. The dotted circle $C(o, t)$ in which $t = d(o, a) = d(o, a')$, is inner tangent to both $C(c, r)$ and $C(c', r)$ at the points a and a', respectively.

Lemma 4- 4- Let $f(r)$, as a function of the radius r, be the diameter of the $CENH(S, r)$. $r^*$is the optimal radius of the $(n, 1, 1, \alpha)$-center problem if and only if $f(r^*) = \alpha$.

Proof. Assume for a contradiction that $f(r^*) > \alpha$. There exists a real number $\varepsilon > 0$ such that the diameter of the $CENH(S, r^* - \varepsilon)$ is greater than or equal to α. This means thatr^* is not the optimal radius which is a contradiction.

Conversely, suppose for a contradiction that $f(r^*) = \alpha$, r_0 be the optimal radius, and $r_0 \neq r^*$. Thus, $r_0 < r^*$. Since f is a strictly monotone increasing function, $f(r_0) < f(r^*) = \alpha$. Therefore, r_0 is not feasible which is a contradiction. □

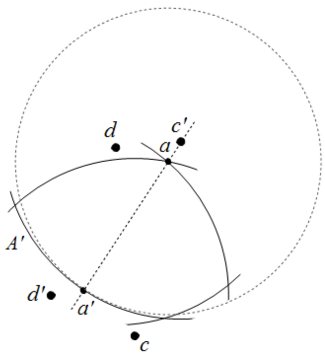

Fig. 5. The center hull of the point set $S = \{c, c', d, d'\}$ with some radius r.$a$is a vertex of the $CENH(S, r)$, a' is a point on the arc A' which is the intersection point of A' and the line passing through a and c', and $d(s_k, a') > d(a, a')$.The dotted circle $C(a, d(a, a'))$ is inner tangent to the circle $C(s_k, r)$ at the point a'.

A critical radius is a radius in which the center hull undergoes a combinatorial change. That is, by increasing the radius by $\varepsilon > 0$, an arc is added to the center hull. If we gradually increase the radius, a critical radius will occur while the center hull intersects with a vertex of the farthest point Voronoi diagram [15].Since the farthest point Voronoi diagram of a set of n points has $O(n)$ vertices [18], the number of critical radii is $O(n)$.The critical radius corresponding to eachVoronoi vertex can be determined by computing the distance between the vertex and its associated farthest neighbor in S, so each critical radius can be determined in constant time. As a result, all critical radii can be determined in linear time.

Theorem 4- 5-The $(n, 1, 1, \alpha)$-center problem can be solved in $O(n \log n)$ time.

Proof. Let $r_1, r_2, \dots, r_k$ be the sequence of critical radii in increasing order, and $d_1, d_2, \dots, d_k$ be its associated diameters sequence. Since the diameter function of the center hull is a strictly monotone increasing function in radius, this sequence is also in increasing order. Let $D(o, r^*)$ and $D(o', r^*)$ be the optimal disks. By applying the binary search on $r_1, r_2, \dots, r_k$, one of the following two cases will happen,

(i) The radius r_t will be found such that $d_t = \alpha$, and so$r_t = r^*$.

(ii) An interval (r_{t-1}, r_t) will be found such that $r_{t-1} < r^* < r_t$, and $d_{t-1} < \alpha < d_t$ (it is possible to r_t be infinity).

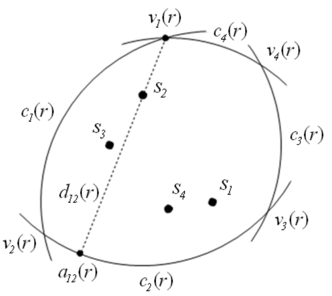

Fig. 6. The center hull of $S = \{s_1, s_2, s_3, s_4\}$ with a radius r. For the antipodal pair $v_1(r)$and $c_2(r)$, $a_{12}(r)$ is the point on the arc $c_2(r)$ farthest from $v_1(r)$, and $d_{12}(r) = d(v_1(r), a_{12}(r))$.

Since the diameter of the center hull of a set of n points can be determined in $O(n)$ time, this procedure can be done in $O(n \log n)$ time. From Lemma 4- 4, if the case (i) occurs, there is nothing more to do. In case (ii), by changing the radius in the interval (r_{t-1}, r_t), the center hull will not undergo any combinatorial change, so the number of arcs and vertices remains unchanged in this interval. Let m be the number of the vertices of the center hull in this interval. Obviously, $m \in O(n)$. Let $v_1(r), c_1(r), v_2(r), c_2(r), \dots, v_m(r), c_m(r)$ be the sequence of the vertices and the arcs of the $CENH(S, r)$ in counterclockwise order, in which $r \in (r_{t-1}, r_t)$, ands_i be the associated center of the arc $c_i(r)$. For any antipodal pair of the vertex $v_i(r)$ and the arc $c_j(r)$, we define $a_{ij}(r)$

to be a point on the arc $c_j(r)$, farthest from $v_i(r)$, and $d_{ij}(r)$ to be the distance between $v_i(r)$ and $a_{ij}(r)$ (See Fig. 6). For any antipodal pair of vertices, $v_i(r)$ and $v_j(r)$, we define $d'_{ij}(r)$ to be the distance between $v_i(r)$ and $v_j(r)$. The number of antipodal pairs is $O(n)$. On the other hand $d_{ij}(r)$ and $d'_{ij}(r)$ each can be computed in constant time. So, all $d_{ij}(r)$s and $d'_{ij}(r)$s can be computed in $O(n)$ time. It is sufficient to find the smallest radius r in the interval (r_{t-1}, r_t) such that $d_{ij}(r) = \alpha$or $d'_{ij}(r) = \alpha$, for some i and j. The equations $d_{ij}(r) = \alpha$and $d'_{ij}(r) = \alpha$ can be solved in constant time for each i and j, so all these equations can be solved in $O(n)$ time and also finding the smallest radius in the solution set can be done in $O(n)$ time. This smallest radius is the optimal radius and the endpoints of its associated diameter are the optimal positions for the centers. Thus, the $(n,1,1,\alpha)$-center problem can be solved in $O(n \log n)$ time. □

We have implemented our algorithm for the $(n,1,1,\alpha)$-center problem. Five experimental results are shown in Table. 1. Let $S = \{s_1, s_2, \dots, s_{10}\}$ be a set of ten points in the plane, o and o' be the optimal centers, and r be the optimal radius for the $(n,1,1,\alpha)$-center problem for S.

5. CONSTRAINED$(n,1,1,\alpha)$-CENTER PROBLEM

Let S be a set of n points in the plane, α be a constant and L be a line, the constrained $(n,1,1,\alpha)$-center problem is the $(n,1,1,\alpha)$-center problem in which the centers are forced to lie on L.In this section, we first propose an $O(n \log n)$ time algorithm for solving the constrained $(n,1,1,\alpha)$-center problem, then we improve the time to $O(n)$.

Lemma 5- 1- If r and r' are the optimal radii of the constrained $(n,1,1,\alpha)$-center problem, then $r = r'$.

Proof. The proof is similar to Lemma 4-1. □

Lemma 5- 2- If $D(o,r)$ and $D(o',\ r')$ are the optimal disks of the constrained $(n,1,1,\alpha)$-center problem, then $d(o,o') = \alpha$.

Proof. The proof is similar to Lemma 4-2. □

From now on, assume without loss of generality that L is the x-axis.

Observation 5- 3- Let S be a set of points in the plane, and o_L and o_R be two points on L such that $o_L$lies on the left side of o_R. For any point in the plane such as p, let $F_S(p)$ be the point in S farthest from p. It can easily be seen that if $d(o_R, F_S(o_R)) > d(o_L, F_S(o_L))$, by moving o_L and o_R to the right on L such that their distance remains unchanged, again $d(o_R, F_S(o_R)) > d(o_L, F_S(o_L))$. It can also be shown that if $d(o_L, F_S(o_L)) > d(o_R, F_S(o_R))$, by moving o_L and o_R to the left on L such that their distance remainsunchanged, again $d(o_L, F_S(o_L)) > d(o_R, F_S(o_R))$.

Theorem 5- 4- The constrained $(n,1,1,\alpha)$-center problem can be solved in $O(n)$ time.

Proof. Let $D(o,r)$ and $D(o',\ r')$ be the optimal disks of the constrained $(n,1,1,\alpha)$-center problem. From Lemmas 5-1 and 5-2, $r = r'$and $d(o,o') = \alpha$. On the other hand, From the Observation 5-3, there exists just one pair of points, say o_L and o_R, on L such that $d(o_L, o_R) = \alpha$ and $d(o_L, F_S(o_L)) = d(o_R, F_S(o_R))$, which are the optimal centers of the constrained $(n,1,1,\alpha)$-center problem. Let $t_1, t_2, \dots, t_m$ be the increasing sequence of the intersection points of L and the farthest point Voronoi diagram of S. This sequence can be determinedin $O(n \log n)$. Thus,$L = (-\infty, t_1] \cup (t_1, t_2] \cup \dots \cup (t_{m-1}, t_m] \cup (t_m, +\infty)$. There are $O(n)$ pairs of these intervals such that one of them contains o and the other one contains o'. It is sufficient to compute the solution of the constrained $(n,1,1,\alpha)$-center problem for each pair of the farthest neighbors corresponding to these pairs of the intervals. Notice that these two intervals, and so these two points can be the same. This takes $O(n)$ time. The solution with the smallest radius is the optimal radius. Now, we want to improve the time to $O(n)$.

Let $D(o,r)$ and $D(o',r)$ be the optimal disks of the constrained $(n,1,1,\alpha)$-center problem. We can write $o' = o + \vec{v}$, in which $\vec{v}$ is a vector of length α in the direction of L. The constrained $(n,1,1,\alpha)$-center problem can be modeled as follows:

$$\min r$$

Subject to:

$$\begin{aligned} & d(o,s) \le r \qquad \forall s \in S \\ & d(o',s) \le r \qquad \forall s \in S \\ & o, o' \in L \end{aligned} \tag{3}$$

On the other hand, for any $s \in S$, we have:

$$d(o',s) = d(o + \bar{v}, s) = d(o, s - \vec{v}). \tag{4}$$

Therefore, the constrained $(n,1,1,\alpha)$-center problem can be written as follows:

$$\min r$$

TABLE 1. EXPERIMENTAL RESULT. LET $\boldsymbol{S} = \{\boldsymbol{s_1}, \boldsymbol{s_2}, \dots, \boldsymbol{s_{10}}\}$ BE A SET OF TEN POINTS IN THE PLANE, $\boldsymbol{o}$ AND $\boldsymbol{o'}$ BE THE OPTIMAL CENTERS, AND$\boldsymbol{r}$ BE THE OPTIMAL RADIUS FOR THE $(\boldsymbol{n}, \mathbf{1}, \mathbf{1}, \boldsymbol{\alpha})$-CENTER PROBLEM FOR $\boldsymbol{S}$.

Expriment	*Point set*		α	o	o'	r
1	s_1	(371.4941, 403.0961)	40	(323.0507, 286.2636)	(295.8908, 256.8980)	233.2640
	s_2	(397.2530, 256.8908)				
	s_3	(362.1912, 168.4747)				
	s_4	(382.4184, 112.7331)				
	s_5	(210.7692, 118.4686)				
	s_6	(138.8527, 429.3831)				
	s_7	(377.9314, 226.8398)				
	s_8	(480.0888, 113.7784)				
	s_9	(275.4977, 252.6234)				
	s_{10}	(406.2067, 418.0800)				
2	s_1	(174.7490, 295.9058)	40	(319.4417, 287.4060)	(312.0549, 248.0939)	172.2570
	s_2	(278.2345, 358.5252)				
	s_3	(383.7459, 401.8747)				
	s_4	(210.4100, 371.8811)				
	s_5	(362.0392, 165.0447)				
	s_6	(147.5991, 299.3456)				
	s_7	(483.8976, 236.1543)				
	s_8	(334.1071, 189.5248)				
	s_9	(400.5068, 202.0380)				
	s_{10}	(302.3828, 379.6307)				
3	s_1	(65.5000, 562.5624)	15	(61.2098, 509.3685)	(74.8002, 515.7167)	90.2079
	s_2	(57.5000, 526.6223)				
	s_3	(98.5000, 526.6223)				
	s_4	(28.5000, 462.7288)				
	s_5	(87.5000, 475.7072)				
	s_6	(138.5000, 485.6905)				
	s_7	(127.5000, 570.5491)				
	s_8	(73.5000, 430.7820)				
	s_9	(105.5000, 430.7820)				
	s_{10}	(17.5000 585.5241)				
4	s_1	(466.8775, 214.3356)	80	(270.4197, 212.1689)	(288.3998, 290.1223)	231.0602
	s_2	(402.8801, 401.4916)				
	s_3	(252.1783, 327.1287)				
	s_4	(130.3417, 121.5800)				
	s_5	(312.3190, 411.6669)				
	s_6	(473.6043, 151.9625)				
	s_7	(327.5295, 287.7563)				
	s_8	(104.7608, 234.8491)				
	s_9	(164.8729, 417.7138)				
	s_{10}	(224.4860, 311.4133)				
5	s_1	(166.2595 340.7928)	80	(306.6794, 346.3628)	(244.5113, 296.0127)	214.4731
	s_2	(205.1885 361.6316)				
	s_3	(375.6858 399.2606)				
	s_4	(280.2166 133.5286)				
	s_5	(191.5908 465.3349)				
	s_6	(160.9512 430.3268)				
	s_7	(315.3370 498.4539)				
	s_8	(131.2702 277.0713)				
	s_9	(142.6611 484.7592)				
	s_{10}	(101.8537 409.9642)				

Subject to:

$$d(o,s) \le r \qquad \forall s \in S$$
$$d(o,s-\vec{v}) \le r \qquad \forall s \in S \tag{5}$$
$$o \in L$$

Obviously, this problem is equivalent to the smallest enclosing circle of the point set $S \cup \{s-\vec{v} | s \in S\}$ with this constraint that the center is forced to lie on L. In 1983, N. Megiddo presented a linear time algorithm for finding the center of the smallest enclosing circle on a given line [5]. Hence, the constrained $(n,1,1,\alpha)$-center problem for a set of n points in the plane can be solved in $O(n)$ (See Algorithm 1). □

Algorithm 1: A linear time algorithm for the constrained $(n,1,1,\alpha)$-center problem.

Input: A set S of n points in the plane, a constant α, and a line L.
Output: Two congruent disks centered at o and o' with the smallest radius such that their intersection covers S, o and o' locate on L, and $d(o,o') \le \alpha$.
begin

$S' \leftarrow \{s - \vec{v} | s \epsilon S\}$.
Find the smallest enclosing circle of $S \cup S'$.
$o \leftarrow$ the center of this circle.
$o' \leftarrow o + \vec{v}$
return o and o'
end

6. CONCLUSION

In this study, we introduced $(n, 1, 1, \alpha)$-center problem, and also a version of this problem which is called constrained $(n, 1, 1, \alpha)$-center problem. We presented an $O(n \log n)$ algorithm for finding two closed disks each of which covers the whole S, the diameter of the bigger one is minimized, and the distance of the two centers is at least α.We also gave an $O(n)$ algorithm for solving this problem, provided the centers lie on a given line.

REFERENCES

[1] Rezaei, M., and FazelZarandi, M.H., "Facility location via fuzzy modeling and simulation," Applied soft computing, Vol. 11, pp. 5330– 5340, 2011.

[2] Megiddo, N., and Supowit, K., "On the complexity of some common geometric location problems," SIAM J. Comput., Vol. 13, pp. 1182– 1196, 1984.

[3] Agarwal, P. K., and Procopiuc, C. M., "Exact and approximation algorithms for clustering," Algorithmica, Vol. 33, pp. 201.226, 2002.

[4] Sylvester, J. J., "A question in the geometry of situation," Quart. J. Math., pp. 79, 1857.

[5] Megiddo, N., "Linear time algorithms for linear programming in $\mathbb{R}^3$," Society for Industrial and Applied Mathematics, Vol. 12, No. 4. November 1983.

[6] Drezner, Z., "The planar two-center and two-median problems," Transportation Science, Vol. 18, pp. 351- 361, 1984.

[7] Hershberger, J., and Suri, S., "Finding tailored partitions," J. Algorithms, Vol.12, pp. 431– 463, 1991.

[8] Agarwal, P. K., and Sharir, M., "Planar geometric location problems," Algorithmica, Vol.11, pp. 185– 195, 1994.

[9] Megiddo, N., "Applying parallel computation algorithms in the design of serial algorithms," J. ACM, Vol.30, pp. 852– 865, 1983.

[10] Eppstein, D., "Dynamic three-dimensional linear programming," ORSA J. Computing, Vol.4, pp. 360– 368, 1992.

[11] Jaromczyk, J., and Kowaluk, M., "An efficient algorithm for the euclidean two-center problem," Proc. 10th ACM Sympos. Comput. Geom, pp. 303– 311, 1994.

[12] Sharir, M., "A near linear algorithm for the planar 2-center problem," Discrete Comput. Geom., Vol.18, pp. 125– 134, 1997.

[13] Eppstein, D., "Faster construction of planar two-centers," Proc. 8th ACM-SIAM Sympos. Discrete Algorithms, pp. 131– 138, 1997.

[14] Chan, T. M., "More planar two-center algorithms," Comput. Geom. Theory Appl., Vol.13, pp. 189– 198, 1999.

[15] Huang, P. H., Tsai, Y. T., and Tang, C. Y., "A near-quadratic algorithm for the alpha-connected two-center problem," Journal Of Information Science and Engineering, Vol. 22, pp. 1317- 1324, 2006.

[16] Huang, P. H., Tsai, Y. T., and Tang, C. Y., "A fast algorithm for the alpha-connected two-center decision problem," Information Processing Letters, Vol. 85, pp. 205- 210, 2003.

[17] Rao, A. S., Goldberg, K., "Manipulating algebraic parts in the plane," Technical Report RUU-CS-93-43, 1993.

[18] Preparata, F. P., and Shamos, M. I., "Computational Geometry: An Introduction," Springer Verlag, New York, 1985.