AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 6(3) (2025) 279-287
https://doi.org/10.22060/AJMC.2024.22985.1211

Original Article

Left ϕ-biflatness and ϕ-biprojectivity of certain Banach algebras with applications

Solaleh Salimi[a], Amin Mahmoodi[a], Mehdi Rostami[b], Amir Sahami*[c]

[a] *Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran*
[b] *Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran*
[c] *Department of Mathematics, Faculty of Basic Sciences, P.O. Box 69315-516, Ilam University, Ilam, Iran*

ABSTRACT: This paper continues the investigation initially begun in [24]. We show that left ϕ-biflatness and left ϕ-biprojectivity are closely related to the notions of left ϕ-amenability and ϕ-inner amenability. We characterize left ϕ-biprojectivity and left ϕ-biflatness of certain semigroup algebras and some algebras related to a locally compact group. We discuss non left ϕ-biflatness of some specified triangular Banach algebras.

Review History:

Received:08 February 2024
Revised:14 April 2024
Accepted:14 April 2024
Available Online:01 July 2025

Keywords:

ϕ-biflatness
ϕ-biprojectivity
ϕ-inner amenability

MSC (2020):

43A07; 46M10; 43A20

1. Introduction and Preliminaries

Biprojectivity and biflatness are two important homological notions that arise naturally in Helemskii's works in the 1980s, interested readers are referred to his comprehensive book [7]. We begin with recalling their definitions. Given a Banach algebra $\mathcal{A}$, we let $\pi_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ denote the *multiplication operator*, i.e., $\pi_{\mathcal{A}}(a \otimes b) = ab$ for $a, b \in \mathcal{A}$. It is known that the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ becomes a Banach $\mathcal{A}$-bimodule in a canonical way, turning $\pi_{\mathcal{A}}$ into a $\mathcal{A}$-bimodule morphism. A Banach algebra $\mathcal{A}$ is *biprojective* if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho\colon \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho(a) = a$. Further, $\mathcal{A}$ is *biflat* if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho\colon \mathcal{A} \to \left(\mathcal{A}\widehat{\otimes}\mathcal{A}\right)^{**}$ such that $\pi_{\mathcal{A}}^{**} \circ \rho(a) = a$ for $a \in \mathcal{A}$. These concepts are closely related to the notion of amenability introduced by Johnson [11, 12].

The notion of ϕ-amenability was introduced in [13] and independently in [16]. Let $\mathcal{A}$ be a Banach algebra. We write $\Delta(\mathcal{A})$ for the set of all nonzero multiplicative linear functionals on $\mathcal{A}$. We call $\mathcal{A}$ *left ϕ-amenable* if $\mathcal{A}$ possess a *ϕ-mean*, i.e., a bounded linear functional m on $\mathcal{A}^*$ satisfying $m(\phi) = 1$ and $m(f \cdot a) = \phi(a)m(f)$ for all $a \in \mathcal{A}$

**Corresponding author.*
E-mail addresses: s.salimi28@yahoo.com, a_mahmoodi@iauctb.ac.ir, mross@aut.ac.ir, a.sahami@ilam.ac.ir

and $f \in \mathcal{A}^*$. Here, we remind that $\mathcal{A}$ is *ϕ-inner amenable* if and only if there exists a bounded net (a_α) in $\mathcal{A}$ such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) = 1$ for all α and $a \in \mathcal{A}$ [10, Theorem 2.1].

The notions of left (right) ϕ-biprojectivity and left (right) ϕ-biflatness, motivated by above considerations, were introduced in [24].

Definition 1.1 ([24]). *Let $\mathcal{A}$ be a Banach algebra, and let $\phi \in \Delta(\mathcal{A})$. Then*

(i) *$\mathcal{A}$ is left ϕ-biprojective if there exists a bounded linear map $\rho\colon \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b)$, and $\phi \circ \pi_{\mathcal{A}} \circ \rho(a) = \phi(a)$ for all $a, b \in \mathcal{A}$;*

(ii) *$\mathcal{A}$ is left ϕ-biflat if there exists a bounded linear map $\rho\colon \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ such that $\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b)$, and $\widetilde{\phi} \circ \pi_{\mathcal{A}}^{**} \circ \rho(a) = \phi(a)$ for all $a, b \in \mathcal{A}$, where $\widetilde{\phi}$ is the unique extension of ϕ on $\mathcal{A}^{**}$.*

The reader may also see [22] for definition of φ-biprojective/biflat Banach algebras.

In this paper, we continue the previous studies started in [21, 23, 24]. Firstly in Section 2, among other things, we shall find some relations between left ϕ-biprojectivity and left ϕ- biflatness of Banach algebras and their ϕ-amenability and ϕ-inner amenability.

In Section 3, we study left ϕ-biprojectivity and left ϕ-biflatness of measure algebras and Clifford semigroup algebras. We prove that $\ell^1(\mathbb{N}_{\min})$ is left ϕ-biprojective, however $\ell^1(\mathbb{N}_{\max})$ fails to be left ϕ-biprojective, where ϕ is the augmentation character.

Finally in Section 4, we will show that a triangular Banach algebra $\mathcal{T}$ is not left ϕ-biflat for some certain $\phi \in \Delta(\mathcal{T})$.

2. Some Properties and relations

The following first result shows that left ϕ-biflatness together with ϕ-inner amenability forces a Banach algebra to be left ϕ-amenable.

Proposition 2.1. *Let $\mathcal{A}$ be a left ϕ-biflat Banach algebra, and let $\phi \in \Delta(\mathcal{A})$. If $\mathcal{A}$ is ϕ-inner amenable, then $\mathcal{A}$ is left ϕ-amenable.*

Proof. Since $\mathcal{A}$ is left ϕ-biflat, there exists a bounded linear map $\rho\colon \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ such that

$$\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a), \quad \widetilde{\phi} \circ \pi_{\mathcal{A}}^{**} \circ \rho(a) = \phi(a), \quad (a \in \mathcal{A}).$$

On the other hand, ϕ-inner amenability of $\mathcal{A}$ implies the existence of a bounded net $(a_\alpha)_{\alpha \in I}$ in $\mathcal{A}$ such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) = 1$, for all $a \in \mathcal{A}$. If we set $m_\alpha = \rho(a_\alpha) \in (A\widehat{\otimes}\mathcal{A})^{**}$, then for each $a \in \mathcal{A}$ we have

$$a \cdot m_\alpha - \phi(a)m_\alpha = a \cdot \rho(a_\alpha) - \phi(a)\rho(a_\alpha) = \rho(aa_\alpha - a_\alpha a) \to 0$$

and

$$\widetilde{\phi} \circ \pi_{\mathcal{A}}^{**}(m_\alpha) = \widetilde{\phi} \circ \pi_{\mathcal{A}}^{**} \circ \rho(a_\alpha) = \phi(a_\alpha) = 1.$$

By Goldstine's Theorem, there exists a bounded net $(n_\alpha^\gamma)_{\gamma \in \Gamma} \subseteq \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $w^* - \lim_\gamma n_\alpha^\gamma = m_\alpha$ in $(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$. For each $a \in \mathcal{A}$ then we have

$$w^* - \lim_\gamma a \cdot n_\alpha^\gamma - a \cdot m_\alpha = 0 \quad \text{and} \quad w^* - \lim_\gamma \phi(a)n_\alpha^\gamma - \phi(a)m_\alpha = 0$$

in $(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$. Thus for $a \in \mathcal{A}$

$$\lim_\alpha \; w^* - \lim_\gamma a \cdot n_\alpha^\gamma - \phi(a)n_\alpha^\gamma = \lim_\alpha a \cdot m_\alpha - \phi(a)m_\alpha = 0 \; .$$

Also w^*-continuity of $\pi_{\mathcal{A}}^{**}$ yields that

$$\lim_\alpha \; w^* - \lim_\gamma \phi \circ \pi_{\mathcal{A}}(n_\alpha^\gamma) = \lim_\alpha \widetilde{\phi} \circ \pi_{\mathcal{A}}^{**}(m_\alpha) = 1 \; .$$

Set $\Lambda = I \times \Gamma^I$, where Γ^I is denoted for the set of all functions from I into Γ. Define the product ordering by

$$(\alpha, \gamma) \preccurlyeq_\Lambda (\alpha', \gamma') \Leftrightarrow \alpha \preccurlyeq_I \alpha', \gamma \preccurlyeq_{\Gamma^I} \gamma'$$

where $\gamma \preccurlyeq_{\Gamma^I} \gamma'$ means $\gamma(d) \preccurlyeq_\Gamma \gamma'(d)$ for each $d \in I$. Take $\lambda = (\alpha, (\gamma_\alpha)) \in \Lambda$. By Iterated limit theorem [14, p. 69] we obtain a bounded net (n_λ) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that for all $a \in \mathcal{A}$

$$w^* - \lim_\lambda a \cdot n_\lambda - \phi(a)n_\lambda = 0 \;\text{ in } (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**} \quad \text{and} \quad \lim_\lambda \phi \circ \pi_{\mathcal{A}}(n_\lambda) - 1 = 0$$

or equivalently

$$wk - \lim_{\lambda} a \cdot n_\lambda - \phi(a) n_\lambda = 0 \ \text{ in } \mathcal{A}\widehat{\otimes}\mathcal{A} \quad \text{and} \quad \lim_{\lambda} \phi \circ \pi_{\mathcal{A}}(n_\lambda) - 1 = 0 \ .$$

Applying Mazur's Lemma, we may assume that

$$\lim_{\lambda} a \cdot n_\lambda - \phi(a) n_\lambda = 0 \quad \text{and} \quad \lim_{\lambda} \phi \circ \pi_{\mathcal{A}}(n_\lambda) = 1 \quad (a \in \mathcal{A}) \ .$$

Putting $u_\lambda = \pi_{\mathcal{A}}(n_\lambda) \in \mathcal{A}$, we see that

$$\lim_{\lambda} a \cdot u_\lambda - \phi(a) u_\lambda = 0 \quad \text{and} \quad \lim_{\lambda} \phi(u_\lambda) = 1 \quad (a \in \mathcal{A})$$

so that $\mathcal{A}$ is left ϕ-amenable, by [13, Theorem 1.4]. □

For a Banach algebra $\mathcal{A}$, we write $\Upsilon_{\mathcal{A}}$ for the *flip map* on $\mathcal{A}\widehat{\otimes}\mathcal{A}$ given by $\Upsilon_{\mathcal{A}}(a \otimes b) = b \otimes a$ for $a, b \in \mathcal{A}$.

The following example shows that the ϕ-inner amenability assumption in Proposition 2.1 can not be dropped.

Example 2.1. *We give a left ϕ-biflat Banach algebra which is not left ϕ-amenable. Let $\mathcal{V}$ be a Banach space with $\dim \mathcal{V} \geq 1$, and let $f \in \mathcal{V}^*$ be a non-zero element such that $||f|| \leq 1$. It is known that $\mathcal{V}$ equipped with either products defined by $a * b = f(a)b$ and $a \bullet b = f(b)a$ for $a, b \in \mathcal{V}$, is a Banach algebra denoted by ${}_f\mathcal{V}$ and $\mathcal{V}_f$, respectively. It is easy to see that $\Delta(\mathcal{V}_f) = \Delta({}_f\mathcal{V}) = \{f\}$. Put $\mathcal{A} = {}_f\mathcal{V}\widehat{\otimes}\mathcal{V}_f$. Clearly the map ϕ defined by $\phi(a \otimes b) = f(a)f(b)$ for all $a, b \in \mathcal{V}$, is a non-zero multiplicative linear functional on $\mathcal{A}$. Choose $a_0 \in \mathcal{V}$ such that $f(a_0) = 1$. We can easily obtain that the map $\rho\colon \mathcal{V}_f \to \mathcal{V}_f\widehat{\otimes}\mathcal{V}_f$ given by $\rho(a) = a \otimes a_0$ for all $a \in \mathcal{V}_f$, is a bounded $\mathcal{V}_f$-bimodule morphism such that $\pi_{\mathcal{V}_f} \circ \rho(a) = a$ for all $a \in \mathcal{V}_f$. It follows that $\mathcal{V}_f$ is biprojective. Also the composition map $\Upsilon_{{}_f\mathcal{V}} \circ \rho\colon {}_f\mathcal{V} \to {}_f\mathcal{V}\widehat{\otimes}{}_f\mathcal{V}$ is a bounded ${}_f\mathcal{V}$-bimodule morphism such that $\pi_{{}_f\mathcal{V}} \circ \Upsilon_{{}_f\mathcal{V}} \circ \rho(a) = a$ for all $a \in_f \mathcal{V}$. So ${}_f\mathcal{V}$ is also biprojective. Applying [20, Proposition 2.4], we see that $\mathcal{A}$ is biprojective. Hence $\mathcal{A}$ is left ϕ-biprojective and whence it is left ϕ-biflat.*

We suppose in contradiction that $\mathcal{A}$ is left ϕ-amenable. Then by [13, Theorem 3.3] $\mathcal{V}_f$ is left f-amenable. So by [13, theorem 1.4] there is a bounded net (a_α) in $\mathcal{V}_f$ such that $a \bullet a_\alpha - f(a)a_\alpha \to 0$ and $f(a_\alpha) = 1$, for each $a \in \mathcal{V}$. It follows that

$$a - f(a)a_\alpha = af(a_\alpha) - f(a)a_\alpha \to 0, \quad (a \in \mathcal{V}_f).$$

Pick $b_0 \in \mathcal{V}_f$ such that $f(b_0) = 1$. Putting $a = b_0$ in above equation, we obtain that $a_\alpha \to b_0$. Combining with $a \bullet a_\alpha - f(a)a_\alpha \to 0$ implies that $a = a \bullet b_0 = f(a)b_0$. It follows that $\dim \mathcal{V} = 1$ which is impossible. Thus $\mathcal{A}$ is not left ϕ-amenable.

From [9] we recall that a Banach algebra $\mathcal{A}$ is *left ϕ-contractible* if there exists an element $m \in \mathcal{A}$ such that $a \cdot m = \phi(a)m$ and $\phi(m) = 1$ for all $a \in \mathcal{A}$.

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra, and let $\phi \in \Delta(\mathcal{A})$.*
(i) If $\mathcal{A}$ is left ϕ-amenable, then $\mathcal{A}$ is left ϕ-biflat;
(ii) If $\mathcal{A}$ is left ϕ-contractible, then $\mathcal{A}$ is left ϕ-biprojective.

Proof. The proofs are similar, so we only prove the clause (i).

Suppose that $\mathcal{A}$ is left ϕ-amenable. Then there exists an element $m \in \mathcal{A}^{**}$ such that $a \cdot m = \phi(a)m$ and $\widetilde{\phi}(m) = 1$ for all $a \in \mathcal{A}$. Set $\eta\colon \mathcal{A} \to \mathcal{A}^{**}\widehat{\otimes}\mathcal{A}^{**}$ by $\eta(a) = a \cdot m \otimes m$. By [6, Lemma 1.7], there exists a bounded linear map $\psi\colon \mathcal{A}^{**}\widehat{\otimes}\mathcal{A}^{**} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ satisfying

$$\psi(a \otimes b) = a \otimes b, \qquad \psi(n) \cdot a = \psi(n \cdot a), \qquad a \cdot \psi(n) = \psi(a \cdot n), \qquad \pi_{\mathcal{A}}^{**}(\psi(n)) = \pi_{\mathcal{A}^{**}}(n)$$

for $a, b \in \mathcal{A}$ and $n \in \mathcal{A}^{**}\widehat{\otimes}\mathcal{A}^{**}$. Setting $\rho = \psi \circ \eta$, it is routinely checked that

$$\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b), \quad \widetilde{\phi} \circ \pi_{\mathcal{A}}^{**} \circ \rho(a) = \phi(a) \quad (a, b \in \mathcal{A})$$

so $\mathcal{A}$ is left ϕ-biflat. □

The clause (i) of Lemma 2.2 is a converse for [10, Corollary 2.2].

The following describe the connection between left ϕ-biflatness of a Banach algebra and its second dual under ϕ-inner amenability.

Proposition 2.3. *Let $\mathcal{A}$ be a Banach algebra, and let $\phi \in \Delta(\mathcal{A})$. Suppose that $\mathcal{A}$ is ϕ-inner amenable. Then $\mathcal{A}^{**}$ is left $\widetilde{\phi}$-biflat if and only if $\mathcal{A}$ is left ϕ-biflat.*

Proof. Let $\mathcal{A}^{**}$ be left $\widetilde{\phi}$-biflat. Then there exists a bounded linear map $\rho\colon \mathcal{A}^{**} \to (\mathcal{A}^{**}\widehat{\otimes}\mathcal{A}^{**})^{**}$ such that

$$\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a), \quad \widetilde{\widetilde{\phi}} \circ \pi^{**}_{\mathcal{A}^{**}} \circ \rho(a) = \widetilde{\phi}(a),$$

for all $a \in \mathcal{A}^{**}$. Here ϕ-inner amenability of $\mathcal{A}$ guarantees the existence of a bounded net (a_α) in $\mathcal{A}$ such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) = 1$, for all $a \in \mathcal{A}$. By [6, Lemma 1.7], there exists a bounded linear map $\psi\colon \mathcal{A}^{**}\widehat{\otimes}\mathcal{A}^{**} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ satisfying

$$\psi(a \otimes b) = a \otimes b, \qquad \psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot n), \qquad \pi^{**}_{\mathcal{A}}(\psi(m)) = \pi_{\mathcal{A}^{**}}(m)$$

for $a, b \in \mathcal{A}$ and $m \in \mathcal{A}^{**}\widehat{\otimes}\mathcal{A}^{**}$. Set $m_\alpha = \pi^{****}_{\mathcal{A}} \circ \psi^{**} \circ \rho_{\big|\mathcal{A}}(a_\alpha)$. Clearly (m_α) is a bounded net in $\mathcal{A}^{****}$. Putting things together, we obtain

$$\begin{aligned} am_\alpha - \phi(a)m_\alpha &= a\pi^{****}_{\mathcal{A}} \circ \psi^{**} \circ \rho_{\big|\mathcal{A}}(a_\alpha) - \phi(a)\pi^{****}_{\mathcal{A}} \circ \psi^{**} \circ \rho_{\big|\mathcal{A}}(a_\alpha) \\ &= \pi^{****}_{\mathcal{A}} \circ \psi^{**} \circ \rho_{\big|\mathcal{A}}(aa_\alpha - a_\alpha a) \to 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \widetilde{\widetilde{\phi}}(m_\alpha) = \widetilde{\widetilde{\phi}} \circ \pi^{****}_{\mathcal{A}} \circ \psi^{**} \circ \rho_{\big|\mathcal{A}}(a_\alpha) &= \widetilde{\widetilde{\phi}} \circ \pi^{**}_{\mathcal{A}^{**}} \circ \rho_{\big|\mathcal{A}}(a_\alpha) \\ &= \widetilde{\phi}(a_\alpha) = \phi(a_\alpha) = 1. \end{aligned} \tag{2}$$

Using Goldstine's Theorem (twice), we may assume that m_α's are in $\mathcal{A}$. Thus $\mathcal{A}$ is left ϕ-amenable. Now by Lemma 2.2 (i), $\mathcal{A}$ is left ϕ-biflat.

Conversely, if we suppose that $\mathcal{A}$ is left ϕ-biflat, it admits a bounded linear map $\rho\colon \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ such that

$$\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b), \quad \widetilde{\phi} \circ \pi^{**}_{\mathcal{A}} \circ \rho(a) = \phi(a),$$

for all $a, b \in \mathcal{A}$. Let the net (a_α) be as above. Set $m_\alpha = \pi^{**}_{\mathcal{A}} \circ \rho(a_\alpha)$. One can see that (m_α) is a bounded net in $\mathcal{A}^{**}$ such that $a \cdot m_\alpha - \phi(a)m_\alpha \to 0$ and $\widetilde{\phi}(m_\alpha) = 1$ for all $a \in \mathcal{A}$. Using Goldstine's Theorem we can assume that m_α's belong to $\mathcal{A}$. Thus $\mathcal{A}$ is left ϕ-amenable. According to [13, Proposition 3.4], $\mathcal{A}^{**}$ is left $\widetilde{\phi}$-amenable. Thus A^{**} is left $\widetilde{\phi}$-biflat, again by Lemma 2.2 (i). □

The following may be compared with [19, Corollary 3.3].

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity, and let $\Delta(\mathcal{A})$ be non-empty. If $\mathcal{A}$ is a left ϕ-biprojective for all $\phi \in \Delta(\mathcal{A})$, then $\Delta(\mathcal{A})$ is discrete with respect to the w^*-topology.*

Proof. Suppose that $\mathcal{A}$ is left ϕ-biprojective for all $\phi \in \Delta(\mathcal{A})$. Since $\mathcal{A}$ has a left approximate identity, $\mathcal{A}$ is left ϕ-contractible for all $\phi \in \Delta(\mathcal{A})$ by [23, Proposition 2.4]. From [3, Proposition 2.3], we conclude that $\Delta(\mathcal{A})$ is discrete with respect to the w^*-topology. □

3. Application to algebras related to locally compact groups and discrete semigroups

A discrete semigroup $\mathcal{S}$ is an *inverse* semigroup if for each $s \in \mathcal{S}$ there exists a unique element $s^* \in \mathcal{S}$ such that $ss^*s = s$ and $s^*ss^* = s^*$. There exists a partial order on each inverse semigroup $\mathcal{S}$, that is, $s \le t \iff s = ts^*s$ for all $s, t \in \mathcal{S}$.

Let $(\mathcal{S}, \le)$ be an inverse semigroup. For each $s \in \mathcal{S}$, set $(x] = \{y \in \mathcal{S} | \, y \le x\}$. We say $\mathcal{S}$ is *uniformly locally finite* if $\sup\{|(x]| : x \in \mathcal{S}\} < \infty$. We write $E(\mathcal{S})$ for the set of all idempotents of $\mathcal{S}$. For every $e \in E(\mathcal{S})$, it is known that $\mathcal{G}_e = \{s \in \mathcal{S} | ss^* = s^*s = e\}$ is a maximal subgroup of $\mathcal{S}$ with respect to e. Moreover, $\mathcal{G}_{e_1} \bigcap \mathcal{G}_{e_2} = \emptyset$ for all $e_1, e_2 \in \mathcal{S}$ with $e_1 \ne e_2$. An inverse semigroup $\mathcal{S}$ is a *Clifford* semigroup if $ss^* = s^*s$ for all $s \in \mathcal{S}$. See [8] as a main reference of semigroup theory.

Left ϕ-biflatness of semigroup algebras related to Clifford semigroups has been studied in [24]. We now characterize left ϕ-biprojectivity of Clifford semigroup algebras.

Proposition 3.1. *Let $\mathcal{S} = \bigcup_{e \in E(\mathcal{S})} \mathcal{G}_e$ be a Clifford semigroup such that $E(\mathcal{S})$ is uniformly locally finite. Then the following are equivalent:*

(i) *$\ell^1(\mathcal{S})$ is left ϕ-biprojective for all $\phi \in \Delta(\ell^1(\mathcal{S}))$;*

(ii) *Each maximal subgroup $\mathcal{G}_e$ is finite;*
(iii) *$\ell^1(\mathcal{S})$ is biprojective.*

Proof. $(i) \Rightarrow (ii)$ Let $\ell^1(\mathcal{S})$ be left ϕ-biprojective for all $\phi \in \Delta(\ell^1(\mathcal{S}))$. It is known that $\ell^1(\mathcal{S})$ is isometrically isomorphic to $\bigoplus_{e\in E(\mathcal{S})} \ell^1(\mathcal{G}_e)$, see [20, Theorem 2.18]. Thus $\Delta(\ell^1(\mathcal{S})) = \bigcup_{e\in E(\mathcal{S})} \Delta(\ell^1(\mathcal{G}_e))$. Let $\phi \in \Delta(\ell^1(\mathcal{G}_e))$. Since each $\ell^1(\mathcal{G}_e)$ has an identity element, there exists an element x in $\mathcal{Z}(\ell^1(\mathcal{S}))$ (the center of $\ell^1(\mathcal{S})$) such that $\phi(x) = 1$. Applying [23, Lemma 2.2], we observe that $\ell^1(\mathcal{S})$ is left ϕ-contractible. So there exists an element a_1 in $\ell^1(\mathcal{S})$ such that

$$aa_1 = \phi(a)a_1, \quad \phi(a_1) = 1, \quad (a \in \ell^1(\mathcal{S})).$$

Pick $a_0 \in \ell^1(\mathcal{G}_e)$ such that $aa_0 = a_0a$ and $\phi(a_0) = 1$ for all $a \in \ell^1(\mathcal{S})$. Since $\ell^1(\mathcal{G}_e)$ is a closed ideal of $\ell^1(\mathcal{S})$, element $b = a_1a_0$ is in $\ell^1(\mathcal{G}_e)$ and satisfies

$$ab = \phi(a)b, \quad \phi(b) = 1, \quad (a \in \ell^1(\mathcal{G}_e)).$$

Then $\ell^1(\mathcal{G}_e)$ is left ϕ-contractible. Then $\mathcal{G}_e$ is compact by [1, Theorem 3.3] . Whence $\mathcal{G}_e$ is finite.

$(ii) \Rightarrow (iii)$ This is proved in [20, Theorem 3.7].

$(iii) \Rightarrow (i)$ This is trivial. □

Remark 3.2. *Notice that every discrete group $\mathcal{G}$ is uniformly locally finite. Therefore, as a consequence of Proposition 3.1, the group algebra $\ell^1(\mathcal{G})$ is left ϕ-biprojective for all $\phi \in \Delta(\ell^1(\mathcal{G}))$ if and only if $\mathcal{G}$ is finite.*

Let $\mathbb{N}_{\min}$ and $\mathbb{N}_{\max}$ be the semigroup $\mathbb{N}$ with products $m *_{\min} n = \min\{m, n\}$ and $m *_{\max} n = \max\{m, n\}$, respectively. Take $\ell^1(\mathbb{N}_{\min})$ and $\ell^1(\mathbb{N}_{\max})$ with convolution products. We write δ_n for the *point mass* at $\{n\}$. For every $n \in \mathbb{N}$ we consider a homomorphism $\phi_n : \ell^1(\mathbb{N}_{\min}) \longrightarrow \mathbb{C}$ defined by $\phi_n\left(\sum_{i=1}^{\infty} \alpha_i\delta_i\right) = \sum_{i=n}^{\infty} \alpha_i$. There is also a homomorphism $\psi_n : \ell^1(\mathbb{N}_{\max}) \longrightarrow \mathbb{C}$ with the formula $\psi_n\left(\sum_{i=1}^{\infty} \alpha_i\delta_i\right) = \sum_{i=1}^{n} \alpha_i$ for each $n \in \mathbb{N} \cup \{\infty\}$. It is known that $\Delta(\ell^1(\mathbb{N}_{\min})) = \{\phi_n : n \in \mathbb{N}\}$ and $\Delta(\ell^1(\mathbb{N}_{\max})) = \{\psi_n : n \in \mathbb{N} \cup \{\infty\}\}$. Notice that $\psi_n = \phi_1 - \phi_{n+1}$ $(n \in \mathbb{N})$, and that $\phi_1 = \psi_\infty$ is the *augmentation character*, see [2].

Proposition 3.3. *(i) $\ell^1(\mathbb{N}_{\min})$ is left ϕ_1-biprojective.*
(ii) $\ell^1(\mathbb{N}_{\max})$ is not left ψ_∞-biprojective.

Proof. (i) Define $\rho: \ell^1(\mathbb{N}_{\min}) \to \ell^1(\mathbb{N}_{\min})\widehat{\otimes}\ell^1(\mathbb{N}_{\min})$ by $\rho(f) = \phi_1(f)\delta_1 \otimes \delta_1$ for all $f \in \ell^1(\mathbb{N}_{\min})$. Clearly $\rho(fg) = f \cdot \rho(g) = \phi_1(g)\rho(f)$, and $\phi_1 \circ \pi_{\ell^1(\mathbb{N}_{\min})} \circ \rho(f) = \phi_1(f)$ for $f, g \in \ell^1(\mathbb{N}_{\min})$. So $\ell^1(\mathbb{N}_{\min})$ is left ϕ_1-biprojective.

(ii) Towards a contradiction, suppose that $\ell^1(\mathbb{N}_{\max})$ is left ψ_∞-biprojective. Since $\ell^1(\mathbb{N}_{\max})$ is unital, it is left ψ_∞-contractible [23, Proposition 2.4]. Define $m = \delta_n - \delta_{n+1}$, clearly $m \in \ell^1(\mathbb{N}_{\max})$. Thus $a \cdot m = \psi_n(a)m$, and $\psi_n(m) = 1$ for all $a \in \ell^1(\mathbb{N}_{\max})$ and $n \in \mathbb{N}$. So $\ell^1(\mathbb{N}_{\max})$ is left ψ_n-contractible for all $n \in \mathbb{N} \cup \{\infty\}$. It follows from [3, Corollry 2.2] that $\Delta(\ell^1(\mathbb{N}_{\max})) = \mathbb{N} \cup \{\infty\}$ is discrete with respect to the w^*-topology. On the other hand by Gelfand representation theorem $\Delta(\ell^1(\mathbb{N}_{\max})) = \mathbb{N} \cup \{\infty\}$ is compact. So $\Delta(\ell^1(\mathbb{N}_{\max})) = \mathbb{N} \cup \{\infty\}$ is finite which is impossible. □

Let S be a locally compact space. A compact space is called Stone-Čech-compactification of S (denoted by βS) if satisfying the following universal property:

$(\star)$ For each compact Hausdorff space $\mathcal{K}$ and each continuous mapping $f : S \longrightarrow \mathcal{K}$, there exists a uniquely determined continuous mapping $\widetilde{f} : \beta S \longrightarrow \mathcal{K}$ such that $\widetilde{f}_{|S} = f$.

Let $\mathcal{S}$ be a discrete semigroup. By the above characterization we have

$$\ell^1(\mathcal{S})^{**} \cong \ell^\infty(\mathcal{S})^* \cong C(\beta\mathcal{S})^* \cong M(\beta\mathcal{S}).$$

For more information see [2, Chapter 6].

We recall that a Banach algebra $\mathcal{A}$ is *ϕ-pseudo-amenable* if there exists a net (a_α) in $\mathcal{A}$ such that $aa_\alpha - \phi(a)a_\alpha \to 0$ and $\phi(a_\alpha) \to 1$ for all $a \in \mathcal{A}$, see [17, Proposition 2.3].

Proposition 3.4. *Let $\mathcal{S}$ be an infinite, commutative and cancellative semigroup. Then $\ell^1(\mathcal{S})^{**} = M(\beta\mathcal{S})$ is not left $\widetilde{\phi}$-biflat, where ϕ is the augmentation character on $\ell^1(\mathcal{S})$.*

Proof. We assume in contradiction that $\ell^1(\mathcal{S})^{**} = M(\beta\mathcal{S})$ is left $\widetilde{\phi}$-biflat. Since $\mathcal{S}$ is commutative, $\ell^1(\mathcal{S})$ is ϕ-inner amenable. So by Proposition 2.3, is $\ell^1(\mathcal{S})$ left ϕ-amenable. Using [13, Proposition 3.4], $\ell^1(\mathcal{S})^{**} = M(\beta\mathcal{S})$ is left $\widetilde{\phi}$-amenable. Then $\ell^1(\mathcal{S})^{**} = M(\beta\mathcal{S})$ is left $\widetilde{\phi}$-pseudo-amenable. By [17, Proposition 2.8], $\ell^1(\mathcal{S})^{**} = M(\beta\mathcal{S})$ doesn't have a non-trivial bounded point derivation at the augmentation character and this is in contradiction to [2, Theorem 11.15]. □

Remark 3.5. *It should be stressed that Proposition 3.4 without cancellativity condition does not hold. To see this, consider the semigroup algebra $\ell^1(\mathbb{N}_{\min})$ with the augmentation character ϕ. It is easily checked that (δ_n) is a bounded approximate identity for $\ell^1(\mathbb{N}_{\min})$, and thus it is ϕ-inner amenable. Hence $\ell^1(\mathbb{N}_{\min})^{**}$ is left $\widetilde{\phi}$-biflat, by Propositions 2.3 and 3.3(i).*

Proposition 3.6. *Let $\mathcal{G}$ be a locally compact group. Then $L^1(\mathcal{G})$ is left ϕ-biflat for all $\phi \in \Delta(L^1(\mathcal{G}))$ if and only if $\mathcal{G}$ is amenable.*

Proof. We first notice that $L^1(\mathcal{G})$ is ϕ-inner amenable for all $\phi \in \Delta(L^1(\mathcal{G}))$, because it has a bounded approximate identity. If $L^1(\mathcal{G})$ is left ϕ-biflat for all $\phi \in \Delta(L^1(\mathcal{G}))$, then it is left ϕ-amenable for all $\phi \in \Delta(L^1(\mathcal{G}))$ by Proposition 2.1. Now by [16, Corollary 2.4], $\mathcal{G}$ is amenable.

Conversely if $\mathcal{G}$ is amenable, then $L^1(\mathcal{G})$ is left ϕ-amenable for all $\phi \in \Delta(L^1(\mathcal{G}))$ again by [16, Corollary 2.4]. Hence, the result follows from Lemma 2.2(i). □

Recall that $M(\mathcal{G})$ is denoted for the measure algebra of a locally compact group $\mathcal{G}$.

Proposition 3.7. *Let $\mathcal{G}$ be a locally compact group. Then $M(\mathcal{G})$ is left ϕ-biprojective for all $\phi \in \Delta(M(\mathcal{G}))$ if and only if $\mathcal{G}$ is finite.*

Proof. Suppose that $M(\mathcal{G})$ is left ϕ-biprojective for all $\phi \in \Delta(M(\mathcal{G}))$. Since $M(\mathcal{G})$ is unital, its left ϕ-biprojectivity is equivalent to its left ϕ-contractibility. So $\mathcal{G}$ must be finite, by [18, Corollary 6.2].

The converse is trivial. □

Proposition 3.8. *Let $\mathcal{G}$ be a locally compact group. Then the following are equivalent:*

(i) *$M(\mathcal{G})^{**}$ is left $\widetilde{\phi}$-biflat for all $\phi \in \Delta(M(\mathcal{G}))$;*
(ii) *$M(\mathcal{G})$ is left ϕ-biflat for all $\phi \in \Delta(M(\mathcal{G}))$;*
(iii) *$\mathcal{G}$ is discrete and amenable.*

Proof. $(i) \iff (ii)$ It is immediate from Proposition 2.3, just note that $M(\mathcal{G})$ is unital and so it is ϕ-inner amenable for all $\phi \in \Delta(M(\mathcal{G}))$.
$(ii) \iff (iii)$ Left ϕ-biflatness of $M(\mathcal{G})$ is equivalent to its left ϕ-amenability, because $M(\mathcal{G})$ is unital. The proof completes by [16, Corollary 2.5]. □

4. Applications to some triangular Banach algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be a Banach algebras and let $\mathcal{X}$ be a Banach $(\mathcal{A}, \mathcal{B})$-module, that is, $\mathcal{X}$ is a Banach space, a left $\mathcal{A}$-module and a right $\mathcal{B}$-module with the compatible module action that satisfies $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ and $||a \cdot x \cdot b|| \leq ||a||||x||||b||$ for every $a \in \mathcal{A}, x \in \mathcal{X}, b \in \mathcal{B}$. With the usual 2×2 matrix operations and the norm

$$\left\|\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}\right\| = ||a|| + ||x|| + ||b||,$$

$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix}$ becomes a Banach algebra which is called a *triangular Banach algebra*. One may see [4, 5, 15] for more information and properties on these algebras.

Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix}$ be a triangular Banach algebra. For every $\phi \in \Delta(\mathcal{B})$, we may consider an element $\Psi_\phi \in \Delta(\mathcal{T})$ defined by

$$\Psi_\phi\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}\right) = \phi(b), \quad (a \in \mathcal{A}, x \in \mathcal{X}, b \in \mathcal{B}) .$$

Theorem 4.1. *Let $\mathcal{A}$ be a Banach algebra, and let $\phi \in \Delta(\mathcal{A})$. If $\mathcal{A}$ is ϕ-inner amenable, then $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$ is not left Ψ_ϕ-biflat.*

Proof. Assume towards a contradiction that $\mathcal{T}$ is left Ψ_ϕ-biflat. Since $\mathcal{A}$ is ϕ-inner amenable, there exists a bounded net (a_α) in $\mathcal{A}$ such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) = 1$, for each $a \in \mathcal{A}$. Set $t_\alpha = \begin{bmatrix} a_\alpha & 0 \\ 0 & a_\alpha \end{bmatrix}$. It is easy to see that (t_α) is a bounded net in $\mathcal{T}$ such that

$$tt_\alpha - t_\alpha t \to 0, \quad \Psi_\phi(t_\alpha) = \phi(a_\alpha) = 1, \qquad (t \in \mathcal{T}).$$

So $\mathcal{T}$ is Ψ_ϕ-inner amenable. Thus by Proposition 2.1, $\mathcal{T}$ is left Ψ_ϕ-amenable. Clearly $\mathcal{I} = \begin{bmatrix} 0 & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$ is a closed two-sided ideal of $\mathcal{T}$ for which $\Psi_\phi \neq 0$ on $\mathcal{I}$. One can easily see that $\mathcal{I}$ is left Ψ_ϕ-amenable. So there exists a bounded net $m_\alpha = \begin{bmatrix} 0 & u_\alpha \\ 0 & v_\alpha \end{bmatrix}$ in $\mathcal{I}$ such that

$$mm_\alpha - \Psi_\phi(m)m_\alpha \to 0, \quad (m \in \mathcal{I}) \tag{3}$$

and $\Psi_\phi(m_\alpha) = \phi(v_\alpha) = 1$. Take an element $a \in \mathcal{A}$ with $\phi(a) = 1$, and take $b \in \ker\phi$. Substitute $m_0 = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ for m in (3), we obtain $m_0 m_\alpha \to 0$. It follows that $av_\alpha \to 0$. Therefore $\phi(v_\alpha) = \phi(av_\alpha) \to 0$, a contradiction. □

Definition 4.2. *Let $\mathcal{B}$ be a Banach algebra, let $\phi \in \Delta(\mathcal{B})$, and let $\mathcal{X}$ be a Banach right $\mathcal{B}$-module. A non-zero linear functional $\psi \in \mathcal{X}^*$ is a right ϕ-character for $\mathcal{X}$ if $\psi(x \cdot b) = \phi(b)\psi(x)$ for each $b \in \mathcal{B}$ and $x \in \mathcal{X}$.*

To see an example satisfying conditions of Definition 4.2, consider a Banach right $\mathcal{B}$-module $\mathcal{X}$ for which $x \cdot b = \phi(b)x$, $b \in \mathcal{B}, x \in \mathcal{X}$. Then every $0 \neq \psi \in \mathcal{X}^*$ is a right ϕ-character for $\mathcal{X}$.

Theorem 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with bounded left approximate identities, let $\phi \in \Delta(\mathcal{B})$, and let $\mathcal{X}$ be a Banach $(\mathcal{A}, \mathcal{B})$-module with a right ϕ-character. Then $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix}$ is not left Ψ_ϕ-biflat.*

Proof. Assume towards a contradiction that $\mathcal{T}$ is left Ψ_ϕ-biflat. Since $\mathcal{A}$ and $\mathcal{B}$ have bounded left approximate identities, $\mathcal{T}$ also has a bounded left approximate identity, by Cohen-Hewit factorization theorem. So $\overline{\mathcal{T}\ker\Psi_\phi}^{||\cdot||} = \ker\Psi_\phi$. It follows from [23, Lemma 2.1] that $\mathcal{T}$ is left Ψ_ϕ-amenable. Clearly $\mathcal{I} = \begin{bmatrix} 0 & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix}$ is a closed ideal of $\mathcal{T}$ for which $\Psi_{\phi}|_{\mathcal{I}} \neq 0$. It follows from [13, Lemma 3.1] that $\mathcal{I}$ is left $\Psi_{\phi}|_{\mathcal{I}}$-amenable. So by [13, Theorem 1.4] there exists a bounded net $n_\alpha = \begin{bmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{bmatrix}$ in $\mathcal{T}$ such that

$$tn_\alpha - \Psi_\phi(t)n_\alpha \to 0, \quad \Psi_\phi(n_\alpha) = \phi(b_\alpha) = 1, \qquad (t \in \mathcal{I}).$$

Take $t = \begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix}$ for arbitrary elements $x \in \mathcal{X}$ and $b \in \mathcal{B}$. Then we have

$$\begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{bmatrix} - \Psi_\phi\left(\begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix}\right)\begin{bmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{bmatrix} \to 0.$$

It gives that

$$x \cdot b_\alpha - \phi(b)x_\alpha \to 0, \quad bb_\alpha - \phi(b)b_\alpha \to 0.$$

Let ψ be a right ϕ-character on $\mathcal{X}$. We then have

$$\psi(x) - \phi(b)\psi(x_\alpha) = \phi(b_\alpha)\psi(x) - \phi(b)\psi(x_\alpha) = \psi(x \cdot b_\alpha - \phi(b)x_\alpha) \to 0, \quad (b \in \mathcal{B}, x \in \mathcal{X}).$$

Hence $\psi(x) = \lim_\alpha \phi(b)\psi(x_\alpha)$ for all $b \in \mathcal{B}, x \in \mathcal{X}$, which is not true. To see this, take $b \in \ker\phi$ and $x \in \mathcal{X}$ with $\psi(x) \neq 0$. □

Acknowledgements

The authors are grateful to the referees for their useful comments which improved the manuscript. The corresponding author is thankful to Ilam university, for it's support.

References

[1] M. Alaghmandan, R. Nasr-Isfahani, and M. Nemati, *Character amenability and contractibility of abstract Segal algebras*, Bull. Aust. Math. Soc., 82 (2010), pp. 274–281.

[2] H. G. Dales, A. T.-M. Lau, and D. Strauss, *Banach algebras on semigroups and on their compactifications*, Mem. Amer. Math. Soc., 205 (2010), pp. vi+165.

[3] M. Dashti, R. Nasr-Isfahani, and S. Soltani Renani, *Character amenability of Lipschitz algebras*, Canad. Math. Bull., 57 (2014), pp. 37–41.

[4] B. E. Forrest and L. W. Marcoux, *Derivations of triangular Banach algebras*, Indiana Univ. Math. J., 45 (1996), pp. 441–462.

[5] ——, *Weak amenability of triangular Banach algebras*, Trans. Amer. Math. Soc., 354 (2002), pp. 1435–1452.

[6] F. Ghahramani, R. J. Loy, and G. A. Willis, *Amenability and weak amenability of second conjugate Banach algebras*, Proc. Amer. Math. Soc., 124 (1996), pp. 1489–1497.

[7] A. Y. Helemskii, *The homology of Banach and topological algebras*, vol. 41 of Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1989. Translated from the Russian by Alan West.

[8] J. M. Howie, *Fundamentals of semigroup theory*, vol. 12 of London Mathematical Society Monographs. New Series, The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.

[9] Z. Hu, M. S. Monfared, and T. Traynor, *On character amenable Banach algebras*, Studia Math., 193 (2009), pp. 53–78.

[10] A. Jabbari, T. M. Abad, and M. Z. Abadi, *On ϕ-inner amenable Banach algebras*, Colloq. Math., 122 (2011), pp. 1–10.

[11] B. E. Johnson, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math., 94 (1972), pp. 685–698.

[12] B. E. Johnson, *Cohomology in Banach algebras*, Memoirs of the American Mathematical Society, No. 127, American Mathematical Society, Providence, RI, 1972.

[13] E. Kaniuth, A. T. Lau, and J. Pym, *On ϕ-amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc., 144 (2008), pp. 85–96.

[14] J. L. Kelley, *General topology*, D. Van Nostrand Co., Inc., Toronto-New York-London, 1955.

[15] A. R. Medghalchi and M. H. Sattari, *Biflatness and biprojectivity of triangular Banach algebras*, Bull. Iranian Math. Soc., 34 (2008), pp. 115–120, 162.

[16] M. S. Monfared, *Character amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc., 144 (2008), pp. 697–706.

[17] R. Nasr-Isfahani and M. Nemati, *Character pseudo-amenability of Banach algebras*, Colloq. Math., 132 (2013), pp. 177–193.

[18] R. Nasr-Isfahani and S. Soltani Renani, *Character contractibility of Banach algebras and homological properties of Banach modules*, Studia Math., 202 (2011), pp. 205–225.

[19] A. Pourabbas and A. Sahami, *On character biprojectivity of Banach algebras*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 78 (2016), pp. 163–174.

[20] P. Ramsden, *Biflatness of semigroup algebras*, Semigroup Forum, 79 (2009), pp. 515–530.

[21] A. Sahami, *On left ϕ-biprojectivity and left ϕ-biflatness of certain Banach algebras*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 81 (2019), pp. 97–106.

[22] A. Sahami and A. Pourabbas, *On φ-biflat and φ-biprojective Banach algebras*, Bull. Belg. Math. Soc. Simon Stevin, 20 (2013), pp. 789–801.

[23] A. Sahami and M. Rostami, *Some cohomological notions in Banach algebras based on maximal ideal space*, Iran. J. Sci. Technol. Trans. A Sci., 46 (2022), pp. 173–179.

[24] A. Sahami, M. Rostami, and A. Pourabbas, *On left φ-biflat Banach algebras*, Comment. Math. Univ. Carolin., 61 (2020), pp. 337–344.

Please cite this article using:

Solaleh Salimi, Amin Mahmoodi, Mehdi Rostami, Amir Sahami, Left ϕ-biflatness and ϕ-biprojectivity of certain Banach algebras with applications, AUT J. Math. Comput., 6(3) (2025) 279-287
https://doi.org/10.22060/AJMC.2024.22985.1211