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Short Contribution

A new characterization of Chevalley groups $\mathrm{G}_2(3^n)$ by the order of the group and the number of elements with the same order

Behnam Ebrahimzadeh*, Hamid Shahbandarzadeh

Department of Mathematics, Persian Gulf University, Bushehr, Iran

ABSTRACT: In this paper, we prove that Chevalley groups $G_2(q)$, where $q = 3^n$ and q^2+q+1 is a prime number, can be uniquely determined by the order of group and the number of elements with the same order.

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1. Introduction

Throughout this paper G be a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of orders of elements in G. If $k \in \pi_e(G)$, then we denote the set of the number of elements of order k in G by $m_k(G)$ and the set of the number of elements with the same order in G by $\mathrm{nse}(G)$. In other words,

$$\mathrm{nse}(G) = \{m_k(G) | k \in \pi_e(G)\}.$$

Also we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of a group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we assume that $2 \in \pi_1$.

The group characterization by $\mathrm{nse}(G)$ pertains to Thompson's problem ([14]) which professor Shi posed it in [17]. The first time, this type of characterization was studied by Shao and Shi. In [16], they proved that if S

**Corresponding author.*
E-mail addresses: behnam.ebrahimzadeh@gmail.com, shabandarzadeh@gmail.com

is a finite simple group with $|\pi(S)| = 4$, then S is characterizable by nse(S) and $|S|$. Following this result, in [1, 3, 4, 5, 6, 7, 8, 9, 12, 13, 15], it is proved that sporadic simple groups, suzuki groups $Sz(q)$, where $q-1$ is a prime number and also Ree groups ${}^2G_2(q)$ where $q \pm \sqrt{3q}+1$ are prime numbers, Suzuki groups, the symplectic groups $C_2(3^n)$, the projective special unitary groups $U_3(3^n)$, the projective special linear groups $L_3(q)$, the Chevalley groups $G_2(2^n)$, the orthogonal groups $B_2(2^{4n})$, the projective special linear groups $L_2(q)$ and certain finite simple groups can be uniquely determined by order of group and nse(G). In this paper, we prove that Chevalley groups $G_2(q)$, where $q = 3^n$ and q^2+q+1 is a prime number can be uniquely determined by nse($G_2(q)$) and $|G_2(q)|$. In fact, we prove the following main theorem.

Main Theorem. Let G be a group with nse(G) = nse($G_2(q)$) and $|G| = |G_2(q)|$, where $q = 3^n$ and q^2+q+1 is a prime number. Then $G \cong G_2(q)$.

2. Notations and Preliminaries

Lemma 2.1 ([11]). *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(1) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(2) *$|H|$ divides $|K|-1$;*
(3) *K is nilpotent.*

Definition 2.2. *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.*

Lemma 2.3 ([2]). *Let G be a 2-Frobenius group of even order. Then*

(1) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
(2) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.4 ([19]). *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(1) *G is a Frobenius group;*
(2) *G is a 2-Frobenius group;*
(3) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.5 ([10]). *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.6. *Let G be a finite group. Then for every $i \in \pi_e(G)$, $\varphi(i)$ divides $m_i(G)$, and i divides $\sum_{j|i} m_j(G)$. Moreover, if $i > 2$, then $m_i(G)$ is even.*

Proof. By Lemma 2.5, the proof is straightforward. □

Lemma 2.7 ([18]). *Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:*

(a) *$L_n(q)$, $n = 2, 3$, $q \equiv \pm 2 \pmod 5$;*
(b) *$G_2(q)$, $q \equiv \pm 2 \pmod 5$;*
(c) *$U_3(q)$, $q \equiv \pm 2 \pmod 5$;*
(d) *${}^3D_4(q)$, $q \equiv \pm 2 \pmod 5$;*
(e) *${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$.*

Lemma 2.8 ([20]). *Let q, k, l be natural numbers. Then*

(1) $(q^k-1, q^l-1) = q^{(k,l)}-1$.
(2) $(q^k+1, q^l+1) = \begin{cases} q^{(k,l)}+1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$
(3) $(q^k-1, q^l+1) = \begin{cases} q^{(k,l)}+1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$ the inequality $(q^k-1, q^k+1) \leq 2$ holds.

3. Main parts of manuscrips

In this section, we prove the main theorem by the following lemmas. For this purpose, we denote the Chevalley groups $G_2(q)$, where $q = 3^n$ and prime number q^2+q+1 by B and p, respectively. Recall that G is a group with $|G| = |B|$ and nse(G) = nse(B). First we prove the following lemma.

Lemma 3.1. *Let B be a Chevalley groups $G_2(q)$, where q^2+q+1 is a prime numbers. Then $m_p(B) = (p-1)|B|/(6p)$ and for every $i \in \pi_e(B) - \{1, p\}$, p divides $m_i(B)$.*

Proof. First, since that $|B_p| = p$, we deduce that B_p is a cyclic group of order p. Thus $m_p(B) = \varphi(p)n_p(B) = (p-1)n_p(B)$. Now it is enough to show $n_p(B) = |B|/(6p)$. By [19], p is an isolated vertex of $\Gamma(G)$. Hence $|C_B(B_p)| = p$ and $|N_B(B_p)| = xp$ for a natural number x. On the other hand, $N_B(B_p)/C_B(B_p)$ embed in $Aut(B_p)$, which implies $x \mid p-1$. Furthermore, by Sylow's Theorem, $n_p(B) = |B : N_B(B_p)|$ and $n_p(B) \equiv 1 \pmod p$. Hence p divides $|G|/(xp)-1$ it follows that q^2+q+1 divides $\frac{q^6(q^6-1)(q^2-1)}{q^2+q+1} - x$. So $q^2+q+1 \mid q^{12}-q^{11}-q^{10}+2q^9-q^8-q^7+q^6-x$ hence $q^2+q+1 \mid (q^2+q+1)(q^{10}-2q^9+4q^7-5q^6+6q^4-6q^2+6q-6)+(6-x)$. It follows that $p \mid 6-x$ and since $x \mid p-1$, we deduce that $x = 6$.
Let $i \in \pi_e(B) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(B)$, we conclude that $p \nmid i$ and $pi \notin \pi_e(B)$. Thus B_p acts fixed point freely on the set of elements of order i by conjugation and hence $|B_p| \mid m_i(B)$. So, we conclude that $p \mid m_i(B)$. □

Lemma 3.2. *$m_2(G) = m_2(B)$, $m_p(G) = m_p(B)$, $n_p(G) = n_p(B)$, p is an isolated vertex of $\Gamma(G)$ and $p \mid m_k(G)$ for every $k \in \pi_e(G) - \{1, p\}$.*

Proof. By Lemma 2.6, for every $2 = i \in \pi_e(G)$, if and only if $m_i(G)$ is odd. Thus we deduce that $m_2(G) = m_2(R)$. According to Lemma 2.6, $(m_p(G), p) = 1$. Thus $p \nmid m_p(G)$ and hence Lemma 2.6 implies that $m_p(G) \in \{m_1(B), m_2(B), m_p(B)\}$. Moreover, $m_p(G)$ is even, so we conclude that $m_p(G) = m_p(B)$. Since G_p and B_p are cyclic groups of order p and $m_p(G) = m_p(B)$, we deduce that $m_p(G) = \varphi(p)n_p(G) = \varphi(p)n_p(B) = m_p(B)$, so $n_p(G) = n_p(B)$.

Now we prove that p is an isolated vertex of Γ(G). On opposite, there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $m_{tp}(G) = \varphi(tp)n_p(G)k$, where k is the number of cyclic subgroups of order t in $C_G(G_p)$ and since $n_p(G) = n_p(B)$, it follows that $m_{tp}(G) = (t-1)(p-1)|R|k/(6p)$. If $m_{tp}(G) = m_p(B)$, then $t = 2$ and $k = 1$. Furthermore, Lemma 2.5 yields $p \mid m_2(G) + m_{2p}(G)$ and since $m_2(G) = m_2(B)$ and $p \mid m_2(B)$, we have $p \mid m_{2p}(G)$ which is a contradiction. So Lemma 3.1 implies that $p \mid m_{tp}(G)$. Hence $p \mid t-1$ and since $m_{tp}(G) < |G|$, we deduce that $p-1 \leq 6$. But this is impossible because $p = q^2+q+1$ and $q = 3^n$.
Let $k \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, $p \nmid k$ and $pk \notin \pi_e(G)$. Thus G_p acts fixed point freely on the set of elements of order k by conjugation and hence $|G_p| \mid m_k(G)$. So we conclude that $p \mid m_k(G)$. □

Lemma 3.3. *The group G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 3.3, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K|-1$. Now by Lemma 3.2, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K|-1$, we conclude that the last case can not occur. So $|H| = p$ and $|K| = |G|/p$, hence $q^2+q+1 \mid \frac{q^6(q^6-1)(q^2-1)}{q^2+q+1} - 1$.In result $q^2+q+1 \mid q^{14}-q^{12}-q^8+q^6-q^2-q-1$,so$q^2+q+1 \mid q^{12}-q^{11}-q^{10}+2q^9-q^8-q^7+q^6-1$,in finally $q^2+q+1 \mid q^{10}-2q^9+4q^7-5q^6+6q^4-6q^3+6q-6)+5$. Thus $p \mid 5$ which is impossible. □

Lemma 3.4. *G is not a 2-Frobenius group.*

Proof. We show that G is not a 2-Frobenius group. On opposite, assume G be a 2-Frobenius group so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups by kernels K/H and H respectively. Set $|G/K| = x$. Since p is an isolate vertex of $\Gamma(G)$, we have $|K/H| = p$ and $|H| = |G|/(xp)$. By Lemma 3.4, $|G/K|$ divides $|Aut(K/H)|$. Thus $x \mid q^2+q$ and $|H| = q^5(q-1)^2(q^3+1)$.Now we consider 2-Sylow subgroup of H where has order of $(q-1)^2$. Hence, $H_2 \rtimes K/H$ is a Frobenius group with kernel H_2 and complement K/H.So $|K/H|$ divides $|H_2|-1$. It implies that $q^2+q+1 \mid (q-1)^2-1$, but this is a contradiction. □

Lemma 3.5. *The group G is isomorphic to the group B.*

Proof. By Lemma 3.2, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 2.4. Now Lemma 3.3 and Lemma3.4 implies that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of Lemma 2.4 occure. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. On the other hand, we know that $5 \nmid |G|$. Thus K/H is isomorphic to one of the groups in Lemma 2.7.

(1) $K/H \ncong {}^3D_4(q')$, where $q' \equiv \pm 2 \pmod 5$. Then by[19, Table Ic], $\pi({}^3D_4(q')) = q'^4 - q'^2 + 1$. So we consider $q^2+q+1 = q'^4 - q'^2 + 1$, in result $q(q+1) = q'^2(q'^2-1)$. Now since that $(q, q+1) = 1$, so we deduce $q'^2 = q+1$. In finally, since $|{}^3D_4(q')| \nmid |G|$, which is a contradiction.

(2) If $K/H \cong U_3(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [19, Table Ic], $\pi(U_3(q') = (q'^2 - q' + 1)/(3, q' + 1)$. If $(3, q' + 1) = 1$, then we consider $q^2 + q + 1 = q'^2 - q' + 1$ it follows that $q(q + 1) = q'(q' - 1)$. Now since $(q, q + 1) = 1$, we deduce $q' = q + 1$. But $|U_3(q')| \nmid |G|$, where this is a contradiction. Now we assume $(3, q' + 1) = 3$. Then we consider $q^2 + q + 1 = q'^2 - q' + 1)/3$, so $3q(q + 1) = (q' - 2)(q' + 1)$. Now we deduce that $(q' - 2, q' + 1) = 1$ or 3. First if $(q' - 2, q' + 1) = 1$, then $3q \mid q' + 1$ and $q' - 2 \mid q + 1$. But $q'^3(q'^3 + 1)(q'^2 - 1) \mid q^6(q^6 - 1)(q^2 - 1)$, so we deduce $q'^3 \mid q^2 - 1$. On the other hand, we have $3q \leq q' + 1$,so $q'^3 \leq q^2 - 1 \leq 3q \leq q' + 1$, which is a contradiction. Now if $(q' - 2, q' + 1) = 3$, then to similarly the last proof we have a contradiction.

(3) If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2m+1}$, then by [19, Table Id], $\pi({}^2G_2(q')) = q' \pm \sqrt{3q'} + 1$. For this purpose, we consider $q^2 + q + 1 = q' \pm \sqrt{3q'} + 1$, as a result $q(q+1) = q' \pm \sqrt{3q'}$. Hence $3^n(3^n + 1) = 3^{m+1}(3^m \pm 1)$, it follows that $3^n = 3^m \pm 1$ and also $3^n + 1 = 3^{m+1}$. First, if $3^n = 3^m \pm 1$, then $3^m + 1(3^m + 2) = 3^{m+1}(3^m + 1)$. As a result, we deduce $3^m + 2 = 3^{m+1}$, where this is a contradiction. Now, if $3^n + 1 = 3^{m+1}$, then $(3^{m+1} - 1)(3^{m+1} = 3^{m+1}(3^m + 1)$, where this a contradiction.

(4) If $K/H \cong L_n(q')$, where $n = 2$ and $q' \equiv \pm 2 \pmod 5$. Then by [19, Table Ib], $\pi(L_n(q')) = q', \frac{q'+1}{(2,q'-1)}$. First, we consider $q^2 + q + 1 = q'$, then since that $|L_2(q')| \nmid |G|$, so we have a contradiction. Now if $q^2 + q + 1 = q' + 1$, where $(2, q' - 1) = 1$, then we deduce $q(q + 1) = q'$ that this is a contradiction, because $q' = p'^m$. Next, we consider $q^2 + q + 1 = \frac{q'+1}{2}$, then we have $2q^2 + 2q + 2 = q' + 1$. So $2q^2 + 2q + 1 = q'$. Now since $|L_2(q')| \nmid |G|$, which is a contradiction.

(5) If $K/H \cong L_n(q')$, where $n = 3$ and $q' \equiv \pm 2 \pmod 5$. Then by [19, Table Ib], $\pi(L_3(q')) = \frac{q'^2+q'+1}{(3,q'-1)}$. First, we consider $q^2 + q + 1 = q'^2 + q' + 1$, then $q(q + 1) = q'(q' + 1)$. As a result $q = q'$. On the other hand, we know $q^2+q+1 \mid q'^3(q'^3-1)(q'^2-1)$ as a result $q^2+q+1 \leq q'^2-1$. But we have $q = q'$ so $q^2+q+1 \leq q^2-1$, where this is a contradiction. Now if $q^2+q+1 = \frac{q'^2+q'+1}{3}$, then $3q^2+3q+3 = q'^2+q'+1$. Hence, $3q(q+1) = (q'+2)(q'-1)$, now by proof (**2**) we have a contradiction, similarly.

So we deduce that $K/H \cong G_2(q')$, where $q' = 3^{n'}$. Now since $\pi(G_2(q')) = q'^2 + q' + 1$, we conclude that, $p = q'^2 + q' + 1$. Thus $q^2 + q + 1 = q'^2 + q' + 1$ and hence $n = n'$ and $q = q'$ and $K/H \cong B$. Now since $|K/H| = |B| = |G|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we have $H = 1$, $G = K \cong B$ and the proof is completed □

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