AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 6(3) (2025) 205-215
https://doi.org/10.22060/AJMC.2024.22768.1193

Original Article

Power study of goodness of fit tests for the Rayleigh distribution based on the empirical distribution function with application to real data

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ABSTRACT: The Rayleigh distribution is widely used to model events that occur in different fields such as medicine and natural sciences. In this article, we suggest some test statistics for examining the Rayleigh goodness of fit based on the empirical distribution function. Critical points and power of the tests are obtained by Monte Carlo simulation. We show that the proposed tests have a good performance against different alternatives and therefore these tests can be confidently used in practice. Finally, the proposed tests are illustrated by real data examples.

Review History:

Received:26 October 2023
Revised:28 April 2024
Accepted:28 April 2024
Available Online:01 July 2025

Keywords:

Rayleigh distribution
Goodness of fit tests
Monte Carlo simulation
Critical values
Power of test

MSC (2020):

62G10; 62G30; 62P12

1. Introduction

The Rayleigh distribution is an important statistical model in communications theory, engineering, physical sciences, clinical studies and medical imaging. This distribution was proposed by Lord Rayleigh [25] in the field of acoustics and then it was used to model wind speed, wave heights and sound/light radiation and multiple paths of dense scattered signals reaching a receiver. Also, it appears as a suitable model in life testing and reliability theory. For more details on the Rayleigh distribution the reader is referred to [18].

A random variable X follows the Rayleigh distribution if and only if it has probability density function

$$f_0(x;\theta) = \frac{x}{\theta^2} \exp\left\{-\frac{x^2}{2\theta^2}\right\}, \qquad x > 0, \quad \theta > 0.$$

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where θ is a scale parameter.
The cumulative distribution function of Rayleigh distribution is as

$$F_0(x;\theta) = 1 - \exp\left\{-\frac{x^2}{2\theta^2}\right\}, \qquad x > 0.$$

The mean and variance of this distribution are

$$\mu = E(X) = \theta\sqrt{\frac{\pi}{2}},$$

and

$$\sigma^2 = \mathrm{Var}(X) = \frac{4-\pi}{2}\theta^2,$$

respectively.
A detailed study about various properties of Rayleigh distribution is conducted by [28] and [29]. Inferences for this distribution have been discussed by several authors. Dyer and Whisenand [13] demonstrated the importance of this distribution in communication engineering. Bhattacharya and Tyagi [6] mentioned that in some clinical studies dealing with cancer patients, the survival pattern follows the Rayleigh distribution. [9] obtained the best invariant estimator and the Bayes estimator of the parameter of Rayleigh distribution under entropy loss. Fernandez [14] addressed the problems of estimating the parameter, hazard rate and reliability function of the Rayleigh distribution on the basis of sample quantiles. Dey and Maiti [11] derived Bayes estimator of the Rayleigh parameter and its associated risk based on extended Jeffrey's prior.
Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Rayleigh distribution, the estimator for the maximum likelihood estimate (MLE) of the parameter θ is

$$\hat{\theta} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}\,.$$

The estimator $\hat{\theta}$ of θ is a biased estimator and also consistent. We will use the ML estimator for the proposed statistics.
Because of the mentioned applications, it is meaningful to construct new goodness of fit tests for this distribution. Toward this end, we construct some distribution-free goodness of fit tests for the Rayleigh distribution, which are based on the empirical distribution function.
An important problem in statistics is to obtain information about the form of the population from which the sample is drawn. Goodness of fit (GOF) tests are designed to measure how well the observed sample data fit some proposed model. One class of GOF tests that can be used consists of tests based on the distance between the empirical and hypothesized distribution functions. Five of the known tests in this class are Cramer-von Mises (W^2), Kolmogorov-Smirnov (D), Kuiper (V), Watson (U^2), and Anderson-Darling (A^2). For more details about these tests, see [10].
Inferences for the Rayleigh distribution based on progressively Type II censored data have been done by some authors such as [19, 22], and [24]. Moreover, recently some researchers suggested tests for Rayleigh distribution, see for example, [1, 16, 17, 23, 26, 32, 35].
Recently, Torabi el al. [30] proposed a new test statistic based on the empirical distribution function and then constructed a test of fit for the normal distribution and show their test is powerful against some alternatives. Also, Torabi el al. [31] again used their test statistic and suggested a test for the exponential distribution. Here, we investigate the behavior of Torabi el al.'s test for the Rayleigh distribution and propose some test statistics for test of fit for the Rayleigh model.
The paper is organized as follows. In Section 2, some goodness of fit tests for the composite Rayleigh hypothesis based on the empirical distribution function are suggested. Also, properties of the test statistics are given. The critical values and the power values of the new tests are presented in Section 3. A real data example is provided to examine the performance of the proposed tests in Section 4. Section 5 contains conclusions.

2. The proposed test statistics

Given X and Y two absolutely continuous random variables with cdfs F_0 and F, respectively, [30] defined the following discrepancy measure:

$$D(F_0, F) = \int_{-\infty}^{\infty} h\left(\frac{1+F_0(x)}{1+F(x)}\right) dF(x) = E_F\left[h\left(\frac{1+F_0(x)}{1+F(x)}\right)\right],$$

where $E_F[.]$ is the expectation under F and $h:(0,\infty)\to R^+$ is a continuous function, decreasing on (0,1) and increasing on $(1,\infty)$ with an absolute minimum at $x=1$ such that $h(1)=0$. For this measure, $D(F,F_0)=0$ if and only if $F=F_0$, almost everywhere.

Torabi et al. [30] proposed to use this measure as a criterion of goodness of fit of an iid sample $X_1,\dots,X_n$ with empirical distribution function F_n, to a given distribution F_0. It is clear that $D(F,F_0)$ can be estimated by

$$H_n = D(F_0,F_n) = \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_{(i)})}{1+F_n(X_{(i)})}\right) = \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_{(i)})}{1+i/n}\right),$$

and we can consider it as a test statistic.

Let $X_1,\dots,X_n$ be a random sample from an unknown continuous cumulative distribution function F with a density $f(x)$. We want to test the hypothesis

$$H_0:\ f(x)=f_0(x;\theta)=\frac{x}{\theta^2}\exp\left\{-x^2/2\theta^2\right\},\qquad x>0,\ \ for\ some\ \theta\in\Theta,$$

where θ is unspecified and $\Theta=R^+$. The alternative to H_0 is

$$H_1:\ f(x)\neq f_0(x;\theta),\qquad for\ any\ \theta\in\Theta.$$

Here, we construct tests for the Rayleigh distribution based on H_n as follows. Let $X_{(1)}\le X_{(2)}\le\cdots\le X_{(n)}$ are the order statistics based on the random sample $X_1,\dots,X_n$. Applying the Torabi et al. [30] distance we have

$$H_n = \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_{(i)})}{1+F_n(X_{(i)})}\right) = \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_{(i)};\hat\theta)}{1+i/n}\right),$$

where $F_0(x)$ is the cumulative distribution function of the Rayleigh distribution and $\hat\theta$ is the maximum likelihood estimate of the parameter θ. It can be shown that the maximum likelihood estimate (MLE) of the parameter θ is

$$\hat\theta=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}x_i^2}\,.$$

The test statistic H_n is expected to take values close to zero when H_0 is true. Hence, the null hypothesis is rejected for large values of H_n. Here, we consider the following functions for h.

$$\begin{aligned}
h_1(x) &= x\log(x)-x+1,\\
h_2(x) &= \left(\frac{x-1}{x+1}\right)^2,\\
h_3(x) &= h_2(x)I_{[0,1]}(x)+h_1(x)I_{[1,\infty)}(x),\\
h_4(x) &= h_1(x)I_{[0,1]}(x)+h_2(x)I_{[1,\infty)}(x).
\end{aligned}$$

The first and second functions are suggested by [30].

Note that $h_k:[0,\infty)\to R^+$ is a non-negative function with the absolute minimum at $x=1$, such that $h_k(1)=0$, $k=1,2,3,4$. Under H_0, we expect that $F_n(x)\approx F_0(x)$. Hence $h_k\left(\frac{1+F_0(X)}{1+F_n(X)}\right)\approx 0$. Thus the value of test statistic is expected to be near zero when H_0 is true. Therefore, it is justifiable to reject H_0 for large values of H_n. Finally, we can write the proposed test statistic as follow.

$$H_n^{(k)}=\frac{1}{n}\sum_{i=1}^{n} h_k\left(\frac{1+F_0(X_{(i)};\hat\theta)}{1+i/n}\right),$$

where $F_0(x;\hat\theta)=1-\exp\left\{-\frac{x^2}{2\hat\theta^2}\right\}$.

Proposition 2.1. *The support of statistics $H_n^{(k)}$, $k=1,\dots,4$, are given by*

$$\mathrm{supp}(H_n^{(1)})=[0,0.38629],\qquad \mathrm{supp}(H_n^{(2)})=[0,0.11111],$$

$$\mathrm{supp}(H_n^{(3)})=[0,0.38629],\qquad \mathrm{supp}(H_n^{(4)})=[0,0.15342].$$

Proof. From Proposition 2.3 of [30], we have that for all $x \in R$

$$0 \le h_k\left(\frac{1+F_0(X)}{1+F_n(X)}\right) \le \max\left(h_k(1/2), h_k(2)\right) = \begin{cases} 0.38629 & k=1 \\ 0.11111 & k=2 \\ 0.38629 & k=3 \\ 0.15342 & k=4 \end{cases}$$

Finally, since $H_n^{(k)}$ is the mean of $h_k(.)$ over the transformed data, the results are obtained. □

Proposition 2.2. *Let F_1 be an arbitrary continuous cdf in H_1. Then under the assumption that the observed sample have cdf F_1, the test based on H_n is consistent.*

Proof. Based on Glivenko-Cantelli theorem, for enough large n, we have that $F_n(x) \approx F_1(x)$, for all $x \in R$. Also, $\hat{\theta}$ is MLE of θ, and hence is consistent. Therefore,

$$\begin{aligned} H_n &= \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_{(i)};\hat{\theta})}{1+F_n(X_{(i)})}\right) = \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_i;\hat{\theta})}{1+F_n(X_i)}\right) \\ &\approx \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_i;\hat{\theta})}{1+F_1(X_i)}\right) \approx \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{1+F_0(X_i;\hat{\theta})}{1+F_1(X_i)}\right) \\ &\to E_{F_1}\left[h\left(\frac{1+F_0(X_i;\hat{\theta})}{1+F_1(X_i)}\right)\right] = D(F_0,F_1), \;\; as\ n\to\infty. \end{aligned}$$

Note that the convergence holds by the law of large numbers and $D(F_0, F_1)$ is a divergence between F_0 and F_1. So, the test based on H_n is consistent. □

Remark 2.3. *As θ is a scale parameter of Rayleigh distribution we consider a scale transformation group as $G = \{g_c : g_c(x) = cx,\ c > 0\}$. Since*

$$H_n(g(x)) = H_n(x) \qquad \forall g \in G,$$

therefore, H_n is a scale-free statistic or invariant under G.

Remark 2.4. *Since the test statistic H_n is invariant under the scale transformations and the parameter space is transitive, the distribution of the proposed test statistic H_n is free of θ. Then, the test is exact and the critical values does not depend on θ. Therefore, the critical values of the test statistic can be obtained by simulation when $\theta = 1$.*

According to the mentioned properties, the H_n test is a reasonable test for the Rayleigh distribution which has some good properties such as to be scale-free, invariancy and consistency. In the next section, we investigate the power values of the proposed tests and then compared them with the power values of the competing tests.

3. Simulation study

In this section, we first obtain the critical points of the proposed tests by Monte Carlo method. Then power of the tests is computed and compared with the power of the competing tests.

3.1. Critical points

The null hypothesis H_0, at the significance level α is rejected if $\left(H_n^{(1)}, H_n^{(2)}, H_n^{(3)}, H_n^{(4)}\right) \ge C(\alpha)$, where the critical point $C(\alpha)$ is determined by the $\alpha-$quantile of the distribution of the test statistics under the null hypothesis.
In order to obtain the critical points of the test statistics, 100,000 samples of size n were generated from the Rayleigh distribution with the parameter one. For each sample ($n = 10, 20, 30, 40, 50, 75, 100$) the test statistics was computed and by using these values the critical points $C(\alpha)$, were determined. The critical points of the statistics $H_n^{(1)}$, $H_n^{(2)}$, $H_n^{(3)}$, and $H_n^{(4)}$ are presented in Table 1. Also, in Figure 1, we show the behavior of the critical values of the proposed tests. From Table 1 and Figure 1, we can see that when the sample size increases the critical values decreases.
Figures 2 and 3 show the empirical probability density functions of the proposed test statistics with Monte Carlo samples. From these figures, it is evident that $H_n^{(2)}$ has closer values to 0 than the other statistics. Then the bias of $H_n^{(2)}$ is smallest. Also, we can see that the test statistic $H_n^{(2)}$ has the smallest variance.

Table 1: Critical values of the proposed test statistics for $\alpha = 0.05$

n	$H_n^{(1)}$	$H_n^{(2)}$	$H_n^{(3)}$	$H_n^{(4)}$
10	0.00579	0.00309	0.00391	0.00537
20	0.00295	0.00153	0.00207	0.00262
30	0.00198	0.00101	0.00141	0.00172
40	0.00148	0.00076	0.00107	0.00128
50	0.00118	0.00060	0.00086	0.00100
75	0.00079	0.00040	0.00058	0.00066
100	0.00059	0.00030	0.00044	0.00049

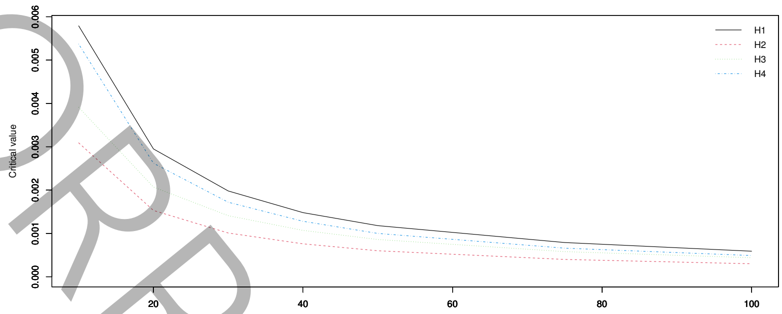

Figure 1: Critical values of the proposed tests for different values of sample sizes.

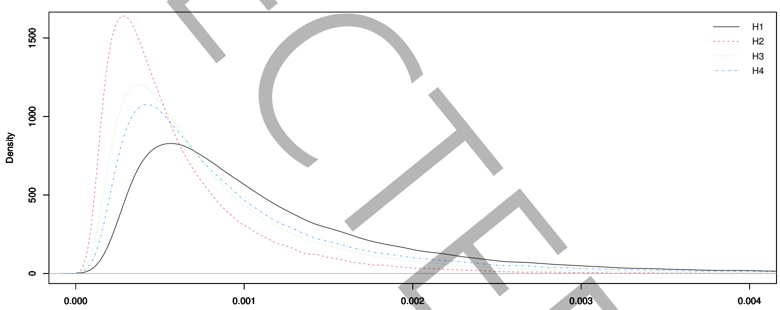

Figure 2: Empirical densities of the test statistics based on 100,000 simulations under the Rayleigh hypothesis and $n = 20$.

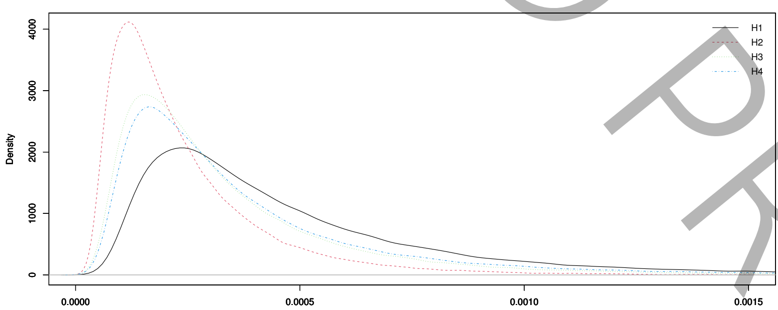

Figure 3: Empirical densities of the test statistics based on 100,000 simulations under the Rayleigh hypothesis and $n = 50$.

3.2. Power comparison

For power comparison, we consider the well-known tests based on the empirical distribution function (EDF) that used widely in practice. These tests are Cramer von Mises W^2, Kolmogorov-Smirnov D, Anderson-Darling A^2, Kuiper V, and Watson U^2. The test statistics of the EDF-tests are briefly described as follows. For more details about these tests, see [10].

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics based on the random sample $X_1, \ldots, X_n$.

1) The Cramer-von Mises statistic [33]:

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - F_0(X_{(i)}; \hat{\theta}) \right)^2 .$$

2) The Watson statistic [34]:

$$U^2 = W^2 - n\big(\bar{P} - 0.5\big)^2,$$

where $\bar{P}$ is the mean of $F_0(X_{(i)}; \hat{\theta}),\ \ i = 1, \ldots, n$.

3) The Kolmogorov-Smirnov statistic [20]:

$$D = \max(D^+, D^-)$$

where

$$D^+ = \max_{1 \le i \le n} \left\{ \frac{i}{n} - F_0(X_{(i)}; \hat{\theta}) \right\}; \quad D^- = \max_{1 \le i \le n} \left\{ F_0(X_{(i)}; \hat{\theta}) - \frac{i-1}{n} \right\}.$$

4) The Kuiper statistic [21]:

$$V = D^+ + D^-$$

5) The Anderson-Darling statistic [3]:

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log F_0(X_{(i)}; \hat{\theta}) + \log \left[1 - F_0(X_{(n-i+1)}; \hat{\theta}) \right] \right\}.$$

In the above test statistics, $F_0(x)$ is the cumulative distribution function of the Rayleigh distribution and $\hat{\theta}$ are the maximum likelihood estimates of the parameter θ.
Moreover, we consider three recent tests introduced by authors and compare our tests with them in terms of power. These tests are as follows.

- Alizadeh et al. [2] proposed a goodness of fit test based on Kullback-Leibler divergence for Rayleigh distribution. Their proposed statistic is

$$KL_{mn} = -HV_{mn} + 2\log(\hat{\theta}) - \frac{1}{n} \sum_{i=1}^{n} \log(X_i) + 1\,,$$

where $\hat{\theta}$ is the ML estimate of θ and HV_{mn} is Vasicek's estimator of entropy given by

$$HV_{mn} = \frac{1}{n} \sum_{i=1}^{n} \log \left\{ \frac{n}{2m} (X_{(i+m)} - X_{(i-m)}) \right\},$$

where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$ and $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are order statistics based on a random sample of size n.

- Baratpour and Khodadadi [4] based on the cumulative residual entropy suggested a test for Rayleigh distribution which is

$$CK_n = \frac{\sum\limits_{i=1}^{n} \left(1 - \frac{i}{n}\right) \log\left(1 - \frac{i}{n}\right) \left\{ X_{(i+1)} - X_{(i)} \right\} + \sqrt{\frac{\pi}{2}} \sqrt{\sum\limits_{i=1}^{n} X_i^3 \Big/ 3 \sum\limits_{i=1}^{n} X_i}}{\bar{X}}.$$

- Safavinejad et al. [27] proposed a test of fit for Rayleigh distribution based on the empirical likelihood ratio. Their test statistic is

$$R_n = \frac{\underset{1 \le m < n^{\delta}}{min} \prod\limits_{j=1}^{n} \left\{ \frac{n}{2m} \left(X_{(i+m)} - X_{(i-m)} \right) \right\}}{\left(\prod\limits_{i=1}^{n} X_i \Big/ \hat{\theta}^{2n} \right) \exp\left\{ -\sum\limits_{i=1}^{n} X_i^2 \Big/ 2\hat{\theta}^2 \right\}},$$

where $\hat{\theta}$ is the ML estimate of θ and $0 < \delta < 1$.

We compute the power of the considered tests and the proposed tests against various distributions. In power comparison, we considered the following alternatives.

- the Weibull distribution with density $\theta x^{\theta-1}\exp\left(-x^{\theta}\right)$, denoted by $W(\theta)$;
- the gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$, denoted by $\Gamma(\theta)$;
- the lognormal law $LN(\theta)$ with density $(\theta x)^{-1}(2\pi)^{-1/2}\exp\left(-(\log x)^2 \Big/ (2\theta^2)\right)$;
- the half-normal HN distribution with density $\Gamma(2/\pi)^{1/2}\exp\left(-x^2/2\right)$;
- the uniform distribution U with density 1, $\quad 0 \le x \le 1$;
- the modified extreme value $EW(\theta)$, with distribution function $1-\exp\left(\theta^{-1}(1-e^{x})\right)$;
- the linear increasing failure rate law $LF(\theta)$ with density $(1+\theta x)\exp\left(-x-\theta x^2/2\right)$;
- Dhillon's [12] law $DL(\theta)$ with distribution function $1-\exp\left(-(\log(x+1))^{\theta+1}\right)$;
- Chen's [8] distribution $CH(\theta)$, with distribution function $1-\exp\left(2\left(1-e^{x^{\theta}}\right)\right)$.

These alternatives include densities f with decreasing failure rates (DFR), increasing failure rates (IFR) as well as models with unimodal failure rate (UFR) functions and bathtub failure rate (BFR) functions.
To assess the power values of the tests, we generate 100,000 random samples from the alternative hypothesis for different choices of sample sizes and then the test statistics are calculated. Then power of the corresponding test is computed by the frequency of the event "the statistic is in the critical region". Tables 2 and 3 display and compares the power values of the tests for sample sizes $n = 10, 20, 30$ at the significance level $\alpha = 0.05$.
For each sample size and alternative, the bold type in these tables indicates the tests achieving the maximal power.

Table 2: Empirical powers of the tests against IFR alternatives at significance level 5%.

Alternative	n	W^2	D	V	U^2	A^2	CK_n	KL_{mn}	R_n	$H_n^{(1)}$	$H_n^{(2)}$	$H_n^{(3)}$	$H_n^{(4)}$
$W(1.4)$	10	0.224	0.194	0.158	0.156	0.323	0.274	0.101	0.102	0.354	0.368	0.279	**0.383**
	20	0.417	0.368	0.274	0.292	0.524	0.427	0.241	0.263	0.548	0.565	0.431	**0.597**
	30	0.584	0.524	0.412	0.439	0.692	0.572	0.413	0.402	0.684	0.699	0.560	**0.733**
$\Gamma(2)$	10	0.209	0.181	0.140	0.141	0.265	0.256	0.073	0.087	0.308	0.318	0.244	**0.332**
	20	0.367	0.324	0.230	0.250	0.421	0.404	0.172	0.181	0.476	0.492	0.366	**0.527**
	30	0.511	0.460	0.340	0.364	0.560	0.542	0.279	0.290	0.605	0.621	0.483	**0.660**
HN	10	0.282	0.244	0.215	0.226	0.466	0.320	0.206	0.219	0.441	0.458	0.359	**0.472**
	20	0.523	0.469	0.384	0.412	0.703	0.495	0.444	0.450	0.662	0.678	0.553	**0.709**
	30	0.705	0.649	0.558	0.598	**0.847**	0.626	0.635	0.650	0.807	0.817	0.700	0.841
U	10	0.119	0.103	0.163	0.152	**0.268**	0.123	0.172	0.178	0.170	0.178	0.140	0.179
	20	0.181	0.154	0.262	0.281	**0.385**	0.210	0.453	0.449	0.240	0.249	0.193	0.257
	30	0.280	0.222	0.391	0.421	**0.522**	0.367	0.712	0.693	0.320	0.334	0.257	0.335
$CH(1)$	10	0.353	0.312	0.264	0.275	0.545	0.371	0.252	0.253	0.513	0.530	0.429	**0.546**
	20	0.603	0.553	0.461	0.506	0.777	0.569	0.538	0.549	0.747	0.760	0.649	**0.784**
	30	0.795	0.749	0.661	0.698	**0.908**	0.723	0.756	0.693	0.876	0.886	0.794	0.904
$CH(1.5)$	10	0.059	0.054	0.064	0.060	0.099	0.057	0.041	0.052	0.094	0.099	0.071	**0.103**
	20	0.068	0.069	0.076	0.076	0.119	0.045	0.071	0.076	0.111	0.119	0.075	**0.133**
	30	0.081	0.079	0.094	0.099	0.143	0.048	0.093	0.099	0.117	0.125	0.073	**0.143**
$LF(2)$	10	0.237	0.205	0.177	0.188	0.402	0.283	0.161	0.163	0.383	0.399	0.306	**0.413**
	20	0.427	0.379	0.304	0.337	0.606	0.408	0.357	0.354	0.573	0.589	0.456	**0.620**
	30	0.597	0.544	0.453	0.484	**0.766**	0.514	0.533	0.541	0.715	0.731	0.599	0.760
$LF(4)$	10	0.167	0.142	0.133	0.134	0.302	0.195	0.113	0.125	0.285	0.299	0.219	**0.313**
	20	0.292	0.259	0.214	0.225	0.467	0.282	0.239	0.242	0.428	0.445	0.315	**0.477**
	30	0.433	0.379	0.312	0.340	**0.618**	0.382	0.379	0.370	0.547	0.567	0.418	0.607
$EV(0.5)$	10	0.350	0.309	0.271	0.275	0.542	0.385	0.248	0.269	0.518	0.533	0.434	**0.548**
	20	0.609	0.557	0.471	0.497	0.777	0.577	0.541	0.556	0.749	0.763	0.650	**0.787**
	30	0.796	0.749	0.655	0.696	**0.909**	0.719	0.752	0.754	0.876	0.886	0.795	0.904
$EV(1.5)$	10	0.150	0.132	0.138	0.134	**0.310**	0.152	0.126	0.130	0.263	0.275	0.203	0.285
	20	0.251	0.228	0.221	0.240	**0.458**	0.199	0.274	0.279	0.376	0.394	0.273	0.423
	30	0.362	0.332	0.323	0.355	**0.589**	0.227	0.433	0.421	0.496	0.517	0.370	0.553

Table 3: Empirical powers of the tests against UFR, DFR and BFR alternatives at level %.

Alternative	n	W^2	D	V	U^2	A^2	CK_n	KL_{mn}	R_n	$H_n^{(1)}$	$H_n^{(2)}$	$H_n^{(3)}$	$H_n^{(4)}$
$LN(0.8)$	10	0.423	0.380	0.313	0.314	0.453	0.461	0.215	0.227	0.525	0.536	0.459	**0.548**
	20	0.702	0.664	0.545	0.574	0.717	0.741	0.501	0.514	0.766	0.773	0.691	**0.792**
	30	0.861	0.830	0.734	0.754	0.858	0.866	0.689	0.710	0.889	0.896	0.835	**0.909**
$LN(1.5)$	10	0.910	0.890	0.847	0.851	0.956	0.920	0.855	0.856	0.959	0.962	0.944	**0.964**
	20	0.997	0.996	0.989	0.992	0.999	0.996	0.993	0.995	0.999	0.999	0.998	**0.999**
	30	1.000	1.000	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	**1.000**
$DL(1)$	10	0.410	0.363	0.295	0.296	0.482	0.455	0.202	0.216	0.526	0.537	0.458	**0.551**
	20	0.696	0.647	0.530	0.541	0.744	0.712	0.475	0.487	0.768	0.778	0.686	**0.799**
	30	0.856	0.818	0.708	0.723	0.885	0.853	0.681	0.689	0.894	0.901	0.837	**0.915**
$DL(1.5)$	10	0.165	0.144	0.108	0.110	0.185	0.206	0.060	0.065	0.238	0.248	0.186	**0.258**
	20	0.284	0.255	0.171	0.180	0.304	0.352	0.123	0.143	0.363	0.375	0.270	**0.405**
	30	0.406	0.361	0.251	0.270	0.419	0.470	0.200	0.213	0.469	0.486	0.352	**0.524**
$W(0.8)$	10	0.812	0.774	0.706	0.706	**0.925**	0.836	0.764	0.762	0.904	0.910	0.869	0.915
	20	0.985	0.976	0.945	0.951	**0.997**	0.979	0.976	0.981	0.994	0.995	0.989	0.996
	30	0.999	0.998	0.993	0.993	**1.000**	0.997	0.999	0.999	1.000	1.000	0.999	1.000
$\Gamma(0.4)$	10	0.949	0.930	0.898	0.901	**0.992**	0.955	0.967	0.967	0.981	0.983	0.970	0.984
	20	0.999	0.999	0.995	0.996	**1.000**	0.998	1.000	1.000	1.000	1.000	1.000	1.000
	30	1.000	1.000	1.000	1.000	**1.000**	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$CH(0.5)$	10	0.929	0.907	0.865	0.866	**0.984**	0.933	0.941	0.940	0.972	0.975	0.956	0.976
	20	0.999	0.998	0.992	0.995	**1.000**	0.998	0.999	0.999	0.999	0.999	0.999	0.999
	30	1.000	1.000	1.000	1.000	**1.000**	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 4: Powerful tests against different alternatives.

IFR	UFR	DFR-BFR
$H_n^{(4)}$ & A^2	$H_n^{(4)}$	A^2

Based on the power values in Table 2, it is seen that the tests based on $H_n^{(4)}$ and A^2 statistics have the most power against IFR alternatives. The power differences between the proposed test and the other tests are substantial. Tables 3 reveals a superiority of the proposed test $H_n^{(4)}$ to all other tests as we can say that these tests outperform all other tests against UFR alternatives. The power differences between the proposed test $H_n^{(4)}$ and the other tests are substantial.
From Table 3, it is evident that the test based on A^2 statistic has the most power against DFR and BFR alternatives. However, the power differences between this test and the proposed test $H_n^{(4)}$ are small and therefore we can select one of the tests based on A^2 or $H_n^{(4)}$ statistics as a powerful test.
Although there is no uniformly most powerful test against all alternatives, the tests based on A^2 and $H_n^{(4)}$ statistics can be recommended in practice. Generally, we summarized the results in Table 4. This table presents the best test in terms of power against different alternatives.

4. Applications to real data

In this section, to show the behavior of the proposed tests in real cases, two real data set are analyzed.

Example 4.1. *We consider the data set discussed by [7]. The data set consists the failure times of 25 ball bearings in endurance test. The failure times are as follows:*
17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 67.80, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.
In Figure 4, we depict the histogram of this data set. Baratpour and Khodadadi [5] applied a goodness of fit test based on the cumulative residual entropy and concluded that this data set follows on the Rayleigh distribution. Then, Safavinejad et al. [27] considered these data and they found that Rayleigh distribution is fitting the above data quite satisfactorily.
The proposed tests can be used to investigate whether the data come from a Rayleigh distribution. The ML estimator of θ is computed as

$$\hat{\theta} = 56.584\,.$$

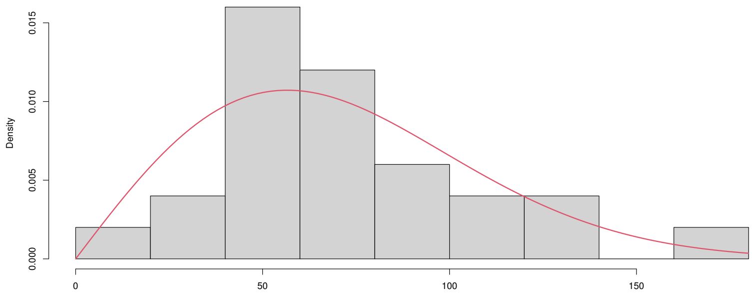

Figure 4: Histogram of failure times and a fitted Rayleigh density function.

The values of the proposed test statistics are

$$H_n^{(1)} = 0.000655,\ H_n^{(2)} = 0.000335,\ H_n^{(3)} = 0.000440,\ H_n^{(4)} = 0.000550,$$

and the critical values at the 5% are obtained as 0.00237, 0.00122, 0.00166, 0.00208, respectively. Since the values of the test statistics are smaller than the corresponding critical values, the Rayleigh assumption is not rejected at the significance level of 0.05. Therefore, we can conclude that the data come from a Rayleigh distribution.

Example 4.2. *Hinkley [15] presented data consist of thirty successive March precipitation (in inches) observations. The data are given in the following:*
0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05.
Histogram of the considered data set is presented in Figure 5. Also, a fitted Rayleigh density function for these data is displayed.
Here, we apply the proposed tests to investigate whether the data come from a Rayleigh distribution. The ML estimator of θ is computed as

$$\hat{\theta} = 1.374 .$$

The values of the test statistics are

$$H_n^{(1)} = 0.000717,\ H_n^{(2)} = 0.000370,\ H_n^{(3)} = 0.000371,\ H_n^{(4)} = 0.000716,$$

and the critical values at the 5% are obtained from Table 1 as 0.00198, 0.00101, 0.00141, 0.00172, respectively. Since the values of the test statistics are smaller than the corresponding critical values, the Rayleigh assumption is not rejected at the significance level of 0.05. Therefore, we can conclude that the data come from a Rayleigh distribution.

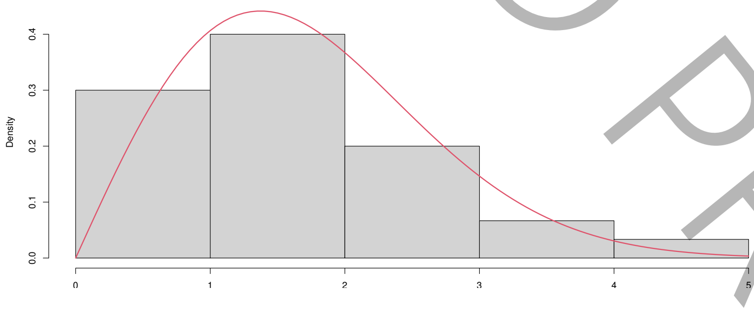

Figure 5: Histogram of observations in Example 2 and a fitted Rayleigh density function.

5. Conclusions

In this paper, we have proposed some goodness of fit test statistics for the Rayleigh distribution. Then, we have presented the properties of these test statistics. We have obtained the power values of the proposed test statistics with Monte Carlo simulation and compared them with the competing test statistics against various alternatives. We have observed that the tests based on $H_n^{(4)}$ and A^2 statistics have the most power against IFR alternatives. Against UFR alternatives, the proposed test $H_n^{(4)}$ has the most power in compared to all other tests so that we can say the new test outperforms all other tests. Moreover, we have observed that the test based on A^2 statistic has the most power against DFR and BFR alternatives and the power differences between the test A^2 and the proposed test $H_n^{(4)}$ were small and so the both tests worked well. Generally, we have concluded that among the proposed and competing tests, the proposed test $H_n^{(4)}$ has a good performance against different alternatives. Therefore, this test can be confidently recommended in practice. Finally, we have illustrated the performance of the new test statistics in real cases and shown that they worked well.

Acknowledgements

The authors are very grateful to anonymous referees and the associate editor for providing some useful comments on an earlier version of this manuscript.

References

[1] V. Ahrari, S. Baratpour, A. Habibirad, and V. Fakoor, *Goodness of fit tests for Rayleigh distribution based on quantiles*, Comm. Statist. Simulation Comput., 51 (2022), pp. 341–357.

[2] R. Alizadeh Noughabi, H. Alizadeh Noughabi, and A. Ebrahimi Moghaddam Behabadi, *An entropy test for the Rayleigh distribution and power comparison*, J. Stat. Comput. Simul., 84 (2014), pp. 151–158.

[3] T. W. Anderson and D. A. Darling, *A test of goodness of fit*, J. Amer. Statist. Assoc., 49 (1954), pp. 765–769.

[4] S. Baratpour and F. a. Khodadadi, *A cumulative residual entropy characterization of the Rayleigh distribution and related goodness-of-fit test*, J. Statist. Res. Iran, 9 (2013), pp. 115–131.

[5] S. Baratpour and A. H. Rad, *Testing goodness-of-fit for exponential distribution based on cumulative residual entropy*, Comm. Statist. Theory Methods, 41 (2012), pp. 1387–1396.

[6] S. K. Bhattacharya and K. T. Ravinder, *Bayesian survival analysis based on the Rayleigh model*, Trabajos de Estadística, 5 (1990), pp. 81–92.

[7] C. Caroni, *The correct "ball bearings" data*, Lifetime Data Anal., 8 (2002), pp. 395–399.

[8] Z. Chen, *A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function*, Statistics & Probability Letters, 49 (2000), pp. 155–161.

[9] Y. Chung, *Estimation of scale parameter from Rayleigh distribution under entropy loss*, J. Appl. Math. Comput., 2 (1995), pp. 33–40.

[10] R. B. D'Agostino and M. A. Stephens, *Goodness-of-fit techniques*, Marcel Dekker, Inc., USA, 1986.

[11] S. Dey and S. Maiti, *Bayesian estimation of the parameter of Rayleigh distribution under the extended Jeffrey's prior*, Electron. J. Appl. Stat. Anal., 5 (2012), pp. 44–59.

[12] B. S. Dhillon, *Life distributions*, IEEE Transactions on Reliability, R-30 (1981), pp. 457–460.

[13] D. D. Dyer and C. W. Whisenand, *Best linear unbiased estimator of the parameter of the Rayleigh distribution*, IEEE Transactions on Reliability, R-22 (1973), pp. 229–231.

[14] A. J. Fernández, *Bayesian estimation and prediction based on Rayleigh sample quantiles*, Quality & Quantity, 44 (2010), pp. 1239–1248.

[15] D. Hinkley, *On quick choice of power transformation*, Journal of the Royal Statistical Society. Series C (Applied Statistics), 26 (1977), pp. 67–69.

[16] S. M. A. Jahanshahi, A. H. Rad, and V. Fakoor, *A goodness-of-fit test for Rayleigh distribution based on hellinger distance*, Annals of Data Science, 3 (2016), pp. 401–411.

[17] S. M. A. Jahanshahi, A. H. Rad, and V. Fakoor, *Some new goodness-of-fit tests for Rayleigh distribution*, Pak. J. Stat. Oper. Res., (2020), pp. 305–315.

[18] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions, Volume 1*, Wiley Series in Probability and Statistics, John Wiley & Sons, Nashville, TN, 2 ed., 1994.

[19] A. Kohansal and S. Rezakhah, *Inference of $R = P(Y < X)$ for two-parameter Rayleigh distribution based on progressively censored samples*, Statistics, 53 (2019), pp. 81–100.

[20] A. Kolmogoroff, *Sulla determinazione empirica di una legge di distribuzione.*, G. Ist. Ital. Attuari, 4 (1933), pp. 83–91.

[21] N. H. Kuiper, *Tests concerning random points on a circle*, Nederl. Akad. Wet., Proc., Ser. A, 63 (1960), pp. 38–47.

[22] H. Liao and W. Gui, *Statistical inference of the Rayleigh distribution based on progressively type-II censored competing risks data*, Symmetry, 11 (2019), p. 898.

[23] S. Liebenberg and J. Allison, *A review of goodness-of-fit tests for the Rayleigh distribution*, Austrian Journal of Statistics, 52 (2023), pp. 1–22.

[24] A. Pak, G. A. Parham, and M. Saraj, *Inference for the Rayleigh distribution based on progressive type-II fuzzy censored data*, J. Mod. Appl. Stat. Methods, 13 (2014), pp. 287–304.

[25] L. Rayleigh, *On the resultant of a large number of vibrations of the same pitch and of arbitrary phase*, Phil. Mag. (5), 10 (1880), pp. 73–78.

[26] J. Ren and W. Gui, *Goodness-of-fit test for Rayleigh distribution based on progressively type-II censored sample*, Commun. Stat., Theory Methods, 50 (2021), pp. 3851–3874.

[27] M. Safavinejad, S. Jomhoori, and H. A. Noughabi, *A density-based empirical likelihood ratio goodness-of-fit test for the Rayleigh distribution and power comparison*, J. Stat. Comput. Simulation, 85 (2015), pp. 3322–3334.

[28] M. M. Siddiqui, *Some problems connected with Rayleigh distributions*, J. Res. Natl. Bur. Stand., Sect. D, 66 (1962), pp. 167–174.

[29] M. M. Siddisqui, *Statistical inference for Rayleigh distributions*, J. Res. Natl. Bur. Stand., Sect. D, 68 (1964), pp. 1005–1010.

[30] H. Torabi, N. H. Montazeri, and A. Grané, *A test for normality based on the empirical distribution function*, SORT, 40 (2016), pp. 55–88.

[31] ———, *A wide review on exponentiality tests and two competitive proposals with application on reliability*, J. Stat. Comput. Simulation, 88 (2018), pp. 108–139.

[32] K. M. Vaisakh, T. Xavier, and E. P. Sreedevi, *Goodness of fit test for Rayleigh distribution with censored observations*, J. Korean Statist. Soc., 52 (2023), pp. 794–815.

[33] R. von Mises, *Vorlesungen aus dem Gebiete der angewandten Mathematik. Bd. 1. Wahrscheinlichkeitsrechnung und ihre Anwendung in der Statistik und theoretischen Physik.* Leipzig u. Wien: Franz Deuticke. X, 574 S. u. 90 Abb. (1931)., 1931.

[34] G. S. Watson, *Goodness-of-fit tests on a circle*, Biometrika, 48 (1961), pp. 109–114.

[35] E. Zamanzade and M. Mahdizadeh, *Goodness of fit tests for Rayleigh distribution based on Phi-divergence*, Rev. Colomb. Estad., 40 (2017), pp. 279–290.

Please cite this article using:

Hadi Alizadeh Noughabi, Mohammad Shafaei Noughabi, Power study of goodness of fit tests for the Rayleigh distribution based on the empirical distribution function with application to real data, AUT J. Math. Comput., 6(3) (2025) 205-215
https://doi.org/10.22060/AJMC.2024.22768.1193