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Original Article

On Zermelo's navigation problem and weighted Einstein Randers metrics

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ABSTRACT: This paper investigates a specific form of weighted Ricci curvature known as the quasi-Einstein metric. Two Finsler metrics, F and $\tilde{F}$ are considered, which are generated by navigation representations (h, W) and (F, V), respectively, where W represents a vector field, and V represents a conformal vector field on the manifold M. The main focus is on identifying the necessary and sufficient condition for the Randers metric F to qualify as a quasi-Einstein metric. Additionally; we establish the relationship between the curvatures of the given Finsler metrics F and $\tilde{F}$.

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1. Introduction

In Finsler geometry, the choice of a measure is not as straightforward as in Riemannian geometry, where there is a unique canonical measure. Consider $(M, F, \mathfrak{m})$ to be a Finsler-measured manifold, where (M, F) is a Finsler manifold with the metric F, and $\mathfrak{m}$ is a positive C^∞-measured on M. For $N \in \mathbb{R} \setminus \{n\}$, Ohta introduced the following Finsler weighted Ricci curvature:

$$\mathrm{Ric}_N(x) := \mathrm{Ric}(x) + \psi''_\eta(0) - \frac{\psi'_\eta(0)^2}{N-n},$$

where ψ_η is C^∞ in $\mathbb{R}$ and η is the geodesic M with $\dot{\eta}(0) = v$, respectively [1]. As $N \to \infty$, we arrive at the following relation:

$$\mathrm{Ric}_\infty(x) = \mathrm{Ric}(x) + \psi''_\eta(0),$$

which is called an ∞-weighted Ricci curvature [3]. When $N \to n$ and $\psi'_\eta(0) = 0$, we have

$$\mathrm{Ric}_n(v) = \mathrm{Ric}(v) + \psi''_\eta(0),$$

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and when $N \to n$ and $\psi'_\eta(0) \neq 0$, we have $\mathrm{Ric}_n = -\infty$. Both cases are known as n-weighted Ricci curvatures. We also assume that $\mathrm{Ric}_N(0) = 0$.

Z. Shen in 1997, introduced a new quantity which is called the S-curvature [15]. Substituting $\psi'_\eta(0)$ with $S(x)$, yields the following equation:

$$\mathrm{Ric}_N(x) = \mathrm{Ric}(x) + \dot{S}(x) - \frac{1}{N-n}S^2, \tag{1}$$

where $\dot{S}$ is the covariant derivative of S along a geodesic of F. Again, as $N \to \infty$,

$$\mathrm{Ric}_\infty(x) = \mathrm{Ric}(x) + \dot{S}(x). \tag{2}$$

This was first studied by Ohta [10]. In the special case, a Finsler metric F is called *quasi-Einstein* (QE) [25]; if it satisfies

$$\mathrm{Ric}_{(QE)} \coloneqq \mathrm{Ric}_\infty = (n-1)cF^2. \tag{3}$$

The projective Ricci curvature introduced by Z. Shen [16], is a specific type of weighted Ricci curvature that possesses the property of projective invariance, when the volume form is fixed [17] (additionally [6][7]), and can be formulated as

$$\mathrm{PRic}(y) = \mathrm{Ric}(y) + (n-1)\left[\frac{\dot{S}}{n+1} + \frac{S^2}{(n+1)^2}\right]. \tag{4}$$

The weighted projective Ricci curvature with respect to a fixed Finsler metric and a volume form, with coefficient σ_0, is defined as

$$\mathrm{WPRic}_0 \coloneqq \mathrm{Ric} + (n-1)\mathcal{S}^2 + \mathcal{S}_{|k}y^k,$$

where $\mathcal{S} \coloneqq \frac{1}{n+1}\left[S + d\ln\left(\frac{\sigma_0}{\sigma}\right)\right]$, and σ is the coefficient of Finsler manifold [22].

Another weighted Ricci curvature is the (a, c)-weighted Ricci curvature in Finsler geometry, that was proposed by Z. Shen and R. Zhao [2], and we express it as

$$\mathrm{Ric}_{(a,c)}(y) = \mathrm{Ric}(y) + a\dot{S} - cS^2, \tag{5}$$

where a and c are constants. Finally; we define the generalized weighted Ricci curvature by

$$\mathrm{Ric}_{(a,c)}(y) = \mathrm{PRic}(y) - \frac{\kappa}{n+1}\left(\dot{S} + \frac{4}{n+1}S^2\right) + \frac{\nu}{(n+1)}S^2, \tag{6}$$

where $\kappa \coloneqq (n-1) - a(n+1)$, and $\nu \coloneqq 3(n-1) - 4a(n+1) - c(n+1)^2$. To find out why we express (a, c)-weighted Ricci curvature in the form (6), see [2, 20].

C. Robles investigated the Randers Einstein metrics in her Ph.D thesis and obtained the necessary and sufficient condition for the Randers metric to be Einstein [14] (see also [23]). B. Rezaei and others in 2007, obtained the necessary and sufficient condition for the Kropina, Matsumoto, and square metrics to be Einstein, when β is a constant Killing form [13]. In 2012 he proved that every n-dimensional ($n \geq 3$) Einstein Matsumoto metric is a Ricci-flat metric with vanishing S-curvature[12]. H. Zhu introduced the notion of quasi-Einstein Finsler metrics and characterized it. He also determined the quasi-Einstein square metrics [25]. In 2014, Shen and Yu classified the Einstein square metrics [19]. The natural question that arises is that, under what conditions is the Randers metric F defined by (h, W), a quasi-Einstein Finsler metric? By answering this question, we can establish the relationship between the curvatures of a Finsler metric F defined by (h, W), and another Finsler metric $\tilde{F}$, defined by (F, V).

Considering the *navigation data* (h, W), and assuming that $\|W\|_h < 1$; we define the Randers metric F as

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \tag{7}$$

where $W_i \coloneqq h_{ij}W^j$ and $\lambda \coloneqq 1 - \|W\|_h^2 > 0$.

Based on the above result, we first prove a characterization as follows in section 3:

Theorem 1.1. *Let $F = \alpha + \beta$ be the Randers metric on an n-dimensional manifold M, defined by the navigation data (h, W), according to (7); with dV as the volume form. Then F will be a quasi-Einstein Finsler metric; if and only if it satisfies the following conditions:*

i) $\mathcal{R}_{00} = -2\sigma h^2$,
ii) $\widetilde{\mathrm{Ric}}_{00} = -2\mathcal{A}\beta - (\alpha^2 + 3\beta^2)(\mathcal{B} - (n-1)c(x)) - 4\beta\mathcal{G}(\alpha^2 + \beta^2) - \mathcal{E} - 2k(x)h^2$,
iii) ${S^i}_{0|i} = \mathcal{A} + 2\beta(\mathcal{B} - (n-1)c(x)) + \mathcal{G}(\alpha^2 + 3\beta^2) - (\mathcal{C} + 4\beta\sigma(x))\lambda$,

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{E}$ are polynomials, and λ is a scalar function on M.

Z. Shen and Q. Xia in 2012, proved the relationship between non-Rimannian quantities, such as the isotropic S-curvature, and the weakly isotropic flag curvature of the Randers metrics F, with $F(x, -V_x) < 1$ and $\tilde{F}$; expressed by the navigation problem (F, V), where V is a conformal vector field on M [18]. The same result was proved for the Kropina metrics F, with $F(x, -V_x) \leq 1$ and $\tilde{F}$ [4]. In theorem 1.2; we prove a similar result for the quasi-Einstein Finsler metrics.

Theorem 1.2. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M with dimensions $n \geq 3$, and V be a conformal vector field on (M, F), with conformal factor $c(x)$. Consider $\tilde{F}$ to be a Randers metric defined from navigation data (F, V) by (7). Then if F is a quasi-Einstein Finsler metric with*

$$\mathrm{Ric}_\infty(x, y) = \mathrm{Ric}(x, y) + \dot{S}(x, y) = (n-1)cF^2(x, y),$$

then $\tilde{F}$ is also a quasi-Einstein Finsler metric with

$$\widetilde{\mathrm{Ric}_\infty}(x, u) = \widetilde{\mathrm{Ric}}(x, u) + \dot{\tilde{S}}(x, u) = (n-1)\tilde{c}\tilde{F}^2(x, u).$$

2. Preliminaries

Let (M, F) be a Finsler manifold. The non-negative function F on TM is a Finsler metric of M (or Finsler structure), if it satisfies three conditions: (i) regularity, (ii) positive 1-homogeneity, and (iii) strong convexity. The *Busemann-Hausdorff* measure on M, which is the most fundamental measure in Finsler geometry, is defined by [11]

$$\mathfrak{m}_{BH}(dx) := \Phi_{BH}(x)dx^1dx^2 \dots dx^n,$$

where the function Φ_{BH} is given by

$$\frac{Vol(\mathbb{B}^n(1))}{\Phi_{BH}(x)} = Vol\left(\left\{(y^i) \in \mathbb{R}^n \middle| F(x, y^i \frac{\partial}{\partial x^i}) < 1\right\}\right).$$

The quantity S, measures the distortions rate of change along the geodesics, where distortions $\tau(x, y)$ are defined as

$$\tau(x, y) := \ln \frac{\sqrt{\det g_{ij}(x, y)}}{\sigma(x)}.$$

The S-curvature and $\dot{S}$ are defined by

$$S(x, y) := \frac{d}{dt}\left[\tau(c(t), \dot{c}(t))\right]\Big|_{t=0} = \tau_{|i}(x, y)y^i,$$
$$\dot{S}(x, y) := \frac{d}{dt}\left[S(c(t), \dot{c}(t))\right]\Big|_{t=0} = S_{|i}(x, y)y^i,$$

where $c = c(x)$ is the geodesic with $c(0) = x$, and $\dot{c} = y$, and "$|$" denotes the horizontal covariant derivative with respect to F. A vector field G, induced by a Finsler metric F on TM_0, is given by [9]

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial x^i},$$

and is called the *spray* of F, and $G^i(x, y)$ are local functions on TM_0, satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, where $\lambda > 0$ is called the *spray coefficient* of F.
Consider F to be a Randers metric defined by (7) and let

$$\mathcal{R}_{ij} := \frac{1}{2}(V_{i|j} + V_{j|i}), \quad \mathcal{S}_{ij} := \frac{1}{2}(V_{i|j} - V_{j|i}),$$

$$\mathcal{R}_i = \mathcal{R}_{ij}V^j, \quad \mathcal{R} = \mathcal{R}_i V^i, \quad \mathcal{S}_i = \mathcal{S}_{ij}V^j, \quad \mathcal{S} = \mathcal{S}_j V^i = 0.$$

The spray coefficients of F can be expressed by [5]

$$G^i = G^i_h + \mathrm{I}^i,$$

where

$$\mathrm{I}^i = -F\mathcal{S}^i{}_0 - \frac{1}{2}F^2(\mathcal{R}^i + \mathcal{S}^i) + \frac{1}{2}\left\{\frac{y^i}{F} - V^i\right\}\Big(2F\mathcal{R}_0 - \mathcal{R}_{00} - F^2\mathcal{R}\Big).$$

Then $R^i{}_j = R^i{}_j(x,y)$ may be written as

$$R^i{}_j = \bar{R}^i{}_j + 2\mathrm{I}^i_{|j} - \mathrm{I}^i_{|m.j}y^m + 2\mathrm{I}^m\mathrm{I}^i_{.m.j} - \mathrm{I}^i_{.m}\mathrm{I}^m_{.j}, \tag{8}$$

where "|" and "." are the horizontal and vertical covariant derivatives with respect to h, respectively. Then $R \coloneqq R^i{}_j$ family is called the Riemann curvature [21, 24].

Let (M,F) be a Finsler manifold, and φ; a diffeomorphism on M. A vector field $V : M \to T_{\varphi(x)}M$ is called a conformal vector field, or an infinitesimal conformal transformation on manifold (M,F), with conformal factor $\rho = \rho(x)$ on M, if a 1-parameter infinitesimal generator group $\{\varphi_t\}$, generated by a vector field V, is a conformal transformation on manifold (M,F). This implies that $F\big(\varphi_t(x), (\varphi_t)_*(y)\big) = e^{2\rho(x)t}F(x,y)$. If ρ is constant, then the vector field V is called *homothetic*; and if ρ is zero, V is called *isometric*, or a *Killing* vector field.
Conformal vector fields have been investigated on Finsler manifolds with (α,β)-metric, and we have the following proposition for their special cases.

Proposition 2.1 ([8]). *Let $F = \alpha + \beta$ be a Randers metric on a manifold M, and (h,W) be its navigation representation. Then a vector field V on (M,F) is conformal; if and only if V satisfies the following system of PDEs:*

i) $V_{i|j} + V_{j|i} = 4\sigma h_{ij}$,
ii) $V^jW_{i|j} + W^jV_{j|i} = 2\sigma W_i$,

where we use h_{ij} to raise and lower the indices of V and W, and ";" is the covariant derivative with respect to the Levi-Civita connection of Riemannian metric h.

3. Proof of the Theorems

Proof of Theorem 1.1 Let $F = \alpha + \beta$ be a Randers metric defined by a vector field W, and a Riemannian metric h, on a manifold M, and consider $R^i{}_i$ to be a Ricci scalar of F. According to (8), we may write

$$R^i{}_i = \bar{R}^i{}_i + 2\mathrm{I}^i_{|i} - \mathrm{I}^i_{|m.i}y^m + 2\mathrm{I}^m\mathrm{I}^i_{.m.i} - \mathrm{I}^i_{.m}\mathrm{I}^m_{.i}, \tag{9}$$

where

$$\begin{aligned}
2\mathrm{I}^i_{|i} &= F^2\Big\{-\big(\mathcal{R}^i+\mathcal{S}^i\big)_{|i}+V^i_{|i}\mathcal{R}+V^i\mathcal{R}_{|i}\Big\}\\
&+F\Big\{-2\mathcal{S}^i{}_{0|i}-2F_{|i}\big(\mathcal{R}^i+\mathcal{S}^i\big)-\mathcal{R}_{|0}-4V^iV^i_{|i}\mathcal{R}_0\mathcal{R}_{0|i}+2V^iF_{|i}\mathcal{R}\Big\}\\
&+\Big\{2\mathcal{R}_{0|0}-2F_{|0}\mathcal{S}^i{}_0-F_{|0}\mathcal{R}+V^i_{|i}\mathcal{R}_{00}-2V^iF_{|i}\mathcal{R}_0+V^i\mathcal{R}_{00|i}\Big\}-\frac{1}{F}\{\mathcal{R}_{00|0}\}+\frac{1}{F^2}\{F_{|0}\mathcal{R}_{00}\},\\
&=A_1F+B_1F^2+E_1+C_1\frac{1}{F}+D_1\frac{1}{F^2},\\
-\mathrm{I}^i_{|m.i}y^m &= F\Big\{2F_{|0.i}\big(\mathcal{R}^i+\mathcal{S}^i\big)-2F_{.i}\big(\mathcal{R}^i+\mathcal{S}^i\big)_{|0}-\frac{(n+3)}{2}\mathcal{R}_{|0}-V^i_{|0}\mathcal{R}_0+V^i\big(F_{|0.i}\mathcal{R}+F_{.i}\mathcal{R}_{|0}\big)\Big\}\\
&+F^2\Big\{-\frac{1}{2}V^i_{|i}\mathcal{R}\Big\}+\frac{1}{F}\{\mathcal{R}_{00|0}\}-\frac{1}{F^2}\{(n+1)F_{|0}\mathcal{R}_{00}\}+E_2,\\
&=A_2F+B_2F^2+E_2+C_2\frac{1}{F}+D_2\frac{1}{F^2},\\
2\mathrm{I}^m\mathrm{I}^i_{.m.i} &= \Big\{-2\mathcal{S}^0{}_m-F^2\big(\mathcal{R}^m+\mathcal{S}^m\big)+y^m\Big(2\mathcal{R}_0-\frac{1}{F}\mathcal{R}_{00}-F\mathcal{R}\Big)-V^m\big(2F\mathcal{R}_0\mathcal{R}_{00}-F^2\mathcal{R}\big)\Big\}\\
&\times\Big\{-F_{.m.i}\mathcal{S}^i{}_0-2F_{.m}F_{.i}\big(\mathcal{R}^i+\mathcal{S}^i\big)-2FF_{.m.i}\big(\mathcal{R}^i+\mathcal{S}^i\big)\\
&+\delta^i{}_m\Big(\mathcal{R}_0+\frac{F_{.i}}{2F^2}\mathcal{R}_{00}+\frac{1}{F}\mathcal{R}_{0i}-\frac{1}{2}F_{.i}\mathcal{R}\Big)+y^i\Big(-\frac{F_{.i}F_{.m}}{F^3}\mathcal{R}_{00}+\frac{F_{.m}}{F^2}\mathcal{R}_{0i}+\frac{F_{.i}}{F^2}\mathcal{R}_{0m}-\frac{1}{F}\mathcal{R}_{im}\Big)\\
&+n\Big(\mathcal{R}_m+\frac{F_{.m}}{2F^2}\mathcal{R}_{00}-\frac{1}{F}\mathcal{R}_{0m}-\frac{F_{.m}}{2}\mathcal{R}\Big)-V^i\big(F_{.m.i}\mathcal{R}_0+F_{.m}\mathcal{R}_i-\mathcal{R}_{im}-F_{.i}F_{.m}\mathcal{R}-FF_{.m.i}\mathcal{R}\big)\Big\},\\
&=A_3F+B_3F^2+G_3F^3+E_3+C_3\frac{1}{F}+D_3\frac{1}{F^2}.
\end{aligned}$$

Similarly; we have

$$-\mathrm{I}^i_{.m}\mathrm{I}^m_{.i}=A_4F+B_4F^2+E_4+C_4\frac{1}{F}+D_4\frac{1}{F^2}.$$

On the other hand; for the Randers metric expressed by (h,W), the S-curvature is of the form

$$S=\frac{(n+1)}{2F}\left\{2F\mathcal{R}_0-\mathcal{R}_{00}-F^2\mathcal{R}\right\}.$$

Consequently; we have

$$\begin{aligned}
\dot{S}&=\frac{(n+1)}{2}\left\{2\mathcal{R}_{0|0}+\frac{F_{|0}}{F^2}\mathcal{R}_{00}-\frac{1}{F}\mathcal{R}_{00|0}-F_{|0}\mathcal{R}-F\mathcal{R}_{|0}\right\},\\
&=A_5F+B_5F^2+E_5+C_5\frac{1}{F}+D_5\frac{1}{F^2}.
\end{aligned}$$

Let F be a quasi-Einstein Finsler metric. Note that the Ricci curvature of F is related to the Ricci curvature ${}^{\alpha}\mathrm{Ric}$ of α by ${}^{\alpha}\mathrm{Ric}+\mathrm{I}^i{}_i$, [25]. Then

$$\begin{aligned}
0&=\mathrm{Ric}+\dot{S}-(n-1)c(x)F^2,\\
&={}^{\alpha}\mathrm{Ric}+\mathrm{I}^i{}_i+\dot{S}-(n-1)c(x)F^2,\\
&={}^{\alpha}\mathrm{Ric}+\mathcal{A}F+\mathcal{B}F^2+\mathcal{G}F^3+\mathcal{E}+\mathcal{C}\frac{1}{F}+\mathcal{D}\frac{1}{F^2}-(n-1)c(x)F^2,
\end{aligned}\tag{10}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{E}$ are polynomials in which

$$\mathcal{A}=\sum_1^5A_i,\quad \mathcal{B}=\sum_1^5B_i,\quad \mathcal{C}=\sum_1^5C_i,\quad \mathcal{D}=\sum_1^5D_i,\quad \mathcal{E}=\sum_1^5E_i.$$

Multiplying (10) by F^2, will yield

$$\mathrm{Rat}+\alpha\,\mathrm{Irrat}=0,$$

where Rat and Irrat given below are both polynomials of y:

$$\begin{aligned}\text{Rat} = &\Big\{5\beta\mathcal{G} + \big(\mathcal{B} - (n-1)c(x)\big)\Big\}\alpha^4 \\ &+ \Big\{\big({}^{\alpha}\text{Ric} + \mathcal{E}\big) + 3\beta\mathcal{A} + 6\beta^2\big(\mathcal{B} - (n-1)c(x)\big) + 10\beta^3\mathcal{G}\Big\}\alpha^2 \\ &+ \Big\{\beta\mathcal{C} + \beta^2\big({}^{\alpha}\text{Ric} + \mathcal{E}\big) + \beta^3\mathcal{A} + \beta^4\big(\mathcal{B} - (n-1)c(x)\big) + \beta^5\mathcal{G} + \mathcal{D}\Big\},\end{aligned} \tag{11}$$

$$\begin{aligned}\text{Irrat} = &\{\mathcal{G}\}\alpha^4 \\ &+ \Big\{\mathcal{E} + \mathcal{A} + 4\beta\big(\mathcal{B} - (n-1)c(x)\big) + 10\beta^2\mathcal{G}\Big\}\alpha^2 \\ &+ \Big\{2\beta\big({}^{\alpha}\text{Ric} + \mathcal{E}\big) + 3\beta^2\mathcal{A} + 4\beta^3\big(\mathcal{B} - (n-1)c(x)\big) + 5\beta^4\mathcal{G} + \mathcal{C}\Big\}.\end{aligned} \tag{12}$$

The necessary and sufficient condition for F to be a quasi-Einstein Finsler metric, is that Rat $= 0$ and Irrat $= 0$. A similar proof can be found in [14], and we omit it. Now;

$$\begin{aligned}0 &= \text{Rat} - \beta\,\text{Irrat}, \\ &= \big(\alpha^2 - \beta^2\big)\Big\{{}^{\alpha}\text{Ric}_{00} + \mathcal{E} - 2\beta\mathcal{A} + 4\beta\big(\alpha^2 + \beta^2\big)\mathcal{G} + \big(\alpha^2 + 3\beta^2\big)\big(\mathcal{B} - (n-1)c(x)\big)\Big\} + \mathcal{D},\end{aligned} \tag{13}$$

in which

$$\mathcal{D} = \left\{\frac{(3-n)}{2}F_{|0} - \left(\frac{3}{2F^2} + \frac{n}{2F^2} - \frac{nV^m F_{.m}}{2F^2}\right)\mathcal{R}_{00}\right\}\mathcal{R}_{00}. \tag{14}$$

It is clear that $\big(\alpha^2 - \beta^2\big)$ divides $\mathcal{D}$, where $\mathcal{D} = \kappa(x)\mathcal{R}_{00}$, and we have

$$\mathcal{R}_{00} = \sigma(x)\big(\alpha^2 - \beta^2\big) = -2\zeta(x)h^2, \tag{15}$$

which proves the first part. We also have

$$\begin{aligned}\mathcal{R}_{ij} &= -2ch_{ij}, \qquad \mathcal{R}_{ij|0} = -2c_{|0}h_{ij}, \\ \mathcal{R} &= -2c\|V\|_h^2, \qquad \mathcal{R}_{|0} = -2\big(c_0\|V\|_h^2 + 2c(\mathcal{R}_0 + \mathcal{S}_0)\big).\end{aligned}$$

For the second part; inserting (15) into (13), and dividing by $\big(\alpha^2 - \beta^2\big)$, we get

$${}^{\alpha}\text{Ric}_{00} = -\mathcal{E} + 2\beta\mathcal{A} - 4\beta\big(\alpha^2 + \beta^2\big)\mathcal{G} - \big(\alpha^2 + 3\beta^2\big)\big(\mathcal{B} - (n-1)c(x)\big). \tag{16}$$

Returning to the expression Irrat $= 0$; replacing ${}^{\alpha}\text{Ric}_{00}$, and wherever possible, using $\mathcal{R}$ and its derivatives, as stated above; and also making use of

$$\begin{aligned}F_{|k} &= \frac{2cF(y_k - FV_k) + F(F\mathcal{S}_k + \mathcal{S}_{k0})}{M}, \\ F_{|0} &= 2cF^2 + \frac{F^2}{M}\mathcal{S}_0, \\ F_{.m|0} &= \left(\frac{h^2}{M^3}\mathcal{S}_0 + 2c\frac{F}{M}\right)y_k - FV_k - \frac{F^2}{M^2}\mathcal{S}_0 V_k - \frac{F}{A}\mathcal{S}_{k0},\end{aligned}$$

where $M = \sqrt{\lambda h^2 + V_0^2}$; we arrive at the following formula:

$$\mathcal{S}^i{}_{0|i} = \mathcal{A} + 2\beta\big(\mathcal{B} - (n-1)c(x)\big) + \mathcal{G}\big(\alpha^2 + 3\beta^2\big) - \big(\mathcal{C} + 4\beta\sigma(x)\big)\lambda, \tag{17}$$

which concludes our proof. □

To prove theorem 1.2, we need the following two propositions:

Proposition 3.1 ([18]). *Let $F(x,y)$ be a Finsler metric on a manifold M, and V_1 be a vector field on M with $F(x,-V_1) \leq 1$. Suppose $\widetilde{F}(x,y)$ is a Finsler metric defined from the navigation data (F,V_1) by (7); and V_2 is a vector field on M with $\tilde{F}(x,-V_2) \leq 1$. Then the Finsler metric $\tilde{\tilde{F}}(x,y)$ defined from $(\tilde{F},V)$ by (7), satisfies the identity*

$$\tilde{\tilde{F}}(x,u) = F\big(x, u - \tilde{\tilde{F}}(x,u)(V_1 + V_2)\big), \tag{18}$$

where $y = u - \widetilde{F}(x,y)V$.

For the special case of the above proposition, we have the following:

Proposition 3.2 ([20]). *Let F be a Randers metric on an n-dimensional manifold M, defined by the navigation data (h, W). F is a weakly (a, b)-Ricci weighted Einstein, satisfying*

$$\mathrm{Ric}_\infty = (n-1)\left(\frac{3\theta}{F} + \sigma\right) F^2,$$

with respect to some volume form dV; if and only if h is a Ricci almost gradient soliton, satisfying ${}^h\mathrm{Ric} + \mathrm{Hess}_h f = (n-1)\mu h^2$, and F is of isotropic S-curvature $\mathcal{R}_{00} = -2ch^2$, for some scalar function c. In this case, $dV = e^{-f} dV_{BH}$, and we have

$$\sigma = \mu - c^2 - 2c_i W^i + \frac{1}{n-1}\Big\{ - f_{i;j} W^i W^j + f_i \mathcal{S}^i \Big\},$$
$$\theta_i = \frac{1}{3(n-1)}\Big\{ (2n-1)c_i + f_{i;j} W^j + f_j \mathcal{S}^j{}_i - c f_i \Big\}.$$

For the special case of $\theta = 0$, we have the following corollary for the quasi-Einstein Finsler metric:

Corollary 3.3. *Let F be a Randers metric on an n-dimensional manifold M, defined by the navigation data (h, W). F is a weakly weighted Einstein, satisfying*

$$\mathrm{Ric}_\infty = (n-1)\sigma F^2, \tag{19}$$

with respect to some volume form dV; if and only if h is a Ricci almost gradient soliton, satisfying ${}^h\mathrm{Ric} + \mathrm{Hess}_h f = (n-1)\mu h^2$, and F is of isotropic S-curvature $\mathcal{R}_{00} = -2ch^2$, for some scalar function c. In this case, $dV = e^{-f} dV_{BH}$, and we have

$$\zeta = \mu - \sigma^2 - 2c_i W^i + \frac{1}{n-1}\Big\{ - f_{i;j} W^i W^j + f_i \mathcal{S}^i \Big\}.$$

Proof of Theorem 1.2 By assumption, V is a conformal vector field on F, with conformal factor $c(x)$. Then, according to the proposition in [18], V must be a conformal vector field of h, with conformal factor $c(x)$. Since F is a quasi-Einstein Finsler metric with $\mathrm{Ric} + \dot{S} = (n-1)\sigma(x)F^2$, according to theorem 1.1, F is of isotropic S-curvature $\sigma(x)$. It follows from corollary 3.3 that h is a Ricci almost gradient soliton with ${}^h\mathrm{Ric} + \mathrm{Hess}_h f = (n-1)\mu h^2$, and $\zeta = \mu - \sigma^2 - 2\sigma_i W^i + \frac{1}{n-1}\left\{-f_{i;j}W^iW^j + f_i\mathcal{S}^i\right\}$. On the other hand; according to the main theorem in [18], $\tilde{F}$ is of isotropic S-curvature, and by proposition 3.1; $\tilde{F}$ defined from (F, V) by (7), satisfies

$$\tilde{F}(x, u) = h\big(x, u - \tilde{F}(x, u)(V + W)\big). \tag{20}$$

Consequently; $\tilde{F}$ can be regarded as a Finsler metric, generated from $(h, V + W)$ by (7). Thus; $(V + W)$ is also a conformal vector field of h, with conformal factor $(\sigma - c)$. Then by corollary 3.3; $\tilde{F}(x, u)$ is a quasi-Einstein Finsler metric, given by $\widetilde{\mathrm{Ric}}(x, u) + \tilde{\dot{S}}(x, u) = (n-1)(c-\sigma)\tilde{F}(x, u)$, and we have

$$\tilde{\zeta} = \zeta - c(c - 2\sigma) + 2(c_i - \sigma_i)V^i + 2c_i W^i - \frac{1}{n-1}\Big\{ - f_{i;j}\big(2W^iV^j + V^iV^j\big) + f_i\mathcal{S}^i \Big\},$$

hereby, completing our proof. ☐

Proposition 3.4 ([20]). *Let F be a Randers metric on an n-dimensional manifold M, defined by the navigation data (h, W); and it is assumed that $\nu \neq 0$. F is a weakly (a, b)-weighted Einstein, satisfying*

$$R_{(a,b)} = (n-1)\left(\frac{3\theta}{F} + \sigma\right) F^2,$$

with respect to a volume form $dV = e^{-f} dV_{BH}$; if and only if h is (a, b)- weighted Einstein, satisfying ${}^h\mathrm{Ric} + a\,\mathrm{Hess}_h f - b(df \otimes df) = (n-1)\mu h^2$ with respect to $dV = e^{(-f)} dV_h$, and W satisfies $W_{i|j} + W_{j|i} = -4ch_{ij}$ for some scalar functions f, c, and μ, on M. In this case; we have

$$\sigma = \mu - c^2 - 2c_i W^i + \frac{1}{n-1}\Big\{ - a f_{i;j} W^i W^j + a f_i \mathcal{S}^i - bc^2\big(n+1\big)^2 + b f_i f_j W^i W^j \Big\},$$
$$\theta_i = \frac{1}{3(n-1)}\Big\{ \big[3(n-1) + a(n+1)\big]c_i + 2a f_{i;j} W^j + 2a f_j \mathcal{S}^j{}_i - 2c f_i\big[a + (n+1)b\big] - 2b f_i f_j W^j \Big\}.$$

Corollary 3.5. *Let $F=\alpha+\beta$ be a Randers metric on a manifold M with dimensions $n\geq 3$, and V be a conformal vector field on (M,F), with conformal factor $c(x)$. Let $\tilde{F}$ be a Randers metric defined from navigation data (F,V) by (7). If F is a weakly (a,b)-weighted Einstein metric with*

$$\mathrm{Ric}_{(a,b)}(x,y)=(n-1)\left(\frac{3\theta}{F(x,y)}+\sigma\right)\tilde{F}^2(x,y),$$

then $\tilde{F}$ is also a weakly (a,b)-weighted Einstein metric with

$$\widetilde{\mathrm{Ric}}_{(a,b)}(x,u)=(n-1)\left(\frac{3\tilde{\theta}}{\tilde{F}(x,u)}+\tilde{\sigma}\right)\tilde{F}^2(x,u),$$

where

$$\begin{aligned}\tilde{\theta}&:=(\theta_i-c_i)u^i,\\ \tilde{\zeta}&:=\zeta-c^2+2c_iV^i,\\ u&:=y+F(x,y)V=y+\tilde{F}(x,u)V,\end{aligned}$$

in which, θ and $\tilde{\theta}$ are 1-forms on M, and σ and $\tilde{\sigma}$ are scalar functions on M.

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