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Original Article

The competition of robust methods in linear and nonlinear regression based on trimming methods and the existence moments of order statistics for heavy-tailed stable errors

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ABSTRACT: Robust regression methods including, least trimmed squares, are among the most important methodologies for computing exact coefficient estimators when data is polluted with outliers. There is interest in generalizing least trimmed squares for regression models with heavy-tailed stable errors. This manuscript, compares estimating coefficients methods with the robust least trimmed squares method in stable errors case. Therefore, we propose stable least trimmed squares and nonlinear stable least trimmed squares methods for linear/nonlinear regression models with stable errors, respectively. The joint distribution of ordered errors is used with the finite variance property of ordered stable errors, whose indexes are defined by cut-off points (Subsection 3.1). We make many comparisons using simulated and real datasets.

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1. Introduction

Least Trimmed Squares (LTS) is the most widespread strategy in robust linear regression and is basic for understanding regression issues containing outlier values. The robust LTS strategies use the Ordinary Least Squares (OLS) fit for linear regression models whose errors are Gaussian. Hence, amplifying the robust LTS strategies to find robust estimations of linear regression coefficients with heavy-tailed stable errors will be imperative.

Heavy-tailed stable data, [20], have a great interest field, [11] and [1]. The reality that stable distributions are upheld by the generalized central limit theorem makes them the most important among other heavy-tailed distributions. Stable distributions have seen wide interest, which energized to spread of the regression models with stable errors. Stable data is characterized by infinite moments, so it would be interesting to search for robust LTS solutions of the regression coefficients.

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In this paper, we are interested in robust LTS estimates of linear and nonlinear regression coefficients when errors are stable. Here, the performance of classic LTS based on OLS fit can be weak. Therefore, robust strategies should be used based on Maximum Likelihood Estimator (MLE) to generalize robust LTS for heavy-tailed error regression.

1.1. Outline

In Section 2, we defined the breakdown point then we mentioned different formulas for its computation using trimming proportion. We defined the robust LTS method and provided a detailed overview of its development and contribution to compute optimal robust estimates for linear regression coefficients. We also explain the procedures for selecting sub-datasets and recall the algorithm of the fast approximate LTS method. Likewise, we mentioned the robust Nonlinear Least Trimmed Squares (NLTS) method in nonlinear regression based on Nonlinear Least Squares (NLS), [4] and recall the NLTS algorithm.

Section 3 defined stable distributions by their characteristic function and mentioned the property of ordered stable errors with finite moments. The MLE method is presented to estimate linear regression coefficients using the probability density function of stable distributions. Finally, methods for computing MLE for error distribution parameters based on stable errors with finite variance and regression residuals are presented.

In Section 4, stable linear regression models are defined, and a brief overview of estimating their coefficients provided. We focus attention on the most important methods, such as the [21] (NOR) and the TLS, [22] methods. Section 5 presented the method proposed by [2] as the robust Stable Least Trimmed Squares (SLTS) method in the linear case and the robust Nonlinear Stable Least Trimmed Squares (NSLTS) method in the nonlinear case. The breakdown point for linear and nonlinear cases was defined, the differences between them were determined, and the algorithms were presented. Finally, examples of simulated and real datasets are provided in Section 6. The conclusion of the manuscript is in Section 7.

1.2. Preliminaries

Consider the multiple linear regression model:

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\theta} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\boldsymbol{Y} = (y_i)_{1\le i\le n}$ is a response variable and, $\boldsymbol{X} = (x_{ij})_{\substack{1\le i\le n\\ 1\le j\le p}}$ is a design matrix with $x_{i,1} = 1$ for all i. Components of error vector $\boldsymbol{\varepsilon} = (\varepsilon_i)_{1\le i\le n}$ are *i.i.d.* Gaussian distribution with zero expectation. The coefficient vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_p)^\top$ to be estimated.

Consider the multiple nonlinear regression model (2):

$$\boldsymbol{Y} = f(\boldsymbol{X}, \boldsymbol{\theta}) + \boldsymbol{\varepsilon}, \tag{2}$$

where $\boldsymbol{X} = (x_{i,j})_{\substack{1\le i\le n\\ 1\le j\le k}}$ is design matrix, $f(\boldsymbol{X}, \boldsymbol{\theta})$ is a nonlinear regression function, and $\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{\varepsilon}$ are defined as above.

Using the estimated regression coefficients, $\widehat{\boldsymbol{\theta}}$, we can compute the predicted response variable $\widehat{\boldsymbol{Y}} = \boldsymbol{X}\widehat{\boldsymbol{\theta}}$ and define the linear regression residuals ($\boldsymbol{e}$) as follows:

$$\boldsymbol{e} := \boldsymbol{Y} - \widehat{\boldsymbol{Y}}.$$

The residuals $\boldsymbol{e} = (e_1, \dots, e_n)^\top$ are key quantities in calculating robust linear regression estimates and a basis for finding the optimal solutions.

2. Robust Least Trimmed Squares (LTS)

Robust regression could be a wide extent of coefficient estimation strategy that overcome a blend of complexities forced by the classical strategies and discover the optimal solutions. The rise of robust regression strategies has profited from recognizing outlier values and achieving broad results by finding robust solutions to many complex problems.

Let $\mathcal{Z}_{n'} = \{(\boldsymbol{Y}, \boldsymbol{X})\} = \{(y_i, x_{i,1}, x_{i,2}, \dots, x_{i,p})\}$, $i = 1, \dots, n'$ is the original dataset, $\mathcal{Z}_m = \{(\boldsymbol{Y}, \boldsymbol{X})\} = \{(y_i, x_{i,1}, x_{i,2}, \dots, x_{i,p})\}$, $i = n'+1, \dots, n'+m$ is the all possible corrupted samples that are obtained by any m of arbitrary values (outliers), and consider $\mathcal{Z}_n = \{(\boldsymbol{Y}, \boldsymbol{X})\} = \{(y_i, x_{i,1}, x_{i,2}, \dots, x_{i,p})\}$, $i = 1, \dots, n', n'+1, \dots, n'+m = n$ is the dataset polluted with m outlier, where $(x_{i,1}, x_{i,2}, \dots, x_{i,p})$ is the ith row in the design matrix $\boldsymbol{X}$ and (y_i) is the ith row in the response variable $\boldsymbol{Y}$. Applying some regression method to

$\mathcal{Z}_{n'}$ gives estimated coefficients $\widehat{\boldsymbol{\theta}}_{n'} = (\hat{\theta'}_1, \ldots, \hat{\theta'}_p)^\top$ and to $\mathcal{Z}_n$ gives $\widehat{\boldsymbol{\theta}}_n = (\hat{\theta}_1, \ldots, \hat{\theta}_p)^\top$. Therefore, the breakdown point $\mathcal{B}^*_n(\widehat{\boldsymbol{\theta}}_n, \mathcal{Z}_n)$ can be defined as follows:

$$\mathcal{B}^*_n(\widehat{\boldsymbol{\theta}}_n, \mathcal{Z}_n) = \min_{0\leq m<n} \left\{ \frac{m}{n};\ \sup_{\mathcal{Z}_m} ||\widehat{\boldsymbol{\theta}}_{n'} - \widehat{\boldsymbol{\theta}}_n|| = \infty \right\}, \tag{3}$$

where $||\cdot|| = ||\cdot||_p$ is the L^p norm. Therefore, as the outlier values increase, we say that the estimate collapses. Consequently, the breakdown point prevents the estimate from collapsing and leads to determining the optimal solution corresponding to the minimum sum of the ordered squared residuals.

[13] provided the first definition of the breakdown point, which has been developed by [14]. [8] believed that the definition of breakdown point must be rearranged and reflect the behavior of the measurable strategies for the bounded sample. [6] expanded this concept to other cases, such as deriving an upper bound for the breakdown points of equivalence statistical functional and proving that the connection between breakdown and equivalence is fragile. The positive breakdown points strategies in the linear regression proposed by [26] are the most robust linear regression procedures and have different applications, [17]. The most important strategy is the robust LTS presented by [25].

The computation of the robust LTS estimator depends on detecting the breakdown point h, setting all sub-datasets, and performing OLS fit for all sub-datasets. The final robust LTS solution has the minimum sum of ordered squared residuals. The breakdown point is communicated in a few formulas such as $h = [n(1-\phi)]$, [28], where ϕ is the trimming proportion. The breakdown point $h = [n/2] + [(p+1)/2]$ is the hieghest breakdown point in linear regression, [28], page 132, where p is the number of coefficients, and n is the sample size. They assigned sub-datasets in two ways; $h-$subset, [24], and $p-$subset, [30] pages 185-194. The $p-$subsets strategy gives exact and solid results, particularly with datasets polluted with outlier values. The estimator $\widehat{\boldsymbol{\theta}}_{\text{LTS}}$ based on $p-$subsets exists and can be obtained analytically. The long mathematical operations within the classic strategy are driven by the robust FAST-LTS strategy proposed by [27]. They consider $p-$subsets as the essential strategy for computing the estimator of robust LTS, whether the data is polluted with outliers or not. Quantile regression is considered a robust strategy that is not affected by outliers, making it comparable with robust regression strategies, [15]. In this paper, we denoted quantile regression with $RQ(\cdot)$. For example, the $RQ(0.5)$ refers to median regression.

2.1. Linear Computation

To compute $\widehat{\boldsymbol{\theta}}_{\text{LTS}}$ of coefficients model (1) based on breakdown point $h = [n/2] + [(p+1)/2]$, it is necessary to recall the C$-$Steps Algorithm 1, convergence, and Algorithm 2 proposed by [27].

Algorithm 1: C-Steps for robust FAST-LTS in linear regression models

1. Take $J = 1$ and an arbitrary $H_J \subset \{1, \ldots, n\}$, called $H_J = \{J_1, \ldots, J_h\}$ with $\#H_J = h$.
2. Compute OLS fit using data with index in H_J.
3. Compute $S_J = \sum_{i=1}^{h} e^2_{J_i}$ using the residuals $\boldsymbol{e} = (e_1, e_2, \ldots, e_n)^\top$.
4. Sort the absolute values of residuals $|e|_{\pi(1)} \leq \cdots \leq |e|_{\pi(n)}$, where π is the permutation that gives their ordered indexes, i.e., $\pi(i)$ is the index of ith absolute ordered residuals.
5. Put $J = J+1$, $H_J = \{J_1 = \pi(1), J_2 = \pi(2), \ldots, J_h = \pi(h)\}$. Go to step 2.

Convergence

The C$-$Steps loop has iterative steps for H_1. It gives many sets $H_2, H_3, \ldots$ and the sums of the squared residuals corresponding to them $S_1, S_2, S_3, \ldots$, where $S_1 \geq S_2 \geq S_3 \geq \cdots$. If $S_m = S_{m-1}$, we say the convergence has occurred for H_1 and the solution here is H_m, $m \geq 2$.

2.2. Nonlinear Computation

For nonlinear regression model, many discuss Robust Nonlinear Least Trimmed Squares (NLTS) with breakdown points in different ways. For example, the unbounded breakdown point and its upper and lower bounds were determined by [31]. Also, the behavior of the robust NLTS under mixing conditions was discussed by [33] and others. The NLS has several pillars: a nonlinear model, initial values, and convergence. So, the ϕ could be at most 20% for nonlinear regression, [23]. To compute $\hat{\boldsymbol{\theta}}_{\text{NLTS}}$ of nonlinear regression coefficients (2) based on a dataset of size n and given trimming proportion ϕ, perform the following steps in Algorithm 3.

Algorithm 2: Robust FAST-LTS for linear regression models with Gaussian error

1: For all set $k_l, l = \{1, 2, \dots, \binom{n}{p}\}$, and coefficients number p, repeat:

  I. Draw the set k_l, $k_l \subset \{1, \dots, n\}$ with $\#k_l = p + 1$.
  II. Compute $\widehat{\boldsymbol{\theta}}_{\text{OLS}}$ using k_l, and compute the residuals $(e_1, \dots, e_n)^\top$.
  III. Sort $|e|_{\pi(1)} \leq \cdots \leq |e|_{\pi(n)}$ and put $H_1 = \{\pi(1), \dots, \pi(h)\}$.
  IV. Perform twice C−Steps based on H_1 yield to $S_2^{k_l}$.

2: For the 10 results with the lowest $S_2^{k_l}$, perform C−Steps until convergence.
3: Report $\widehat{\boldsymbol{\theta}}_{\text{LTS}}$ with the minimum S_J^* and get the optimal set of indexes data H_J^*.

Algorithm 3: Robust NLTS of nonlinear regression coefficients with Gaussian error

1: Compute the breakdown point $h = [n(1 - \phi)]$, where ϕ is the trimming proportion.
2: Find all sub-datasets of size h and perform NLS fits of all sub-datasets.
3: Derive the optimal solution $\hat{\boldsymbol{\theta}}_{\text{NLTS}}$ with the minimum sum of the ordered squared residuals.
4: Deduce H^* with $\#H^* = h$ contains the indexes that give the optimal solution.

3. Stable Distribution and their Order Statistics

Stable distributions are a four parameter family: tail index $\alpha \in (0, 2]$, skewness $\beta \in [-1, 1]$, scale $\gamma > 0$, and location $\delta \in (-\infty, \infty)$ denoted by $\mathbb{S}(\alpha, \beta, \gamma, \delta)$. The standard stable distribution has zero location and unit scale, i.e., $\varepsilon \sim \mathbb{S}(\alpha, \beta, 1, 0)$ or simply $\mathbb{S}(\alpha, \beta)$. If ε has a stable distribution, $\mathbb{S}(\alpha, \beta, \gamma, \delta)$, then its characteristic function can be described in (4) and its moments including variance does not exist.

$$\varphi_\varepsilon(t) = E(\exp(it\varepsilon)) = \begin{cases} \exp\left\{-\gamma^\alpha |t|^\alpha \left[1 - i\beta \left(\tan \frac{\pi\alpha}{2}\right)(\operatorname{sign} t)\right] + i\delta t\right\}, & \alpha \neq 1, \\ \exp\left\{-\gamma |t| \left[1 + i\beta \frac{2}{\pi} (\operatorname{sign} t) \log |t|\right] + i\delta t\right\}, & \alpha = 1. \end{cases} \tag{4}$$

There is an interest in estimating the four parameters of a stable distribution, generating stable random variables, and truncated random variables. For example, simulating truncated stable random variables using a nonlinear transformation by [29] and truncated stable random variables on the interval (a, b) for any tail index in $(0, 2)$ and skewness $-1 \leq \beta \leq 1$ by [32].

3.1. Finite Moments of Trimmed Ordered Stable Variable

Let $X_1, \dots, X_n$ be a random sample from $\mathbb{S}(\alpha, \beta, 1, 0)$ and $X_{(1)}, \dots, X_{(n)}$ be its corresponding order statistics. According to Theorem 1 in [18] variance or second moment of the kth ordered stable variable exists, $E(\varepsilon_{(k)}^2) < \infty$, if and only if $\left[\frac{2}{\alpha}\right] + 1 \leq k \leq \left[n + 1 - \frac{2}{\alpha}\right] - 1$, where $[x]$ denotes the integer part of x. The numbers $c = \left[\frac{2}{\alpha}\right] + 1$ and $d = \left[n + 1 - \frac{2}{\alpha}\right] - 1$ are called cut−off points.

The most important uses of finite variance for ordered stable variables is computing regression coefficients in regression models with stable errors. The model (1) can be written with matrix forms as:

$$\overset{\boldsymbol{Y}}{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}} = \begin{pmatrix} \overset{\boldsymbol{X}_1}{1} & \overset{\boldsymbol{X}_2}{x_{1,2}} & \cdots & \overset{\boldsymbol{X}_p}{x_{1,p}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n,2} & \cdots & x_{n,p} \end{pmatrix} \overset{\boldsymbol{\theta}}{\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_p \end{pmatrix}} + \overset{\boldsymbol{\varepsilon}}{\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}},$$

rewrite model (1) by re-ordering $\boldsymbol{X}$ and $\boldsymbol{Y}$ based on ordered residuals $(e_{(1)}, \dots, e_{(n)})$. Trim $\boldsymbol{X}$ and $\boldsymbol{Y}$ by cut−off points c, and d, and define a new model in (5). This model has finite variance,

$$\overset{\boldsymbol{Y}^{\text{Trim}}}{\begin{pmatrix} y_{[c]} \\ \vdots \\ y_{[d]} \end{pmatrix}} = \begin{pmatrix} \overset{\boldsymbol{X}_1^{\text{Trim}}}{1} & \overset{\boldsymbol{X}_2^{\text{Trim}}}{x_{[c,2]}} & \cdots & \overset{\boldsymbol{X}_p^{\text{Trim}}}{x_{[c,p]}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{[d,2]} & \cdots & x_{[d,p]} \end{pmatrix} \overset{\boldsymbol{\theta}}{\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_p \end{pmatrix}} + \overset{\boldsymbol{\varepsilon}^{\text{Trim}}}{\begin{pmatrix} \varepsilon_{(c)} \\ \vdots \\ \varepsilon_{(d)} \end{pmatrix}}, \tag{5}$$

where $(y_{[c]},\ldots,y_{[d]})^\top$, $(x_{[c,2]},\ldots,x_{[d,2]})^\top,\ldots,$ and $(x_{[c,p]},\ldots,x_{[d,p]})^\top$ are trimmed vectors correspond to the trimmed order statistics of residuals $(e_{(c)},\ldots,e_{(d)})^\top$. Then compute the MLE of coefficients using (6), [22]:

$$\ell(\boldsymbol{\theta}) = \sum_{i=c}^{d} \log f_{\varepsilon_{(i)}}\left(y_{[i]} - \left(\theta_1 + \theta_2 x_{[i,2]} + \cdots + \theta_p x_{[i,p]}\right)\right). \tag{6}$$

For nonlinear regression model (2), the coefficients can be estimated using (7).

$$\ell(\boldsymbol{\theta}) = \sum_{i=c}^{d} \log f_{\varepsilon_{(i)}}\left(y_{[i]} - g(x_{[i],j},\boldsymbol{\theta})\right). \tag{7}$$

The density function of order statistic $f_{\varepsilon_{(i)}}(t)$ defined in (8).

$$f_{\varepsilon_{(i)}}(t) = \frac{n!}{(i-1)!(n-i)!}\left(F_\varepsilon(t)\right)^{i-1}\left(1-F_\varepsilon(t)\right)^{n-i} f_\varepsilon(t), \tag{8}$$

where $f_\varepsilon(\cdot)$ and $F_\varepsilon(\cdot)$ are probability density and cumulative distribution functions of $\mathbb{S}(\alpha,\beta)$, respectively.

3.2. Tail index and Skewness Estimates Using MLE & MLEO

The MLE approach is a usual method for estimating stable distribution parameters. Let $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ be random variables from $\mathbb{S}(\alpha,\beta,\gamma,\delta)$ with observed values $t_1,\ldots,t_n$. The MLE of $\alpha,\beta,\gamma,\delta$ called $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}$ can be computed by numerical solution of (9), [19]:

$$\text{MLE: } \ell(\alpha,\beta,\gamma,\delta) = \sum_{i=1}^{n} \log\ f_\varepsilon(t_i|\alpha,\beta,\gamma,\delta). \tag{9}$$

MLE solutions are suitable and logical in most cases. In the absence of variance of stable variables, the MLE is not robust and becomes less effective. Therefore, a new estimator, Maximum Likelihood Estimator Order Statistics (MLEO), is formulated, which is implemented using Algorithm 4.

Algorithm 4: MLEO of stable parameters based on order statistics

1: Let $\varepsilon_1,\ldots,\varepsilon_n$ be a random sample from $\mathbb{S}(\alpha,\beta,1,0)$ with observed values $t_1,\ldots,t_n$.
2: Compute MLE of $\hat{\alpha}$ and $\hat{\beta}$ using all observed values $t_1,\ldots,t_n$ and (9).
3: Define the order statistics $\varepsilon_{(1)},\ldots,\varepsilon_{(n)}$ corresponding to $\varepsilon_1,\ldots,\varepsilon_n$.
4: Compute cut-off points $c = \left[\frac{2}{\hat{\alpha}}\right]+1$ and $d = \left[n+1-\frac{2}{\hat{\alpha}}\right]-1$.
5: Re-estimate parameters $\hat{\alpha},\hat{\beta}$ through the numerical solution of (10).

$$\text{MLEO: } \ell(\alpha,\beta,1,0) = \sum_{i=c}^{d} \log\ f_{\varepsilon_{(i)}}(t_i|\alpha,\beta,1,0). \tag{10}$$

The probability density function of the order statistics $f_{\varepsilon_{(i)}}$ is given in (8). In fact, MLEO estimates have been used by [22] to compute the MLE of regression coefficients, and here, we used them to estimate the parameters of stable distributions in different cases. It is interesting to find the superiority between MLE and MLEO, as in Figure 1.

4. Stable Linear Regression methods

We will be interested in the robust LTS strategy with stable linear regression models, i.e., linear regression models with stable errors. Stable linear regression has a long history, and numerous dialogues around coefficient estimation were displayed. [3] measure the linear coefficients of stable regression with infinite variance for $\alpha > 1$. They have included the MSAE (Mean Square Absolute Errors) estimator for a regression model with stable errors. They demonstrate that the MSAE overcomes OLS for tail index $1 < \alpha < 1.5$, and this predominance will vault with numerous unstable distributions. [9] extend the [3] method to $\alpha \in (0,2)$ through a ranked set sampling design. [10] have applied the regression with stable errors in the presence of some restrictions on the tail index, scale, and skewness parameters. The MLE method was used to estimate coefficients with symmetric stable errors by [16]. [21] clear the way for using robust strategies. They displayed Algorithm 5 to compute MLE of coefficients in linear and nonlinear stable regression models. Consider in the linear regression model (1) the errors $\varepsilon_1,\ldots,\varepsilon_n$ are *i.i.d.*

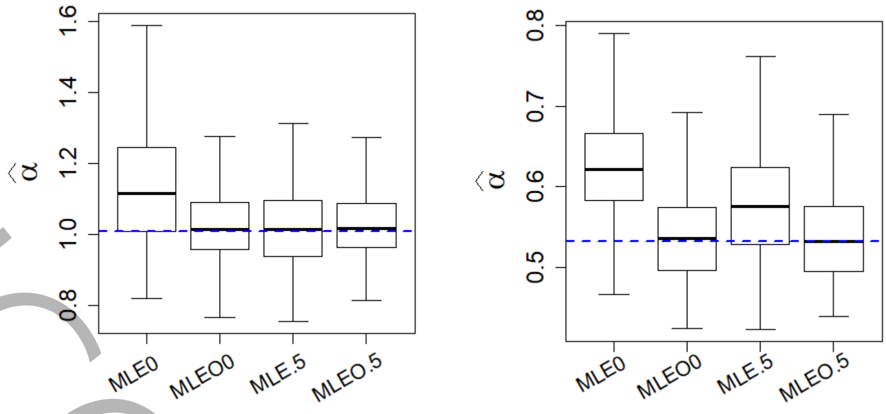

Figure 1: Comparing unbiasedness and dispersion of tail index estimated using MLE and MLEO for $\alpha = 1$ (left) and $\alpha = 0.55$ (right). The blue line is the exact value. The MLE0 and MLEO0 are the symmetric solutions, $\beta = 0$. The MLE.5 and MLEO.5 are the asymmetric solutions, $\beta = 0.5$.

standard stable distribution, at that assumption compute estimated coefficients using numerical MLE (11), [21] as follows:

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f_\varepsilon \left(y_i - (\theta_1 + \theta_2 x_{i,2} + \cdots + \theta_p x_{i,p})\right). \tag{11}$$

Also, with similarity for nonlinear model (2), the coefficients estimated using (12), [21].

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f_\varepsilon \left(y_i - g(x_{i,j}, \boldsymbol{\theta})\right), \tag{12}$$

where $f_\varepsilon(\cdot)$ is the probability density function of the standard stable distribution.

Algorithm 5: NOR estimation method of stable linear and nonlinear regression coefficients

For stable linear regression:

1: Find $\hat{\boldsymbol{\theta}} :=$ OLS fit using all data.
2: Compute residuals corresponding to $\widehat{\boldsymbol{\theta}}$.
3: Reorder the indexes of the data based on ordered residuals.
4: Trim a 10% and 90% of reordered data.
5: Perform the second OLS fit to update $\hat{\boldsymbol{\theta}}$ using trimmed dataset.
6: Using updated residuals, compute MLE of $\widehat{\boldsymbol{\phi}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$.
7: Update MLE of coefficients (11) using all data and $\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\phi}}$ as starting values.

For stable nonlinear regression:

1: Find $\widehat{\boldsymbol{\theta}}$:= NLS fit using all data.
2: Compute residuals corresponding to $\widehat{\boldsymbol{\theta}}$.
3: Use residuals to find MLE of $\widehat{\boldsymbol{\phi}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$.
4: Update MLE of regression coefficients (12) using all data and $\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\phi}}$ as starting values.

[22] improved the method of trimming based on ordered stable errors, calculated the Best Linear Unbiased Estimator (BLUE) of regression coefficients, and provided Algorithm 6 to estimate coefficients based on finite variance of trimmed ordered stable errors.

5. Robust Stable Least Trimmed Squares (SLTS) Method

The model (1) is called a stable multiple linear regression if errors $\varepsilon_1, \ldots, \varepsilon_n$ are *i.i.d.* standard stable distribution. In this case, to compute the LTS estimator of regression coefficients, we propose the Stable LTS (SLTS) method within the MLE framework using (8), [12]. We define a SLTS, [2] as a LTS on truncated data such that order statistics variance of data exist.

5.1. Linear SLTS

The LTS estimator $\hat{\boldsymbol{\theta}}_{\text{LTS}}$ using Least Squares method defined in (13) by [33] as follows:

Algorithm 6: TLS estimation method of stable regression coefficients

1: Compute $\hat{\boldsymbol{\theta}}$:= LS fit of all datasets and compute residuals $\boldsymbol{e}$ corresponding to $\hat{\boldsymbol{\theta}}$.
2: Using residuals $\boldsymbol{e}$ to compute MLE of error parameters, called $\hat{\alpha}$ and $\hat{\beta}$.
3: Find cut-off points, $c = \left[\frac{2}{\hat{\alpha}}\right] + 1$ and $d = \left[n + 1 - \frac{2}{\hat{\alpha}}\right] - 1$.
4: Reorder the indexes of the dataset based on ordered residuals $\boldsymbol{e}$.
5: Trim c and d of the reordered dataset and perform LS fit of new data to update $\hat{\boldsymbol{\theta}}$.
6: Use the updated residuals corresponding to $\hat{\boldsymbol{\theta}}$ to find MLE of $\hat{\boldsymbol{\psi}} = (\hat{\alpha}, \hat{\beta})$, with $\gamma = 1, \delta = 0$.
7: Compute update MLE of coefficients (6), (7) using the joint distribution of order statistics and $\hat{\boldsymbol{\psi}}$, $\hat{\boldsymbol{\theta}}$ as starting values.

$$\hat{\boldsymbol{\theta}}_{\text{LTS}} = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{h} e_{(i)}^2 = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} e_{(i)}^2 \quad I_{\left\{e_{(i)}^2 \leqslant e_{(h)}^2\right\}}, \tag{13}$$

where $\boldsymbol{e} = \boldsymbol{Y} - \hat{\boldsymbol{Y}} = (e_i)_{1 \leq i \leq n}$ is the residuals, $e_{(i)}^2$ is the ith ordered squared residuals, and h is the breakdown point. Consider in multiple linear regression (1) the errors $\varepsilon_1, \ldots, \varepsilon_n$ are *i.i.d.* standard stable distribution, then the traditional LTS is considered insufficient with heavy tailed errors, so we recall SLTS Algorithm 8 presented by [2] and recall LTS Algorithm 7 based on MLE.

5.1.1. Breakdown point for SLTS

The breakdown point in linear regression should not exceed 50%, which is the highest. In the SLTS method, we compute the breakdown point for linear stable regression models under the condition of the presence variance of ordered stable errors. The breakdown point h_s is defined through Lemma 5.1 as a function of tail index α, number of coefficients p, and sample size n.

Lemma 5.1. *The breakdown point for the robust SLTS was computed in (14),*

$$h_s = \left[\frac{n}{2}\right] + \left[\frac{p+1}{2}\right] + \left[1 - \frac{2}{\alpha}\right]. \tag{14}$$

Proof. For proof of Lemma 5.1, see [2]. □

5.1.2. $SLTS_{MLE}$ and LTS_{MLE} Algorithm

Now to compute $\widehat{\boldsymbol{\theta}}_{\text{LTS}}$ and $\widehat{\boldsymbol{\theta}}_{\text{SLTS}}$ using MLE approach for stable linear regression models, perform steps in the following Algorithms, [2]:

Algorithm 7: LTS estimation method based on MLE for stable linear regression coefficients

1: Using the model (1), perform OLS regression, compute residuals $(e_1, \ldots, e_n)$, and use them to compute the MLE of the stable distribution parameters, called $\widehat{\alpha}$, and $\widehat{\beta}$.
2: Use $(\widehat{\alpha}, \widehat{\beta})$ to find cut−off points c, and d. Compute breakdown point h_s in (14).
3: Based on the model (1) and using h_s, compute $\widehat{\boldsymbol{\theta}}_{\text{LTS}}$ using Algorithm 2 and deduce the solution H^* with $\#H^* = h_s$, where H^* is the indexes that gives the optimal solution.
4: Re-estimate MLE of $\widehat{\boldsymbol{\psi}} = (\alpha, \beta)$ using corresponding residuals $\widehat{\boldsymbol{\theta}}_{\text{LTS}}$.
5: Use $\widehat{\boldsymbol{\theta}}_{\text{LTS}}, \widehat{\boldsymbol{\psi}}$ as initial values to find $\widehat{\boldsymbol{\theta}}_{\text{LTS-M}}$ a numerical MLE using (11) for all i of H^*.

5.2. Nonlinear SLTS

Consider in nonlinear regression model (2) the errors $\varepsilon_1, \ldots, \varepsilon_n$ are *i.i.d.* standard stable distribution. The popular method for estimating coefficients is the NLS method. It is considered insufficient with heavy-tailed errors, so this manuscript considered robust NLTS estimators of nonlinear regression coefficients.

Algorithm 8: SLTS estimation method of stable linear regression coefficients

1: Using the model (1), perform OLS regression, compute residuals $(e_1, \ldots, e_n)$ and use them to compute the MLE of the stable distribution parameters, called $\widehat{\alpha}$, and $\widehat{\beta}$.
2: Use $(\widehat{\alpha}, \widehat{\beta})$ to find cut−off points c, and d. Compute breakdown point h_s in (14).
3: Re-order $\boldsymbol{X}$ and $\boldsymbol{Y}$ based on ordered residuals $(e_{(1)}, \ldots, e_{(n)})$.
4: Trim $\boldsymbol{X}$ and $\boldsymbol{Y}$ by cut−off points c, and d, and define the model (5).
5: Based on the model (5) and using h_s, compute $\widehat{\boldsymbol{\theta}}_{\text{LTS}}$ using the LTS Algorithm and deduce the solution H^* with $\#H^* = h_s$, where H^* is the indexes that gives the optimal solution.
6: Re-estimate MLE of $\widehat{\boldsymbol{\psi}} = (\alpha, \beta)$ using corresponding residuals $\widehat{\boldsymbol{\theta}}_{\text{LTS}}$ of the model (5).
7: Use $\widehat{\boldsymbol{\theta}}_{\text{LTS}}, \widehat{\boldsymbol{\psi}}$ as initial values to find $\widehat{\boldsymbol{\theta}}_{\text{SLTS}}$ a numerical MLE using (6) for all i of H^*.

5.2.1. Breakdown Point for Robust NSLTS and NLTS

The breakdown point in the linear regression should not exceed 50%. However, with nonlinear regression, we considered the minimum trimming proportion and tried to adjust to avoid losing the information. Therefore, ϕ must be adjusted accurately because the robust NLTS and NSLTS solutions may be bad in nonlinear regression with high breakdown points. So, the lowest possible trimming proportion must be considered, as in [31]. This paper considered heavy-tailed errors, i.e., stable nonlinear regression models, so it used ordered stable errors with indexes between cut-off points c and d that have a finite variance. Lemma 5.2 illustrates computing the robust NLTS and NSLTS breakdown points for stable nonlinear regression models.

Lemma 5.2. *Consider ϕ the trimming proportion and α the tail index in stable distribution, then the breakdown point of robust NLTS and NSLTS in stable nonlinear regression equals:*

$$h_s = \left[\left(n + 2 - \frac{4}{\alpha} \right) (1 - \phi) \right]. \tag{15}$$

Proof. In a similar way of Lemma 5.1. □

5.2.2. Robust NSLTS$_{MLEO}$ and NLTS$_{MLE}$ Algorithms

Using the breakdown point (15) through Lemma 5.2, we recall the robust NLTS$_{\text{MLE}}$ algorithm based on the MLE and provided the NSLTS$_{\text{MLEO}}$ algorithm.

Algorithm 9: Robust NLTS$_{\text{MLE}}$ estimation method of stable nonlinear regression coefficients

1: Perform NLS fit of all data and compute residuals $e_i = y_i - g(x_i, \hat{\boldsymbol{\theta}}_{\text{NLS}}),\ i = 1, 2, \ldots, n$.
2: Using e_i in step 1, compute MLE of stable parameters, $\hat{\alpha}$, and $\hat{\beta}$.
3: Using $\hat{\alpha}$ in step 2, compute breakdown point h_s in (15).
4: Using h_s, compute $\hat{\boldsymbol{\theta}}_{\text{NLTS}}$ and deduce H^* using Algorithm 3.
5: Consider $\hat{\boldsymbol{\theta}}_{\text{NLTS}}, \hat{\alpha}$, and $\hat{\beta}$ as starting values to compute MLE solution using (12).

Algorithm 10: Robust NSLTS$_{\text{MLEO}}$ method of stable nonlinear regression coefficients

1: Perform NLS fit through all data.
2: Use residuals to compute MLE of $\hat{\alpha}$, $\hat{\beta}$, and cut-off points c and d.
3: Reorder the response variable and covariate vector using ordered residuals.
4: Trim the reordered dataset using c and d and define a new dataset.
5: Compute h_s in (15), compute $\hat{\boldsymbol{\theta}}_{\text{NLTS}}$, and deduce H^* using Algorithm 3.
6: Use residuals corresponding to $\hat{\boldsymbol{\theta}}_{\text{NLTS}}$ to find MLEO of $\hat{\boldsymbol{\psi}} = (\hat{\alpha}, \hat{\beta})$ using equation (10) and using step 2 values as initial values.
7: Use $\hat{\boldsymbol{\theta}}_{\text{NLTS}}, \hat{\boldsymbol{\psi}}$ as started values to find $\hat{\boldsymbol{\theta}}_{\text{NSLTS}}$ estimators numerically using (7).

5.3. Simulation

1. **Linear case**: We compare SLTS with {OLS, TLS, NOR, RQ(0.5), LTS(MLE)} methods described previously. So, we simulate a simple linear regression model $y_i = \theta_1 + \theta_2 x_i + \varepsilon_i,\ i = 1, \ldots, n$, where $x_i \sim u(0, 100)$, $\theta_1 = 5$, $\theta_2 = 2$ and ε_i errors have $\mathbb{S}(\alpha, \beta, 1, 0)$ with $\alpha = \{0.7, 1, 1.5\}, \beta = \{-0.5, 0, 0.5\}$ similar to [21] (NOR) using sample sizes $n = 100$, we repeat each simulation $k = 1000$ times and the estimated values (EST) and errors (MAE) presented in Tables 1 and 2 computed using (16) and (17), respectively.

2. **Nonlinear case**: As [21], consider a nonlinear regression model $y_i = \theta_1 \exp(-\theta_2 x_i) + \varepsilon_i$ where $x_i = (0.2, 20)$ by step=0.2, $\theta_1 = 30$, $\theta_2 = 0.1$ and $\varepsilon_1, \dots, \varepsilon_n$ the errors distributed with a stable distribution with $\alpha = \{0.7, 1, 1.3\}$, $\beta = \{-0.5, 0, 0.5\}, \delta = 0, \gamma = 0.5$. For trimming proportions $\phi = \{10\%, 15\%, 20\%, 25\%\}$. Each case was repeated 500 times, and the estimated EST and MAE errors in Tables 3 and 4 were computed through (16) and (17), respectively. Here, We compare NSLTS with {NLS, TLS, NOR, NLTS(MLE)}

$$\text{EST}\left(\widehat{\theta}\right) = \frac{1}{k}\sum_{i=1}^{k}\widehat{\theta}_i, \tag{16}$$

$$\text{MAE}\left(\widehat{\theta}\right) = \frac{1}{k}\sum_{i=1}^{k}|\widehat{\theta}_i - \theta|. \tag{17}$$

5.4. Regression Simulation Results

For linear regression: Table 1 shows that for an asymmetric case $\beta = 0.5$, MLE of SLTS are unbiased for all cases. For the symmetric case, $\beta = 0$ and $\alpha = 1.5$, the MLE of SLTS are unbiased. Also, the SLTS based on MLE has minimum errors and outperforms all other methods. Table 2 computed stable parameter estimates based on residuals fitted models. In Figure 2, we find the box plots of estimates. They are unbiased, as the range of estimates is short.

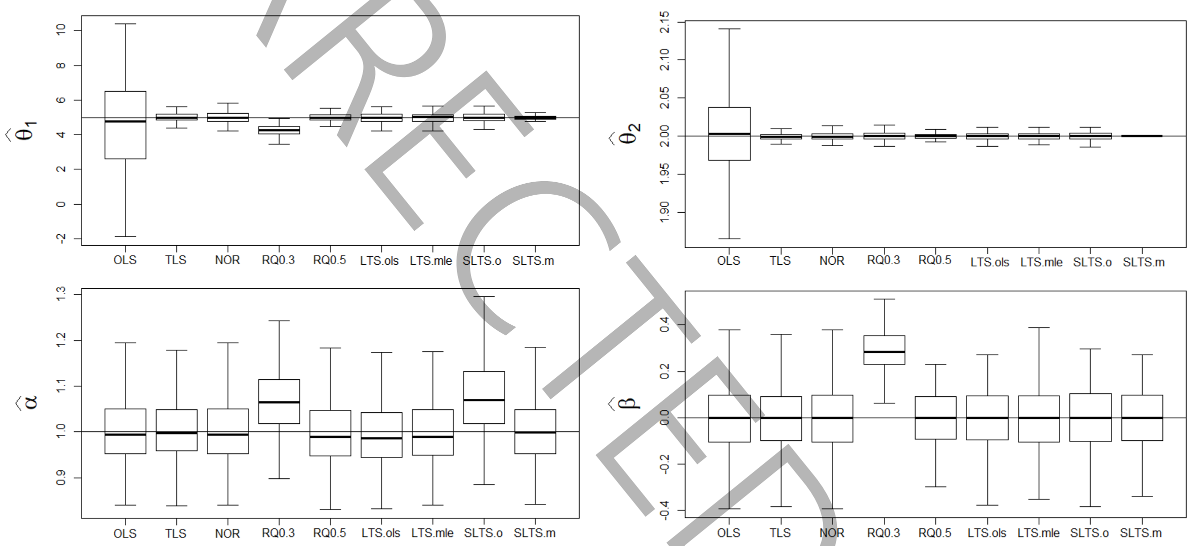

Figure 2: Estimated linear regression coefficients with $\theta_{1_{\text{exact}}} = 5$ and $\theta_{2_{\text{exact}}} = 2$ and stable parameters with $\alpha = 1$ and $\beta = 0$, where SLTS.o= SLTS_{OLS} and SLTS.m= SLTS_{MLE}.

For nonlinear regression: The NSLTS solutions were performed based on the new MLEO methods, and it was noted that MLEO solutions are closer to being unbiased, especially in asymmetric cases. Almost all solutions are unbiased for symmetric cases, but their ranges were wide compared to the $\text{NSLTS}_{\text{MLEO}}$. Also, $\text{NSLTS}_{\text{MLEO}}$ has the minimum errors for the estimated coefficients in all cases, as in Table 3. These results were obtained by selecting the appropriate range $\gamma = 0.5$, as in [21]. It is noted that the estimates of the tail indexes and skewness are similar for all methods, as in Table 4 and Figure 3.

6. Real Datasets

To evaluate SLTS and NSLTS Algorithms for real datasets, we consider **U.S. Federal Reserve Differences (FRD)** and **Ultrasonic Calibration (UC)** datasets. The solutions were evaluated using Bootstrap for computing standard deviation estimators, [7], as in Table 5, using Algorithm 11.

6.1. U.S. Federal Reserve Differences Data (2005-2008)

The Federal Reserve is considered the most important bank in the U.S. As [21] we will be interested in weekly data on interest rates. Particularly, on U.S. Treasury bonds with a fixed entitlement of 10 years ($\boldsymbol{x}$) versus an inflation-indicated ($\boldsymbol{y}$) period between 2005−2008. The differences from week to week were computed, and the

Table 1: The robust estimates of linear regression coefficients with exact values $\theta_1 = 5$, $\theta_2 = 2$. The minimum MAEs are bolded

													SLTS			
			OLS		TLS		NOR		RQ(0.5)		LTS(MLE)		OLS		MLE	
α	β		$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_1$	$\widehat{\theta}_2$
0.7	−0.5	EST	1.326	1.685	4.683	2.003	-22.697	2.349	4.592	2.001	4.872	2.002	5.045	2.001	5.002	2.000
		MAE	118.634	1.932	0.583	0.009	34.434	0.463	0.589	0.009	0.396	0.010	0.391	0.006	**0.181**	**0.001**
	0	EST	49.648	1.399	5.004	2.000	41.521	1.591	5.034	2.000	5.041	1.999	5.055	1.999	5.002	2.000
		MAE	148.244	2.509	0.348	0.006	47.506	0.568	0.349	0.006	0.317	0.006	0.334	0.006	**0.167**	**0.001**
	0.5	EST	-21.787	3.410	5.320	1.997	11.616	1.934	5.333	2.000	5.090	1.999	4.925	2.000	5.003	2.000
		MAE	102.583	2.554	0.556	0.009	7.903	0.107	0.539	0.009	0.420	0.007	0.405	0.007	**0.163**	**0.001**
1	−0.5	EST	4.799	2.001	4.770	2.001	4.856	1.998	4.716	2.001	4.950	1.997	5.052	2.000	5.001	2.000
		MAE	6.658	0.145	0.459	0.007	1.244	0.018	0.492	0.008	0.455	0.011	0.488	0.009	**0.188**	**0.001**
	0	EST	90.265	1.058	5.017	2.000	131.148	0.477	5.009	2.000	5.074	1.999	5.003	2.000	5.008	2.000
		MAE	89.469	1.016	0.383	0.006	127.096	1.537	0.410	0.007	0.532	0.009	0.522	0.009	**0.207**	**0.001**
	0.5	EST	16.249	1.869	5.195	2.000	5.831	1.994	5.238	2.001	5.064	1.999	4.963	2.000	5.033	2.000
		MAE	13.838	0.204	0.431	0.007	1.196	0.016	0.460	0.008	0.542	0.016	0.536	0.009	**0.193**	**0.001**
1.5	-0.5	EST	4.590	1.997	4.870	2.000	4.809	2.000	4.817	2.000	5.053	1.996	5.044	1.999	5.002	2.000
		MAE	0.827	0.012	0.378	0.006	0.436	0.007	0.413	0.007	0.434	0.011	0.676	0.011	**0.213**	**0.001**
	0	EST	5.030	1.999	4.971	2.000	4.990	2.000	5.007	1.999	4.949	2.000	5.067	1.998	5.011	2.000
		MAE	0.675	0.014	0.387	0.007	0.370	0.007	0.406	0.007	0.441	0.007	0.656	0.012	**0.216**	**0.001**
	0.5	EST	4.959	1.999	5.252	2.000	5.196	2.006	4.673	2.000	5.608	2.004	4.528	2.001	5.036	2.000
		MAE	0.760	0.013	0.792	0.007	0.813	0.022	0.494	0.007	1.168	0.011	0.863	0.013	**0.520**	**0.001**

Table 2: The estimates of tail index $\widehat{\alpha}$ and skewness $\widehat{\beta}$ using residuals in linear simulation. The best MAEs are bolded.

													SLTS			
			OLS		TLS		NOR		RQ(0.5)		LTS(MLE)		OLS		MLE	
α	β		$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$
0.7	-0.5	EST	0.716	-0.178	0.722	-0.410	0.716	-0.178	0.760	-0.406	0.709	-0.462	1.054	-0.770	0.705	-0.503
		MAE	0.230	0.901	0.116	0.192	0.142	0.400	0.121	0.194	0.113	0.190	0.358	0.340	0.100	**0.155**
	0	EST	0.708	0.020	0.707	0.032	0.708	0.020	0.705	0.019	0.705	0.029	1.123	0.061	0.714	0.027
		MAE	0.219	0.500	0.098	0.159	0.109	0.253	0.093	**0.135**	0.096	0.186	0.426	0.346	**0.089**	0.172
	0.5	EST	0.719	0.250	0.727	0.424	0.719	0.250	0.756	0.413	0.707	0.466	1.033	0.706	0.703	0.497
		MAE	0.211	0.895	0.102	0.194	0.115	0.337	0.108	0.185	0.103	0.174	0.340	0.337	**0.088**	**0.155**
1	-0.5	EST	1.032	-0.351	1.031	-0.435	1.032	-0.351	1.044	-0.408	1.006	-0.447	1.248	-0.670	1.007	-0.488
		MAE	0.248	0.713	0.136	0.205	0.140	0.272	0.140	0.211	0.138	0.262	0.259	0.324	**0.133**	**0.201**
	0	EST	1.027	-0.006	1.025	0.008	1.027	-0.006	1.018	0.006	1.014	-0.001	1.282	0.004	1.025	0.009
		MAE	0.208	0.386	**0.143**	0.210	0.148	0.243	0.151	**0.192**	0.148	0.272	0.290	0.356	0.145	0.235
	0.5	EST	1.052	0.350	1.041	0.456	1.052	0.350	1.055	0.417	1.017	0.468	1.255	0.677	1.017	0.506
		MAE	0.241	0.740	0.147	0.195	0.153	0.272	0.144	0.197	0.145	0.242	0.274	0.319	**0.140**	**0.193**
1.5	-0.5	EST	1.578	-0.407	1.576	-0.445	1.578	-0.407	1.573	-0.427	1.524	-0.535	1.679	-0.500	1.547	-0.502
		MAE	0.221	0.482	0.182	0.398	0.188	0.418	0.183	**0.391**	0.181	0.424	0.234	0.538	**0.174**	0.396
	0	EST	1.509	0.053	1.507	0.051	1.509	0.053	1.500	0.046	1.491	0.058	1.679	0.089	1.495	0.059
		MAE	0.191	**0.291**	0.181	0.368	0.179	0.372	0.180	0.346	**0.177**	0.469	0.236	0.623	0.178	0.398
	0.5	EST	1.493	0.350	1.547	0.303	1.493	0.350	1.469	0.482	1.519	0.111	1.618	0.580	1.544	0.274
		MAE	0.212	**0.481**	0.209	0.492	0.237	0.509	**0.205**	0.481	0.234	0.608	0.210	0.561	0.208	0.507

Table 3: Estimated nonlinear coefficients (first line) and their errors (second line) using NSLTS and other methods with exact $\theta_1 = 30$, $\theta_2 = 0.1$. The minimum errors are bolded.

				NLS		TLS		NOR		$NLTS_{MLE}$		$NSLTS_{MLEO}$		$NSLTS_{NLS}$	
ϕ	α	β		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$
10%	1.3	-0.5	EST	29.628	0.102	30.005	0.100	29.988	0.100	29.980	0.100	30.021	0.100	29.920	0.101
			MAE	0.771	0.004	0.208	0.002	0.211	0.001	0.207	0.001	**0.046**	**3.5e-4**	0.233	0.002
20%			EST	29.628	0.102	30.005	0.100	29.988	0.100	29.988	0.100	30.011	0.100	29.964	0.100
			MAE	0.7710	0.004	0.208	0.001	0.211	0.001	0.206	0.001	**0.052**	**5e-4**	0.223	0.001
25%			EST	29.628	0.102	30.005	0.100	29.988	0.100	29.990	0.100	30.006	0.100	29.954	0.101
			MAE	0.7710	0.004	0.208	0.001	0.211	0.001	0.206	0.001	**0.056**	**5e-4**	0.236	0.001
10%	1.3	0.00	EST	29.980	0.100	29.994	0.100	30.089	0.099	30.005	0.099	30.001	0.100	30.022	0.100
			MAE	0.814	0.005	0.251	0.002	0.366	0.003	0.247	0.001	**0.094**	**8e-4**	0.220	0.001
20%			EST	29.980	0.100	29.994	0.100	30.089	0.099	30.018	0.100	30.009	0.100	30.013	0.100
			MAE	0.814	0.005	0.251	0.002	0.366	0.002	0.265	0.001	**0.105**	**8e-4**	0.225	0.001
25%			EST	29.980	0.100	29.994	0.100	30.089	0.099	30.005	0.100	30.002	0.100	30.005	0.100
			MAE	0.814	0.005	0.251	0.002	0.366	0.002	0.253	0.001	**0.096**	**9e-4**	0.231	0.001
10%	1.3	0.50	EST	30.577	0.097	30.040	0.100	30.053	0.100	30.066	0.099	30.003	0.100	30.080	0.098
			MAE	1.035	0.005	0.201	0.001	0.209	0.001	0.202	0.001	**0.047**	**3e-4**	0.164	0.001
20%			EST	30.577	0.097	30.040	0.100	30.053	0.100	30.060	0.099	30.005	0.100	30.065	0.098
			MAE	1.035	0.005	0.201	0.001	0.209	0.001	0.203	0.001	**0.050**	**5e-4**	0.151	0.002
25%			EST	30.577	0.097	30.040	0.100	30.053	0.100	30.059	0.099	30.004	0.100	30.049	0.098
			MAE	1.035	0.005	0.201	0.001	0.209	0.001	0.201	0.001	**0.049**	**5e-4**	0.139	0.002

Table 4: Estimates of tail index $\hat{\alpha}$ and skewness $\hat{\beta}$ using residuals for nonlinear simulated data. The minimum MAEs are bolded.

				NLS		TLS		NOR		$NLTS_{MLE}$		$NSLTS_{MLEO}$		$NSLTS_{NLS}$	
ϕ	α	β		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
10%	1.3	-0.5	EST	1.345	-0.440	1.300	-0.470	1.302	-0.470	1.437	-0.561	1.333	-0.485	1.445	-0.554
			MAE	0.110	0.202	0.104	0.161	0.104	0.160	0.164	0.225	**0.102**	**0.155**	0.168	0.229
20%			EST	1.345	-0.440	1.300	-0.470	1.302	-0.470	1.438	-0.547	1.327	-0.483	1.438	-0.556
			MAE	0.110	0.202	0.104	0.161	0.104	0.160	0.165	0.234	**0.101**	**0.147**	0.163	0.226
25%			EST	1.345	-0.440	1.300	-0.470	1.302	-0.470	1.436	-0.564	1.328	-0.488	1.437	-0.547
			MAE	0.110	0.202	0.104	0.161	0.104	0.160	0.164	0.226	**0.101**	**0.143**	0.163	0.224
10%	1.3	0.00	EST	1.349	0.002	1.309	-0.011	1.316	-0.007	1.441	-0.001	1.300	-0.002	1.452	-0.017
			MAE	0.123	0.266	0.120	0.221	0.118	0.222	0.178	0.280	**0.107**	**0.118**	0.185	0.292
20%			EST	1.349	0.002	1.309	-0.011	1.316	-0.007	1.442	-0.009	1.360	-0.007	1.445	-0.007
			MAE	0.123	0.266	0.120	0.221	0.118	0.222	0.178	0.286	**0.114**	**0.148**	0.179	0.282
25%			EST	1.349	0.002	1.309	-0.011	1.316	-0.007	1.440	0.003	1.363	-0.011	1.442	-0.005
			MAE	0.123	0.266	0.120	0.221	0.118	0.222	0.177	0.288	**0.102**	**0.153**	0.178	0.279
10%	1.3	0.50	EST	1.331	0.441	1.285	0.480	1.286	0.479	1.417	0.590	1.315	0.504	1.426	0.592
			MAE	0.114	0.231	0.114	0.180	**0.112**	0.177	0.156	0.243	0.131	**0.169**	0.158	0.242
20%			EST	1.331	0.441	1.285	0.480	1.286	0.479	1.416	0.591	1.317	0.504	1.421	0.590
			MAE	0.114	0.231	0.1143	0.180	**0.112**	0.177	0.156	0.242	0.128	**0.169**	0.156	0.239
25%			EST	1.331	0.441	1.285	0.480	1.286	0.479	1.416	0.591	1.321	0.505	1.421	0.588
			MAE	0.114	0.231	0.114	0.180	**0.112**	0.177	0.156	0.242	0.128	**0.167**	0.156	0.240

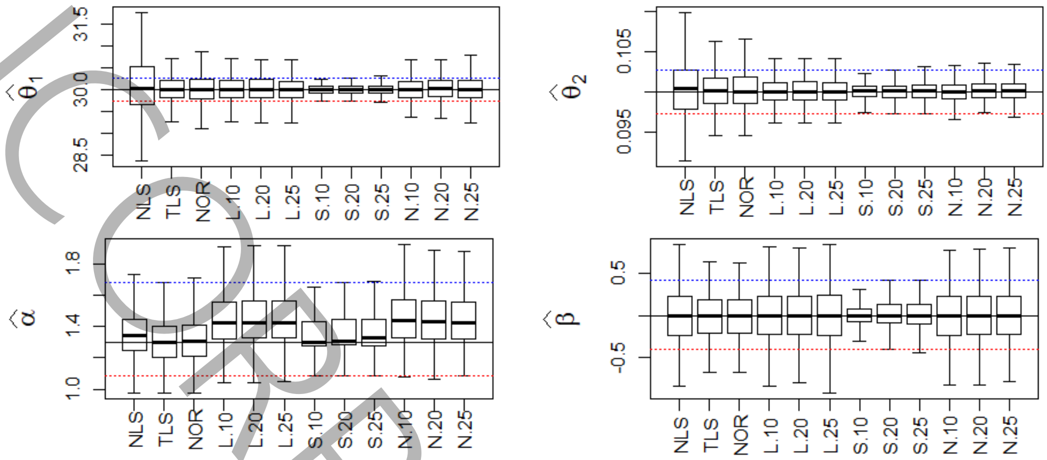

Figure 3: The range of estimated nonlinear coefficients (above) and stable parameters (bottom) for tail index $\alpha =$ 1.3, where L.10=10% of NLTS_{MLE}, S.10=10% of $\text{NSLTS}_{\text{MLEO}}$, and N.10=10% of $\text{NSLTS}_{\text{NLS}}$ and so on. A horizontal solid line donates the coefficient values.

Algorithm 11: Bootstrap to compute estimated coefficients standard deviations

1: Draw bootstrap samples $(Y_1^*, X_1^*)_l, \ldots, (Y_n^*, X_n^*)_l,\ l = 1, \ldots, m,\ m = 1000.$

2: For the given nonlinear regression model, compute $\hat{\boldsymbol{\theta}}_l = (\hat{\theta}_{l1}, \ldots, \hat{\theta}_{lp})^\top$ of the drawn sample in step 1.

3: Compute

$$\text{SD}(\hat{\theta}_j) = \sqrt{\frac{1}{m}\sum_{l=1}^{m}\left(\hat{\theta}_{lj} - \overline{\theta}_j\right)^2},$$

where $\overline{\theta}_j = \frac{1}{m}\sum\limits_{l=1}^{m}\hat{\theta}_{lj},\ j = 1, \ldots, p.$

4: Report $\text{SD}(\hat{\boldsymbol{\theta}}) = (\text{SD}(\hat{\theta}_1), \ldots, \text{SD}(\hat{\theta}_p))^\top$ the standard deviation of estimated coefficients.

linear representation is shown in Figure 4 and model (18). It is suitable for stable studies as its residuals lie within the tail index $\widehat{\alpha}_{\boldsymbol{\varepsilon}} = 1.573$ and the skewness $\widehat{\beta}_{\boldsymbol{\varepsilon}} = 0.351$.

$$\boldsymbol{y} = \beta_1 + \beta_2 \boldsymbol{x} + \boldsymbol{\varepsilon}. \tag{18}$$

6.2. Ultrasonic Calibration Data (UC)

[5] measures how the experimental response of the ultrasonic (y_i) is affected by the metal distance (x_i) by Ultrasonic Calibration Dataset. These data result from NIST fitted through model (19) shown in Figure 4. It is appropriate for stable data, as their errors have tail index $\hat{\alpha}_{\boldsymbol{\varepsilon}} = 0.983$ and the skewness errors terms $\hat{\beta}_{\boldsymbol{\varepsilon}} = 1.000$, using the initial values $(\theta_1 = 0.190, \theta_2 = 0.006, \theta_3 = 0.011)$.

$$y_i = \frac{\exp(-\theta_1 x_i)}{\theta_2 + \theta_3 x_i} + \varepsilon_i. \tag{19}$$

Table 5: Estimating coefficients real datasets models. The estimated value in the first line (Est) and the Bootstrap standard deviation in the second line (SD)

Data	n	p	Trimming	Method		$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_3$	$\widehat{\beta}_4$	$\widehat{\beta}_5$	$\widehat{\beta}_6$
U.S. FRD Data	208	2	50%	SLTS_{OLS}	Est	0.0111	0.8462				
					SD	0.0331	0.0351				
				SLTS_{MLE}	Est	0.0026	0.8246				
					SD	**8e-06**	**0.0001**				
				OLS	Est	0.0092	0.7864				
					SD	0.0048	0.0463				
				NOR	Est	0.0066	0.8156				
					SD	0.0002	0.0021				
				TLS	Est	0.0063	0.8157				
					SD	0.0002	0.0021				
UC Data	214	3	5%	$\text{NSLTS}_{\text{NLS}}$	Est	0.1667	0.0061	0.0112			
					SD	0.0228	0.0004	0.0009			
				$\text{NSLTS}_{\text{MLEO}}$	Est	0.1742	0.0060	0.0112			
					SD	**0.0159**	**0.0002**	**0.0001**			
				TLS	Est	0.1697	0.0059	0.0114			
					SD	0.0200	0.0002	0.0003			
				NOR	Est	0.1633	0.0058	0.0113			
					SD	0.0200	0.0003	0.0005			
				NLS	Est	0.1902	0.0061	0.0105			
					SD	0.0219	0.0003	0.0007			

7. Conclusion

We have provided a review of the LTS method for stable data using SLTS and NSLTS methods. The concept of trimming is basic to robust computations and makes results seem more logical with heavy-tailed, stable data. The LTS method is like data mining under the permutation method and minimum sum of squared residuals. SLTS and NSLTS methods are based on the MLE using the property of finite variance of ordered stable errors, and this makes that very important for estimating stable regression coefficients.

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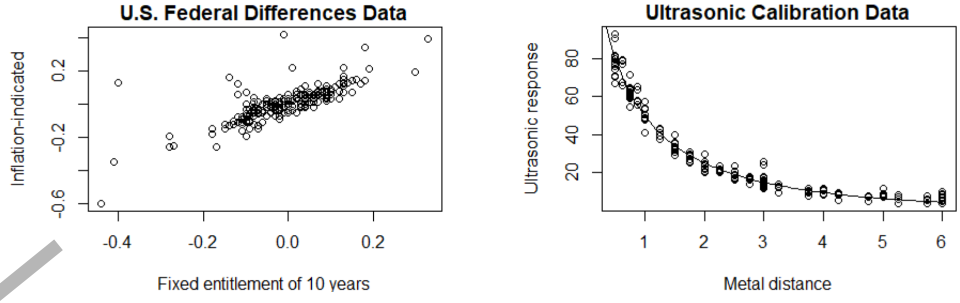

Figure 4: Scatter plots of real datasets fitted with linear and nonlinear regression fits using $\mathrm{SLTS}_{\mathrm{OLS}}$, $\mathrm{SLTS}_{\mathrm{MLE}}$, OLS, NOR, and TLS for U.S. FRD Data and $\mathrm{NSLTS}_{\mathrm{NLS}}$, $\mathrm{NSLTS}_{\mathrm{MLEO}}$, TLS, NOR, and NLS for UD dataset.

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