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Original Article

An existence result for a Robin problem involving $p(x)$-Kirchhoff-type equation with indefinite weight

Mehdi Latifi*[a], Mohsen Alimohammady[b]

[a] *Department of Basic Sciences, Khatam-Ol-Anbia (PBA) University, Tehran, Iran*
[b] *Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar 47416-1468, Iran*

ABSTRACT: This paper discusses the existence of at least two distinct nontrivial weak solutions for a class of $p(x)$-Kirchhoff-type equation plus an indefinite potential under Robin boundary condition. The variable exponent theory of generalized Lebesgue-Sobolev spaces, mountain pass theorem and Ekeland variational principle are used for this purpose.

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1. Introduction

In this study, we examine the existence of a nontrivial weak solution for $p(x)$-Kirchhoff type problem with a source that has critical growth in the sense of the Sobolev embedding and Robin boundary condition:

$$\begin{cases} -M\left(t\right)\Delta_{p(x)}u+\xi\left(x\right)|u|^{q(x)-2}u=\lambda b\left(x\right)|u|^{r(x)-2}u, & \text{in} \quad \Omega; \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}+\beta\left(x\right)|u|^{p(x)-2}u=0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $t=\int\limits_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx+\int\limits_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}d\sigma_x$, $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$, is a bounded smooth domain with cone property, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $d\sigma_x$ is the measure on $\partial\Omega$, $\beta\in L^{\infty}\left(\partial\Omega\right)$ with $\beta^{-}:=\inf\limits_{x\in\partial\Omega}\beta\left(x\right)>0$, $M:\mathbb{R}^{+}\to\mathbb{R}^{+}$ is a continuous map, λ is a positive number and $p,q,r,s:\overline{\Omega}\to\mathbb{R}^{+}$ are continuous maps.

**Corresponding author.*
E-mail addresses: latifi.mmali@gmail.com, m_latifi@khadu.ac.ir, amohsen@umz.ac.ir

The potential function $\xi \in L^{s(x)}(\Omega)$ and in general it is sign-changing. Thus, the left-hand side of (1) is not coercive. On the right-hand side of the problem (the reaction), $b(x) > 0$, a.e. $x \in \overline{\Omega}$ and $b \in L^{\gamma(x)}(\Omega)$, where $\gamma(x) = \frac{p^*(x)}{p^*(x)-r(x)}$ and $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$.

Below, we will use the notations:

$$g^+ := \sup_{x\in\overline{\Omega}} g(x) \qquad \text{and} \qquad g^- := \inf_{x\in\overline{\Omega}} g(x).$$

In addition, in this document, we presume that the following assumptions are valid:

$$1 < q^- < p^- \leq p^+ < r^- \leq r^+ < \left(p^-\right)^* \quad \text{and} \quad q(x) < p(x) \leq p^*(x) < N < s(x).$$

Nonlinear boundary value problem with variable exponent growth condition has been received considerable attention in the last decade. These problems are interesting for modeling a wide range of phenomena, and pose many difficult mathematical problems. For example, the model of motion of electrorheological fluids, stationary thermo-rheological viscous flows of non-Newtonian fluids and the processes filtration of an ideal barotropic gas through a porous medium, image processing, ...; see e.g. [3, 4, 16, 19, 20].

The operator $\Delta_{p(x)}u := div\left(|\nabla u|^{p(x)-2}\nabla u\right)$, is said to be the $p(x)$-Laplacian, which is a natural extension of the classical p-Laplacian, possesses more complicated nonlinearities than the p-Laplacian; mainly due to the fact that it is inhomogeneous.

The questions relating to the extension of the classical D'Alembert wave equation for the free vibrations of elastic rope lead to the Kirchhoff-type equations. Robin boundary conditions are a weighted combination of Dirichlet and Neuman boundary conditions. The problem (1), is a generalization of a model introduced by kirchhoff [19].

This problem is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^1 \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

for $0 < x < L$, $t \geq 0$ presented by Kirchhoff in 1883, as an extension of the classical D'Alambert wave equation for free vibration of elastic strings. In above equation, $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length and P_0 is the initial axial tension. Such nonlinear Kirchhoff model can also be used for describing the dynamics of an axially moving string (see [21]).

These problems are also noted from the standpoint of pure mathematics thus. These kinds of problems have worked in many ways.

Allaoui [1] introduced the $p(x)$-Kirchhoff- type problem involving boundary conditions and examined problem (1), with $M(t) = 1$ and positive coefficients. Driven by the aforementioned document and the results on the $p(x)$-Laplacian operator, the purpose of this article is to examine the Robin problem (1).

We study the nonlinear problem (1), when the function $\xi(x)$, has an indefinite sign in a suitable variable exponent Lebesgue space.

For the Kirchhoff function $M : \mathbb{R}^+ \to \mathbb{R}^+$, we assume it is continuous and

(M1) there exists a constant m_0, such that $0 < m_0 \leq M(t)$ for every $t \in [0, \infty)$.

(M2) there exists $0 \leq t_0$, such that $\hat{M}(t) := \int_0^t M(s)ds \geq tM(t)$ for every $t \in [t_0, \infty)$.

The condition (M1) on M is weaker than the condition consider by [7].

Remark 1.1. *If $\beta \equiv 0$, we recover the Neumann problem. For this reason our current document includes the Neumann problems as a special case.*

The most important result of this document is as follows.

Theorem 1.2. *Under the circumstances referred for problem (1), with (M1) and (M2); there exists $\lambda^* > 0$, such that for every $\lambda \in (0, \lambda^*)$, problem (1), possesses at least two distinct nontrivial weak solutions.*

To achieve this objective, we use the Palais-Smale condition introduced by Ambrosetti-Rabinowitz in [3], and Ekeland variational principal.

The rest of this paper is organized as follows. In Section 2, we will introduce some necessary preliminary knowledge and lemmas on variable exponent Sobolev spaces, and in Section 3 and 4, we give the proof of our main result.

2. Preliminaries

For the convenience of the reader, we recall some necessary basic knowledge and propositions concerning on variable exponent Lebesgue spaces and Sobolev spaces. We refer the reader to [9, 10, 11, 13, 15, 17] for details. Let Ω be a bounded open domain of $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary $\partial\Omega$, and $p \in C_+\left(\overline{\Omega}\right)$ where

$$C_+\left(\overline{\Omega}\right) = \left\{p : p \in C\left(\overline{\Omega}\right), p(x) > 1, \quad \text{for all} \quad x \in \overline{\Omega}\right\}.$$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by,

$$L^{p(x)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function and } \int_\Omega |u|^{p(x)} dx < \infty\right\},$$

under the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$.

It becomes a Banach space. We also define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

equipped with the norm $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{1,p(x)} = ||\nabla u||_{p(x)} + |u|_{p(x)}$.

Moreover, for any $u \in W^{1,p(x)}$, we define $\|u\|_\partial := ||\nabla u||_{p(x)} + |u|_{L^{p(x)}(\partial\Omega)}$, then $\|u\|_\partial$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to the norm $\|u\|$, [8]. Now, we introduce a norm, that will be used in this article. Let $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$, and for $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_\beta = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u}{\lambda}\right|^{p(x)} dx + \int_{\partial\Omega} \beta(x) \left|\frac{u}{\lambda}\right|^{p(x)} d\sigma_x \leq 1\right\}.$$

By theorem 2.1 in [8], $\|.\|_\beta$ is a norm on $W^{1,p(x)}(\Omega)$, which is equivalent to $\|.\|$ and $\|u\|_\partial$.

The following basic properties of the variable exponent Lebesgue and Sobolev spaces are required and listed below [11, 18, 13, 9, 12, 6].

Lemma 2.1. *$\left(L^{p(x)}(\Omega), |.|_{p(x)}\right)$ is Banach space, and Hölder inequality holds; i.e.*

$$\int_\Omega |uv|\, dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. It is reflexive if and only if $1 < p^- \leq p^+ < +\infty$. Moreover if $0 < |\Omega| < +\infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$, a.e. $x \in \Omega$, then there exists the continuous embedding $L^{p_2(x)} \hookrightarrow L^{p_1(x)}$.

Lemma 2.2. *$\left(W^{1,p(x)}(\Omega), \|.\|_\beta\right)$ is a separable and reflexive Banach space. If $s(x) < p^*(x)$ for all $x \in \bar{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$, is compact and continuous. Also if $s \in C_+(\partial\Omega)$ and $s(x) < p_*(x)$ for all $x \in \partial\Omega$, then the trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\partial\Omega)$, is compact and continuous, where $p_*(x) = \frac{(N-1)p(x)}{N-p(x)}$ if $p(x) < N$ or $p_*(x) = +\infty$ if $p(x) > N$.*

So, according to assumption on (1), the embeddings $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and $W^{1,p(x)}(\Omega) \hookrightarrow L^{s'(x)q(x)}(\Omega)$ are compact and continuous, and there exist a positive constant $C > 1$, such that $|u|_{q(x)} \leq C\|u\|_\beta$ and $|u|_{s'(x)q(x)} \leq C\|u\|_\beta$ for all $u \in W^{1,p(x)}(\Omega)$.

The modular, which is the mapping $\rho_{p(x)} : W^{1,p(x)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{p(x)}(u) = \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)} d\sigma_x,$$

is a major tool in generalized Lebesgue Sobolev space studies.

Lemma 2.3 ([8]). *For $u \in W^{1,p(x)}(\Omega)$ and $\beta^- > 0$, we have:*

i) For $u \neq 0$, $\quad \|u\|_\beta = \lambda \quad \Leftrightarrow \quad \rho\left(\frac{u}{\lambda}\right) = 1$;

ii) $\|u\|_\beta < 1\,(=1, >1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1\,(=1, >1)$;

iii) if $\|u\|_\beta \leq 1$ *then* $\|u\|_\beta^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_\beta^{p^-}$;
if $\|u\|_\beta \geq 1$ *then* $\|u\|_\beta^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_\beta^{p^+}$;
(briefly) $\min\left\{\|u\|_\beta^{p^+}, \|u\|_\beta^{p^-}\right\} \leq \rho_{p(x)}(u) \leq \max\left\{\|u\|_\beta^{p^+}, \|u\|_\beta^{p^-}\right\}$;

iv) $\|u_n\|_\beta \to 0\,(\text{ or } +\infty) \Leftrightarrow \rho_{p(x)}(u_n) = 0(\text{or } \infty)$ *as* $n \to \infty$.

Remark 2.4. *These relations show that; topology defined by the norm and that defined by the modular is equivalent.*

Lemma 2.5 ([5]). *For measurable functions p, q with $p \in C_+(\bar{\Omega})$ and $pq \in L^\infty{}_+(\Omega)$. Let $0 \neq u \in L^{q(x)}(\Omega)$, then*

$$|u|_{p(x)q(x)} \leq 1 \qquad \Rightarrow \qquad |u|_{p(x)q(x)}^{P^+} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|_{p(x)q(x)}^{P^-}$$

$$|u|_{p(x)q(x)} \geq 1 \qquad \Rightarrow \qquad |u|_{p(x)q(x)}^{P^-} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|_{p(x)q(x)}^{P^+}.$$

Definition 2.6. *Let X be a Banach space, and $I : X \to \mathbb{R}$ be a differentiable functional.*

1. *A sequence $\{u_n\} \subset X$ is called $(PS)_c$-sequence for I, if $I(u_n) \to c$ and $\|I'(u_n)\|_{X^*} \to 0$, as $n \to \infty$.*
2. *If every $(PS)_c$-sequence for I, has a converging subsequence (in X), we say that I satisfies the $(PS)_c$-conditions.*

Theorem 2.7 (Mountain Pass Theorem. [2]). *Let $(X, \|.\|)$ be a real Banach space and $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function, such that $I(0_X) = 0$ and satisfying the (PS)-condition. Suppose that:*

1. *there exist constants $\rho, \alpha > 0$, such that $I(u) \geq \alpha$, if $\|u\| = \rho$;*
2. *there exist $e \in X$ with $\|e\| > \rho$, such that $I(e) \leq 0$,*

then I possesses a critical value $c \geq \alpha$, which can be characterized as:

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u)$$

where $\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = e\}$.

Here, problem (1) is stated in the framework of the generalized Sobolev space $X := W^{1,p(x)}(\Omega)$.
For $\beta \in L^\infty(\partial\Omega)$ with $\beta^- > 0$, we define $A : X \to \mathbb{R}$, by

$$A(u) = \int\limits_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int\limits_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)} d\sigma_x \qquad \forall u \in X.$$

Lemma 2.8 ([14]). 1. *$A \in C^1(X, \mathbb{R})$ and it's derivative $A' : X \to X^*$, is given by*

$$\langle A'(u), v\rangle = \int\limits_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int\limits_{\partial\Omega} \beta(x) |u|^{p(x)-2} uv d\sigma_x \qquad \forall u, v \in X;$$

2. *A' is a continuous, bounded, strictly monotone and an operator of type $(S)_+$, i.e. if $u_n \rightharpoonup u$ in X, and $\limsup \langle A'(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ (strongly) in X.*

The Euler-Lagrange functional associated with (1), is $\Phi_\lambda : X \to \mathbb{R}$, that defined as:

$$\Phi_\lambda(u) = \hat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}d\sigma_x\right) + \int_\Omega \frac{\xi(x)}{q(x)}|u|^{q(x)}dx - \lambda\int_\Omega \frac{b(x)}{r(x)}|u|^{r(x)}dx.$$

Standard arguments imply that, the functional Φ_λ, is well-defined and of C^1-class on X, with the derivative given by

$$\begin{aligned}\langle \Phi'_\lambda(u), v\rangle =& M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}d\sigma_x\right)\left(\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uvd\sigma_x\right)\\ &+ \int_\Omega \xi(x)|u|^{q(x)-2}uvdx - \lambda\int_\Omega b(x)|u|^{r(x)-2}uvdx,\end{aligned}$$

for all $u, v \in X$.

We say that $u \in X$, is a weak solution of (1), if

$$\begin{aligned}&M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}d\sigma_x\right)\left(\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uvd\sigma_x\right)\\ &+ \int_\Omega \xi(x)|u|^{q(x)-2}uvdx = \lambda\int_\Omega b(x)|u|^{r(x)-2}uvdx\end{aligned}$$

for all $v \in X$. Hence, we can find weak solutions of (1) as the critical points of the functional Φ_λ, in the space X.

3. Existence of first weak solution

For the proof of our theorem, we will use the Mountain Pass Theorem. Let us verify that all the prerequisites of this theorem are fulfilled.

Lemma 3.1. *With assuming condition on M, the functional Φ_λ, satisfies the (PS)-condition.*

Proof. Suppose that $\{u_n\} \subset X$ is a (PS)-sequence; that is $|\Phi_\lambda(u_n)| < c$ and $|\langle \Phi_\lambda(u), v\rangle| \le \varepsilon_n \|v\|_\beta$, for $v \in X$, and $\varepsilon_n \to 0$ as $n \to \infty$.
Step 1. For every $\epsilon > 0$, there exist K_ϵ such that

$$\max\left\{|u|^{q^+}_{s'(x)q(x)}, |u|^{q^-}_{s'(x)q(x)}\right\} \le \varepsilon\rho_{p(x)}(u) + K_\varepsilon \min\left\{|u|^{q^+}_{q(x)}, |u|^{q^-}_{q(x)}\right\}.$$

By contradiction, assume that there exist $\epsilon_0 > 0$, and a sequence $\{u_n\}$ in X, such that $|u|_{s'(x)q(x)} = 1$, and

$$\varepsilon_0\rho_{p(x)}(u_n) + n\min\left\{|u_n|^{q^+}_{q(x)}, |u_n|^{q^-}_{q(x)}\right\} < 1.$$

So by lemma 2.3 and reflection of X, $\{u_n\}$ is a bounded sequence in X, and up to a subsequence, if necessary, it converges weakly to some $u_0 \in X$ and strongly in $L^{s'(x)q(x)}(\Omega)$. Therefore $|u_0|_{s'(x)q(x)} = 1$ and consequently $\min\left\{|u_0|^{q^+}_{q(x)}, |u_0|^{q^-}_{q(x)}\right\} < 0$, that is contradictory.
Step 2. Using the Hölder inequality, lemma 2.5 with choice $T > \frac{2}{q^-}|\xi|_{s(x)}$, we deduce that

$$\left|\int_\Omega \frac{\xi(x)}{q(x)}|u|^{q(x)}dx\right| \le \frac{2}{q^-}|\xi|_{s(x)}\max\left\{|u|^{q^+}_{s'(x)q(x)}, |u|^{q^-}_{s'(x)q(x)}\right\} \le T\varepsilon\rho_{p(x)}(u) + TK_\varepsilon\min\left\{|u|^{q^+}_{q(x)}, |u|^{q^-}_{q(x)}\right\}.$$

Step 3. $\{u_n\}$ is bounded in X.
Suppose the contrary. Then passing to a ubsequence, we may assume $\max\{1, t_0p^+\} < \|u_n\|_\beta \to \infty$, as $n \to \infty$, (t_0 in (M2)), and so

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}d\sigma_x > t_0,$$

and

$$\begin{aligned}
c+1+\varepsilon_n\|u_n\|_\beta \geq\ & \Phi_\lambda(u_n) - \frac{1}{r^-}\left\langle \Phi'_\lambda(u_n), u_n\right\rangle \\
=\ & \hat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}d\sigma_x\right) + \int_\Omega \frac{\xi(x)}{q(x)}|u_n|^{q(x)}dx - \lambda\int_\Omega \frac{b(x)}{r(x)}|u_n|^{r(x)}dx \\
& - \frac{1}{r^-}M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}d\sigma_x\right)\times\left(\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)}d\sigma_x\right) \\
& - \frac{1}{r^-}\int_\Omega \xi(x)|u_n|^{q(x)}dx + \frac{\lambda}{r^-}\int_\Omega b(x)|u_n|^{r(x)}dx
\end{aligned}$$

$$\begin{aligned}
& \geq m_0\left(\frac{1}{p^+}-\frac{1}{r^-}\right)\|u_n\|_\beta^{p^-} - \left(\frac{1}{q^-}+\frac{1}{r^-}\right)\int_\Omega \xi(x)|u_n|^{q(x)}dx \\
& \geq m_0\left(\frac{1}{p^+}-\frac{1}{r^-}\right)\|u_n\|_\beta^{p^-} - \left(\frac{1}{q^-}+\frac{1}{r^-}\right)T\varepsilon\|u_n\|_\beta^{p^-} - \left(\frac{1}{q^-}+\frac{1}{r^-}\right)TK_\varepsilon \min\left\{|u|_{q(x)}^{q^+}, |u|_{q(x)}^{q^-}\right\}.
\end{aligned}$$

Now by choosing ϵ, small enough such that the coefficient of $\|u_n\|_\beta^{p^-}$ become positive, and also considering the compact embedding of $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, we infer

$$c+1+\varepsilon_n\|u_n\|_\beta \geq A\|u_n\|_\beta^{p^-} - TM_\varepsilon \min\left\{C^{q^+}\|u_n\|_\beta^{q^+}, C^{q^-}\|u_n\|_\beta^{q^-}\right\}. \tag{2}$$

Dividing (2), by $\|u_n\|_\beta^{p^-}$, and passing to the limit, as $n\to\infty$, eventuate $0 > A$, it is contradictory.
So $\{u_n\}$ is bounded in X, and up to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u_0 in X and converges strongly in $L^{\omega(x)}(\Omega)$ with $1<\omega(x)<p^*(x)$.
Step 4. $\{u_n\}$ converges strongly to u_0 in X. Since the functional Φ_λ satisfies the (PS)-condition, we obtain

$$\begin{aligned}
\left\langle \Phi'_\lambda(u_n), u_n-u_0\right\rangle = & M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}d\sigma_x\right) \\
& \times\left(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u_0)\,dx + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n-u_0)\,d\sigma_x\right) \\
& + \int_\Omega \xi(x)|u_n|^{q(x)-2}u_n(u_n-u_0)\,dx - \lambda\int_\Omega b(x)|u_n|^{r(x)-2}u_n(u_n-u_0)\,dx \to 0.
\end{aligned}$$

So by lemma 2.3

$$\begin{aligned}
m_0\left\langle A'(u_n), u_n-u_0\right\rangle \leq & \left\langle \Phi'_\lambda(u_n), u_n-u_0\right\rangle - \int_\Omega \xi(x)|u_n|^{q(x)-2}u_n(u_n-u_0)\,dx \\
& + \lambda\int_\Omega b(x)|u_n|^{r(x)-2}u_n(u_n-u_0)\,dx.
\end{aligned}$$

On the other hand, because $1<q(x)<p(x)$, then $u\in X\subset L^{p(x)}(\Omega)\subset L^{q(x)}(\Omega) = L^{(q(x)-1)q'(x)}(\Omega)$, and again $|u|^{q(x)-1}\in L^{q'(x)}(\Omega)$, so by using Sobolev compact embeddings and Hölder inequality, we have

$$\left|\int_\Omega \xi(x)|u_n|^{q(x)-2}u_n(u_n-u_0)\,dx\right| \leq C|\xi|_{s(x)}|u_n|_{q'(x)}|u_n-u_0|_{\alpha(x)} \to 0,$$

as $n\to\infty$, where $\alpha(x) = \frac{s(x)q(x)}{s(x)-q(x)} < p^*(x)$.

Similarly, by applying interpolation inequality with appropriate coefficients and powers, we have

$$\begin{aligned}\left|\int_\Omega b(x)|u_n|^{r(x)-2}u_n(u_n-u_0)\,dx\right| &\le 2|b|_{\gamma(x)}\left||u_n|^{r(x)-1}(u_n-u_0)\right|_{\frac{p^*(x)}{r(x)}}\\ &\le C\left||u_n|^{r(x)-1}\right|^b_{ap(x)}|u_n-u_0|_{\theta p^*(x)}\to 0,\end{aligned}$$

as $n\to\infty$, where $0<\theta<1$, is chosen appropriately. So

$$\limsup\langle A'(u_n),u_n-u_0\rangle\le 0.$$

Finally, since the operator A' has the $(S)_+$ property, it conclusion $u_n\to u_0$ strongly in X, and hence the functional Φ_λ fulfills (PS)-condition.

□

The following lemma shows that the functional Φ_λ has the geometry of the Mountain Pass Theorem.

Lemma 3.2. *There exists $\lambda^*>0$, that for every $\lambda\in(0,\lambda^*)$, there exists $\rho,\tau>0$ such that $\Phi_\lambda(u)\ge\tau$ for every $u\in X$, whit $\|u\|=\rho$.*

Proof. Using Sobolev embedding theorem and equivalent of norms, there exists a positive constant $C\ge 1$, such that $|u|_{p(x)}\le C\|u\|_\beta$, $|u|_{p^*(x)}\le C\|u\|_\beta$ and $|u|_{s'(x)q(x)}\le C\|u\|_\beta$ for all $u\in X$.
set $\rho=\frac{1}{nC}$, that n will determine its value in the future. Thus $\rho\in\,]0,1[$ and $|u|_{p(x)}\le 1$, $|u|_{p^*(x)}\le 1$ and $|u|_{s'(x)q(x)}\le 1$, for all $u\in X$ with $\|u\|_\beta=\rho$.
Also we have

$$\left|\int_\Omega\frac{\xi(x)}{q(x)}|u|^{q(x)}dx\right|\le\frac{2}{q^-}|\xi|_{s(x)}|u|^{q^-}_{q(x)s'(x)}\le\frac{2}{q^-}|\xi|_{s(x)}C^{q^-}\|u\|^{q^-}_\beta$$

$$\left|\int_\Omega\frac{b(x)}{r(x)}|u|^{r(x)}dx\right|\le\frac{2}{r^-}|b|_{\gamma(x)}|u|^{r^-}_{p^*(x)}\le\frac{2}{r^-}|b|_{\gamma(x)}C^{r^-}\|u\|^{r^-}_\beta,$$

hence

$$\begin{aligned}\Phi_\lambda(u)&=\hat M\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}d\sigma_x\right)+\int_\Omega\frac{\xi(x)}{q(x)}|u|^{q(x)}dx-\lambda\int_\Omega\frac{b(x)}{r(x)}|u_n|^{r(x)}dx\\ &\ge\frac{m_0}{p^+}\left(\int_\Omega|\nabla u|^{p(x)}dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)}d\sigma_x\right)-\frac{2C^{q^-}}{q^-}|\xi|_{s(x)}\|u\|^{q^-}_\beta-\frac{2\lambda C^{r^-}}{r^-}|b|_{\gamma(x)}\|u\|^{r^-}_\beta\\ &\ge\frac{m_0}{p^+}\|u\|^{p^+}_\beta-\frac{2C^{q^-}}{q^-}|\xi|_{s(x)}\|u\|^{q^-}_\beta-\frac{2\lambda C^{r^-}}{r^-}|b|_{\gamma(x)}\|u\|^{r^-}_\beta=\frac{m_0}{p^+}\rho^{p^+}-\frac{2|\xi|_{s(x)}}{q^-n^{q^-}}-\frac{2\lambda|b|_{\gamma(x)}}{r^-n^{r^-}}\end{aligned}$$

a straightforward computation shows that the right side for

$$\lambda^*=\frac{m_0r^-n^{r^-}}{2p^+|b|_{\gamma(x)}}-\frac{r^-n^{r^-}|\xi|_{s(x)}}{q^-n^{q^-}|b|_{\gamma(x)}}$$

is zero, and if n is chosen larg enough such that $n^{q^-}>\frac{2p^+|\xi|_{s(x)}}{m_0q^-}$, we have $\lambda^*>0$, and there exist $\tau>0$ such that $\Phi_\lambda(u)\ge\tau$, for every $\lambda\in(0,\lambda^*)$. □

Let us now move on to the proof of the main theorem.

Proof. To apply the Mountain Pass Theorem, we need to prove that there exist $e\in X$, with $\|e\|>\rho$, such that $\Phi_\lambda(e)\le 0$.
First, for $t>t_0$ (t_0 in (M2) condition on M),

$$\hat M(t)\le\frac{M(t_0)}{t_0}t=m_1t.$$

Now for $0 < u_0 \in X$ and $t > 1$, such that $\|tu_0\| > t_0$,

$$\Phi_\lambda(tu_0) = \hat{M}\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla u_0|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)t^{p(x)}}{p(x)}|u_0|^{p(x)}d\sigma_x\right) + \int_\Omega \frac{\xi(x)t^{q(x)}}{q(x)}|u_0|^{q(x)}dx - \lambda\int_\Omega \frac{b(x)t^{r(x)}}{r(x)}|u|^{r(x)}dx$$
$$\leq \frac{m_1 t^{p^+}}{p^-}\|u_0\|_\beta^{p^-} + \int_\Omega \frac{\xi(x)t^{q(x)}}{q(x)}|u_0|^{q(x)}dx - \frac{\lambda t^{r^-}}{r^+}\int_\Omega b(x)|u|^{r(x)}dx.$$

Since $q^-, q^+, p^+ < r^-$, we have $\Phi_\lambda(tu_0) \to -\infty$ as $t \to \infty$.

This shows that there exists $u_0 \in X$, with $\|u_0\|_\beta > \rho$ and $\Phi_\lambda(u_0) < 0$.
Therefore, there exists a critical point $u_1 \in X$ for the functional Φ_λ, with positive energy and characterized by

$$\Phi_\lambda(u_1) = \bar{c} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\lambda(\gamma(t)) > \tau > 0,$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = u_1\}$. □

4. The second nontrivial weak solution of (1)

We establish the existence of the second nontrivial weak solution of (1), by applying the Ekeland variational principle.

Proof. By lemma 3.2, we see that for a fix $\lambda \in (0, \lambda^*)$,

$$\inf_{u\in\partial B_\rho(0)}\Phi_\lambda(u) \geq \tau > 0, \quad (B_\rho(0)\,\text{the ball of radius } \rho \text{ centered at origin in } X).$$

Also, according to the choice λ, $\Phi_\lambda(u)$ is bounded from below. In fact, we have $\Phi_\lambda(u) \geq \frac{-m_0}{p^+} > -\infty$.
It can be seen from the proof of the first part of the theorem that there exists $u_0 \in X$, for small enough t, such that $\Phi_\lambda(tu_0) < 0$, so

$$-\infty < \underline{c} := \inf_{u\in\overline{B_\rho(0)}}\Phi_\lambda(u) < 0.$$

Now if we choose $\varepsilon > 0$, such that

$$0 < \varepsilon < \inf_{u\in\partial B_\rho(0)}\Phi_\lambda(u) - \inf_{u\in\overline{B_\rho(0)}}\Phi_\lambda(u),$$

by applying Ekeland's variational principle to the functional $\Phi_\lambda|_{\overline{B_\rho(0)}}$, there exists $u_\varepsilon \in \overline{B_\rho(0)}$, that

$$\Phi_\lambda(u_\varepsilon) < \inf_{u\in\overline{B_\rho(0)}}\Phi_\lambda(u) + \varepsilon$$

$$\Phi_\lambda(u_\varepsilon) < \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_\beta \qquad u \neq u_\varepsilon,$$

so, $\Phi_\lambda(u_\varepsilon) < \inf_{u\in\partial B_\rho(0)}\Phi_\lambda(u)$, and $u_\varepsilon \in B_\rho(0)$.
For the functional $\Psi_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, with $\Psi_\lambda(u) = \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_\beta$, and u_ε as the minimum point of Ψ_λ,

$$\frac{\Psi_\lambda(u_\varepsilon + t\nu) - \Psi_\lambda(u_\varepsilon)}{t} \geq 0$$

for all $t > 0$, small enough and all $\nu \in B_\rho(0)$, or

$$\frac{\Phi_\lambda(u_\varepsilon + t\nu) - \Phi_\lambda(u_\varepsilon)}{t} + \varepsilon\|\nu\|_\beta \geq 0$$

that when $t \to 0^+$, we deduce that

$$\langle \Phi'_\lambda(u_\varepsilon), \nu\rangle \geq -\varepsilon\|\nu\|_\beta.$$

and by replacing ν by $-\nu$,

$$\langle \Phi'_\lambda(u_\varepsilon), \nu\rangle \leq \varepsilon\|\nu\|_\beta.$$

Therefore $\|\Phi'_\lambda(u_\varepsilon)\|_{X^*} \leq \varepsilon$. So for $\varepsilon = \frac{1}{n}$, there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that $\Phi_\lambda(u_n) \to \underline{c} < 0$ and $\Phi'_\lambda(u_n) \to 0$ in X^*, as $n \to \infty$. This sequence converges strongly to some u_2 and $\Phi'_\lambda(u_2) = 0$. Thus u_2 is a nontrivial weak solution of (1) with $\Phi_\lambda(u_2) = \underline{c} < 0$, and $u_2 \neq u_1$. □

References

[1] M. Allaoui, *Existence results for a class of $p(x)$-Kirchhoff problems*, Studia Sci. Math. Hungar., 54 (2017), pp. 316–331.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis, 14 (1973), pp. 349–381.

[3] S. N. Antontsev and J. F. Rodrigues, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat., 52 (2006), pp. 19–36.

[4] S. N. Antontsev and S. I. Shmarev, *A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions*, Nonlinear Anal., 60 (2005), pp. 515–545.

[5] M. Avci, B. Cekic, and R. A. Mashiyev, *Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory*, Math. Methods Appl. Sci., 34 (2011), pp. 1751–1759.

[6] B. Cekic and R. A. Mashiyev, *Existence and localization results for $p(x)$-Laplacian via topological methods*, Fixed Point Theory Appl., (2010), pp. Art. ID 120646, 7.

[7] N. T. Chung, *Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities*, Electron. J. Qual. Theory Differ. Equ., (2012), pp. No. 42, 13.

[8] S.-G. Deng, *Positive solutions for Robin problem involving the $p(x)$-Laplacian*, J. Math. Anal. Appl., 360 (2009), pp. 548–560.

[9] D. E. Edmunds and J. Rákosník, *Density of smooth functions in $W^{k,p(x)}(\Omega)$*, Proc. Roy. Soc. London Ser. A, 437 (1992), pp. 229–236.

[10] ———, *Sobolev embeddings with variable exponent*, Studia Math., 143 (2000), pp. 267–293.

[11] X. Fan, J. Shen, and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl., 262 (2001), pp. 749–760.

[12] ———, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl., 262 (2001), pp. 749–760.

[13] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., 263 (2001), pp. 424–446.

[14] B. Ge and Q.-M. Zhou, *Multiple solutions for a Robin-type differential inclusion problem involving the $p(x)$-Laplacian*, Math. Methods Appl. Sci., 40 (2017), pp. 6229–6238.

[15] M. K. Hamdani, *On a nonlocal asymmetric Kirchhoff problem*, Asian-Eur. J. Math., 13 (2020), pp. 2030001, 15.

[16] M. K. Hamdani, A. Harrabi, F. Mtiri, and D. D. Repovš, *Existence and multiplicity results for a new $p(x)$-Kirchhoff problem*, Nonlinear Anal., 190 (2020), pp. 111598, 15.

[17] M. K. Hamdani, L. Mbarki, M. Allaoui, O. Darhouche, and D. D. Repovš, *Existence and multiplicity of solutions involving the $p(x)$-Laplacian equations: on the effect of two nonlocal terms*, Discrete Contin. Dyn. Syst. Ser. S, 16 (2023), pp. 1452–1467.

[18] M. K. Hamdani and D. D. Repovš, *Existence of solutions for systems arising in electromagnetism*, J. Math. Anal. Appl., 486 (2020), pp. 123898, 18.

[19] G. R. Kirchhoff, *Vorlesungen über mathematische Physik. I. Mechanik.* Leipzig. Teubner, 1876.

[20] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, vol. 1748 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000.

[21] S. M. Shahruz and S. A. Parasurama, *Suppression of vibration in the axially moving Kirchhoff string by boundary control*, J. Sound Vibration, 214 (1998), pp. 567–575.

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