

Original Article

Co-even domination number of a modified graph by operations on a vertex or an edge

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ABSTRACT: Let $G = (V, E)$ be a simple graph. A dominating set of G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D . The cardinality of a smallest dominating set of G , denoted by $\gamma(G)$, is the domination number of G . A dominating set D is called co-even dominating set if the degree of vertex v is even number for all $v \in V \setminus D$. The cardinality of a smallest co-even dominating set of G , denoted by $\gamma_{coe}(G)$, is the co-even domination number of G . In this paper, we study the co-even domination number of graphs which constructed by some operations on a vertex or an edge of a graph.

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Let $G = (V, E)$ be a graph with vertex set V and edge set E . Throughout this paper, we consider graphs without loops and directed edges. For each vertex $v \in V$, the set $N_G(v) = \{u \in V | uv \in E\}$ refers to the open neighbourhood of v and the set $N_G[v] = N_G(v) \cup \{v\}$ refers to the closed neighbourhood of v in G . The degree of v , denoted by $\deg(v)$, is the cardinality of $N_G(v)$. A set $D \subseteq V$ is a dominating set if every vertex in $V \setminus D$ is adjacent to at least one vertex in D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G . There are various domination numbers in the literature. For a detailed treatment of domination theory, the reader is referred to [6].

A dominating set D is called a co-even dominating set, if the degree of each vertex in $V \setminus D$ is even ([7]). The cardinality of a smallest co-even dominating set of G , denoted by $\gamma_{coe}(G)$, is the co-even domination number of G . Ghanbari in [4], considered binary operations of graphs and presented some bounds for the co-even domination number of join, corona, neighbourhood corona, and Hajós sum of two graphs. In this paper, we present more results for the co-even domination number.

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In the next section, first we mention the definition of vertex and edge removal of a graph and then study the co-even domination number of a graph which constructed by vertex and edge removal and find some bounds for them. In Section 3, we mention the definition of vertex and edge contraction and study the co-even domination number of vertex and edge contraction of a graph. We find some bounds for them, and finally, we present the upper and the lower bounds for the co-even domination number of a graph regarding the vertex (edge) removal and contraction.

1. Vertex and edge removal of a graph

The graph $G - v$ is a graph that is made by deleting the vertex v and all edges connected to v from the graph G and the graph $G - e$ is a graph that obtained from G by simply removing the edge e . Our main results in this section are obtaining some bounds for the co-even domination number of the vertex and the edge removal of a graph. First, we state some known results.

Proposition 1.1 ([7]). *Let $G = (V, E)$ be a graph and D is a co-even dominating set of G . Then,*

- (i) *All vertices of odd or zero degrees belong to every co-even dominating set.*
- (ii) *$\deg(v) \geq 2$, for all $v \in V - D$.*
- (iii) *$\gamma(G) \leq \gamma_{coe}(G)$.*

By the definition of co-even domination number, we have the following easy result:

Proposition 1.2. *Let G be a disconnected graph with components G_1 and G_2 . Then*

$$\gamma_{coe}(G) = \gamma_{coe}(G_1) + \gamma_{coe}(G_2).$$

Now, we consider the vertex removal of a graph and find an upper bound and a lower one for the co-even domination number of the constructed graph.

Theorem 1.3. *Let $G = (V, E)$ be a graph and $v \in V$. Then,*

$$\gamma_{coe}(G) - \deg(v) - 1 \leq \gamma_{coe}(G - v) \leq \gamma_{coe}(G) + \deg(v) - 1.$$

Proof. Suppose that $v \in V$ and $D_{coe}(G)$ is a co-even dominating set of G . First, we find the upper bound for $\gamma_{coe}(G - v)$. We consider the following cases:

- (i) $\deg(v)$ is even and $v \notin D_{coe}(G)$. Then at least one of the neighbours of v should be in $D_{coe}(G)$. Now by adding all other neighbours of v in $D_{coe}(G)$, we have a co-even dominating set for $G - v$. The size of this set is at most $\gamma_{coe}(G) - 1 + \deg(v)$.
- (ii) $\deg(v)$ is even and $v \in D_{coe}(G)$. By removing v and all edges related to it, some of the vertices in its neighbour, may not dominate with any other vertex. So by adding all of the neighbours of v in $D_{coe}(G) - \{v\}$, we have a co-even dominating set for $G - v$. So $\gamma_{coe}(G - v) \leq \gamma_{coe}(G) + \deg(v) - 1$.
- (iii) $\deg(v)$ is odd. Then by Proposition 1.1, $v \in D_{coe}(G)$. Now by the same argument as (ii), we have $\gamma_{coe}(G - v) \leq \gamma_{coe}(G) + \deg(v) - 1$.

Therefore $\gamma_{coe}(G - v) \leq \gamma_{coe}(G) + \deg(v) - 1$. Now, we find the lower bound for $\gamma_{coe}(G - v)$. First, we remove v and all the corresponding edges to it. Now we find a co-even dominating set for $G - v$. Suppose that this set is $D_{coe}(G - v)$. One can easily check that $D_{coe}(G - v) \cup N_G[v]$ is a co-even dominating set of G . So

$$\gamma_{coe}(G) \leq \gamma_{coe}(G - v) + \deg(v) + 1,$$

and therefore we have the result. □

Remark 1.4. *The bounds in Theorem 1.3 are sharp. For the upper bound, it suffices to consider G as shown in Figure 1. The set of black vertices in G is a co-even dominating set of G . Now, by removing vertex v with degree 4, the set of black vertices is a co-even dominating set of $G - v$, and $\gamma_{coe}(G - v) = \gamma_{coe}(G) + \deg(v) - 1$. For the lower bound, it suffices to consider H as shown in Figure 2. The set of black vertices in H and $H - v$ are co-even dominating sets of H and $H - v$, respectively. So $\gamma_{coe}(H - v) = \gamma_{coe}(H) - \deg(v) - 1$.*

The following example shows that $\gamma_{coe}(G)$ and $\gamma_{coe}(G - v)$ can be equal.

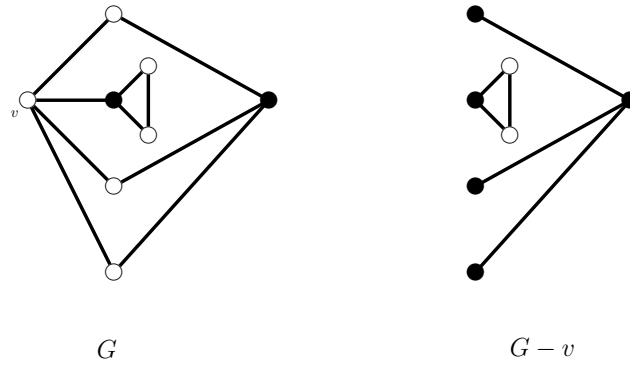


Figure 1: Graphs G and $G - v$

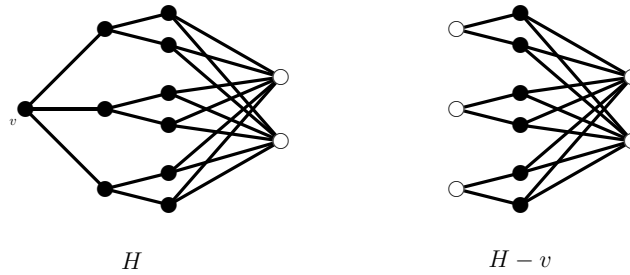


Figure 2: Graphs H and $H - v$

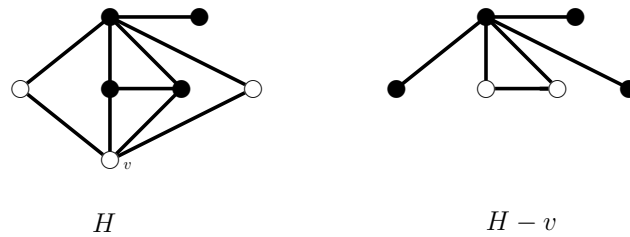


Figure 3: Graphs H and $H - v$ with same co-even domination numbers

Example 1.1. Consider the graphs H and $H - v$ as shown in Figure 3. Obviously, the set of black vertices of each graph is co-even dominating set with smallest size. Therefore, there are some graphs G such that $\gamma_{coe}(G) = \gamma_{coe}(G - v)$.

Now, we consider the edge removing of a graph and present the upper and the lower bound for the constructed graph.

Theorem 1.5. Let $G = (V, E)$ be a graph and $e \in E$. Then,

$$\gamma_{coe}(G) - 2 \leq \gamma_{coe}(G - e) \leq \gamma_{coe}(G) + 2.$$

Proof. Suppose that $e = uv \in E$ and $D_{coe}(G)$ is a co-even dominating set of G . First we find the upper bound for $\gamma_{coe}(G - e)$. We consider the following cases:

- (i) $u, v \notin D_{coe}(G)$. In this case, the degree of these vertices should be even. Now by removing e , the degree of these vertices are odd and they should be in co-even dominating set of $G - e$. Now by considering $D_{coe}(G) \cup \{u, v\}$ as a dominating set of $G - e$, we have a co-even dominating set for that with size $\gamma_{coe}(G) + 2$. So $\gamma_{coe}(G - e) \leq \gamma_{coe}(G) + 2$.
- (ii) $u \in D_{coe}(G)$ and $v \notin D_{coe}(G)$. In this case, the degree of v should be even. Now by removing e and the same argument as previous case, $D_{coe}(G) \cup \{v\}$ is a co-even dominating set of $G - e$ and $\gamma_{coe}(G - e) \leq \gamma_{coe}(G) + 1$.

- (iii) $u, v \in D_{coe}(G)$. By removing edge e and considering $D_{coe}(G)$ as a domination set of $G - e$, we have a co-even dominating set for $G - e$ too. So $\gamma_{coe}(G - e) \leq \gamma_{coe}(G)$.

Now, we find the lower bound for $\gamma_{coe}(G - e)$. First, we remove e . At this step, we find a co-even dominating set for $G - e$. It is easy to see that $D_{coe}(G - e) \cup \{u, v\}$ is a co-even dominating set of G . So

$$\gamma_{coe}(G) \leq \gamma_{coe}(G - e) + 2,$$

and therefore we have the result. \square

We end this section by showing that the bounds are sharp in the Theorem 1.5.

Remark 1.6. The bounds in Theorem 1.5 are sharp. For the upper bound, it suffices to consider G as shown in Figure 4. The set of black vertices in G is a co-even dominating set of G . Now, by removing edge e , the set of black vertices is a co-even dominating set of $G - e$, and $\gamma_{coe}(G - e) = \gamma_{coe}(G) + 2$. For the lower bound, it suffices to consider H as shown in Figure 5. The set of black vertices in H and $H - e$ are co-even dominating sets of H and $H - e$, respectively. So $\gamma_{coe}(H - e) = \gamma_{coe}(H) - 2$.

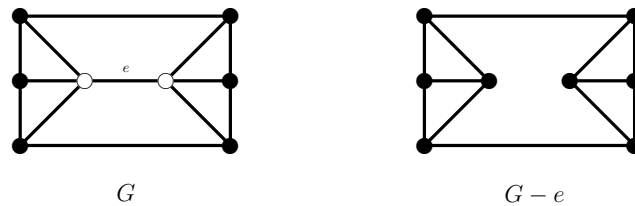


Figure 4: Graphs G and $G - e$



Figure 5: Graphs H and $H - e$

2. Vertex and edge contraction of a graph

Let v be a vertex in graph G . The contraction of v in G denoted by G/v is the graph obtained by deleting v and putting a clique on the open neighbourhood of v . Note that this operation does not create parallel edges; if two neighbours of v are already adjacent, then they remain simply adjacent (see [1, 8]). In a graph G , contraction of an edge e with endpoints u, v is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v . The resulting graph G/e has one less edge than G ([3]). We denote this graph by G/e . In this section, we examine the effects on $\gamma_{coe}(G)$ when G is modified by an edge contraction and vertex contraction. First, we consider the vertex contraction of a graph and find the upper and the lower bound for its co-even domination number.

Theorem 2.1. Let $G = (V, E)$ be a graph and $v \in V$. Then,

$$\gamma_{coe}(G) - \deg(v) - 1 \leq \gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1.$$

Proof. Suppose that $v \in V$ and $D_{coe}(G)$ is co-even dominating set of G . First we find the upper bound for $\gamma_{coe}(G/v)$. We consider the following cases:

- (i) $\deg(v)$ is odd. So $v \in D_{coe}(G)$. Now, by deleting v and putting a clique on the open neighbourhood of v , we have G/v . One can easily check that

$$(D_{coe}(G) - \{v\}) \cup N_G(v)$$

is a co-even dominating set for G/v with size at most $\gamma_{coe}(G) - 1 + \deg(v)$. So in this case, $\gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1$.

- (ii) $\deg(v)$ is even and $v \in D_{coe}(G)$. By the same argument as previous case, we conclude that $\gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1$.
- (iii) $\deg(v)$ is even and $v \notin D_{coe}(G)$. Then at least one vertex in $N_G(v)$ should be in $D_{coe}(G)$. Now $D_{coe}(G) \cup N_G(v)$ is a co-even dominating set for G/v with size at most $\gamma_{coe}(G) - 1 + \deg(v)$, and $\gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1$.

Therefore $\gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1$. Now we find the lower bound for $\gamma_{coe}(G/v)$. First we remove v and put a clique in open neighbourhood of that. Now we find a co-even dominating set for G/v . It is possible that we have t vertices from $N_G(v)$ in $D_{coe}(G/v)$, where $0 \leq t \leq \deg(v)$. Now we keep our dominating set for G/v and remove all the edges we added before and add vertex v and all corresponding edges to that. In any case, $D_{coe}(G/v) \cup N_G[v]$ is a co-even dominating set for G . So

$$\gamma_{coe}(G) \leq \gamma_{coe}(G/v) + \deg(v) + 1,$$

and therefore we have the result. \square

Remark 2.2. The bounds in Theorem 2.1 are sharp. For the upper bound, it suffices to consider G as shown in Figure 6. The set of black vertices in G and G/v are co-even dominating sets of G and G/v , respectively. Hence $\gamma_{coe}(G/v) = \gamma_{coe}(G) + \deg(v) - 1$. For the lower bound, it suffices to consider H as shown in Figure 7. The set of black vertices in H and H/v are co-even dominating sets of them, respectively. So $\gamma_{coe}(H/v) = \gamma_{coe}(H) - \deg(v) - 1$.

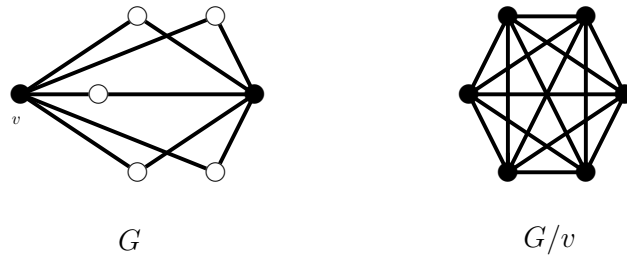


Figure 6: Graphs G and G/v

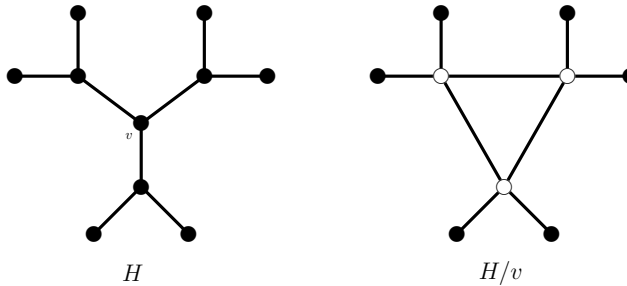


Figure 7: Graphs H and H/v

As an immediate result of Theorems 1.3 and 2.1, we have:

Corollary 2.3. Let $G = (V, E)$ be a graph and $v \in V$. Then,

$$\frac{\gamma_{coe}(G - v) + \gamma_{coe}(G/v)}{2} - \deg(v) + 1 \leq \gamma_{coe}(G) \leq \frac{\gamma_{coe}(G - v) + \gamma_{coe}(G/v)}{2} + \deg(v) + 1.$$

Now we consider the edge contraction of a graph and find upper and lower bound for co-even domination number of that.

Theorem 2.4. Let $G = (V, E)$ be a graph and $e \in E$. Then,

$$\gamma_{coe}(G) - 2 \leq \gamma_{coe}(G/e) \leq \gamma_{coe}(G).$$

Proof. Suppose that $e = uv \in E$ and $D_{coe}(G)$ is co-even dominating set of G . Also let w be the vertex which is replacement of u and v in G/e . First, we find the upper bound for $\gamma_{coe}(G/e)$. We consider the following cases:

- (i) $u, v \notin D_{coe}(G)$. In this case, the degree of these vertices should be even. Now by removing e , the degree of these vertices are odd and therefore the degree of w is even in G/e . Now by considering $D_{coe}(G)$ as a dominating set of G/e too, we have a co-even dominating set for that with size $\gamma_{coe}(G)$. So $\gamma_{coe}(G/e) \leq \gamma_{coe}(G)$.
- (ii) $u \in D_{coe}(G)$ and $v \notin D_{coe}(G)$. In this case, $(D_{coe}(G) - \{u\}) \cup \{w\}$ is a co-even dominating set of G/e and $\gamma_{coe}(G/e) \leq \gamma_{coe}(G)$.
- (iii) $u, v \in D_{coe}(G)$. In this case, $(D_{coe}(G) - \{u, v\}) \cup \{w\}$ is a co-even dominating set of G/e and $\gamma_{coe}(G/e) \leq \gamma_{coe}(G) - 1$.

So $\gamma_{coe}(G/e) \leq \gamma_{coe}(G)$. Now we find the lower bound for $\gamma_{coe}(G/e)$. First we consider G/e and find a co-even dominating set for that. In the worst case, $w \notin D_{coe}(G/e)$ and the degree of u and v are odd in G . So in any case, $(D_{coe}(G) - \{w\}) \cup \{u, v\}$ is a co-even dominating set for G . Hence $\gamma_{coe}(G) \leq \gamma_{coe}(G/e) + 2$, and therefore we have the result. \square

Remark 2.5. The bounds in Theorem 2.4 are sharp. For the upper bound, it suffices to consider G as shown in Figure 8. The set of black vertices in G is a co-even dominating set of G . Also, the set of black vertices in G/e is a co-even dominating set of G/e , and $\gamma_{coe}(G/e) = \gamma_{coe}(G)$. For the lower bound, it suffices to consider H as shown in Figure 9. The set of black vertices in H and H/e are co-even dominating sets of them, respectively. So $\gamma_{coe}(H/e) = \gamma_{coe}(H) - 2$.

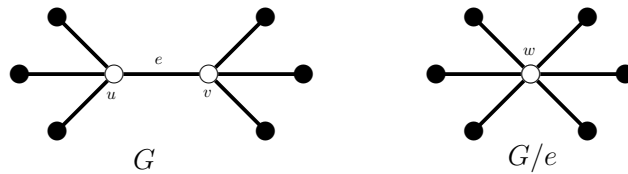


Figure 8: Graphs G and G/e

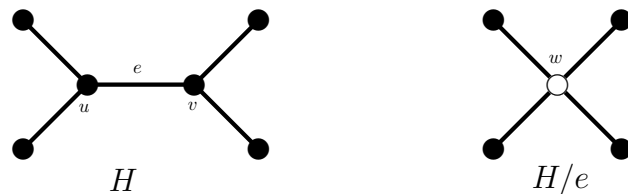


Figure 9: Graphs H and H/e

In Theorem 2.4, we showed that $\gamma_{coe}(G) - 2 \leq \gamma_{coe}(G/e) \leq \gamma_{coe}(G)$, for every $e \in E$. Also in Remark 2.5, we concluded that these bounds are sharp. In the following example, we show that $\gamma_{coe}(G/e)$ can be $\gamma_{coe}(G) - 1$ too:

Example 2.1. Consider G as shown in Figure 10. The set of black vertices in G is a co-even dominating set of G . Also, the set of black vertices in G/e is a co-even dominating set of G/e , and $\gamma_{coe}(G/e) = \gamma_{coe}(G) - 1$.

We end this section by an immediate result of Theorems 1.5 and 2.4:

Corollary 2.6. Let $G = (V, E)$ be a graph and $e \in E$. Then,

$$\frac{\gamma_{coe}(G - e) + \gamma_{coe}(G/e)}{2} - 1 \leq \gamma_{coe}(G) \leq \frac{\gamma_{coe}(G - e) + \gamma_{coe}(G/e)}{2} + 2.$$

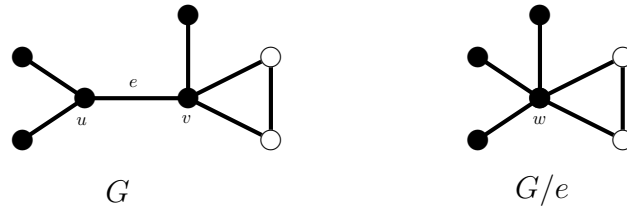


Figure 10: Graphs G and G/e

3. Conclusions

In this paper, we obtained some lower bounds and upper bounds for the co-even domination number of graphs which constructed by the vertex removing, the edge removing, and also vertex and edge contraction regarding the co-even domination number of the main graph. Also we have shown that these bounds are sharp. After that, we presented the upper bounds and the lower bounds for the co-even domination number of a graph regarding vertex (edge) removal and contraction of that as immediate result of our previous results. Future topics of interest for future research include the following suggestions:

- (i) Finding the co-even domination number of other unary operations of graphs such as subdivision of a graph (see [2, 5] for some results in this topic), power of a graph, etc.
- (ii) Finding the co-even domination number of other operations of graphs such as graph rewriting, line graph, Mycielskian, etc.
- (iii) Finding the co-even domination number of interval graphs, intersection graphs, word-representable graphs, etc.

References

- [1] S. ALIKHANI, *On the domination polynomials of non P_4 -free graphs*, Iran. J. Math. Sci. Inform., 8 (2013), pp. 49–55, 142.
- [2] C. S. BABU AND A. A. DIWAN, *Subdivisions of graphs: a generalization of paths and cycles*, Discrete Math., 308 (2008), pp. 4479–4486.
- [3] J. A. BONDY AND U. S. R. MURTY, *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976.
- [4] N. GHANBARI, *Co-even domination number of some binary operations on graphs*, Submitted. Preprint available at <https://arxiv.org/abs/2111.01845>.
- [5] N. GHANBARI, *Secure domination number of k -subdivision of graphs*. Preprint available at <https://arxiv.org/abs/2110.09190>.
- [6] T. W. HAYNES, S. T. HEDETNIEMI, AND P. J. SLATER, *Fundamentals of domination in graphs*, vol. 208 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1998.
- [7] M. MAKKI SHAALAN AND A. OMRAN, *Co-even domination in graphs*, Int. J. Control Automat., 13 (2020), pp. 330–334.
- [8] M. WALSH, *The hub number of a graph*, Int. J. Math. Comput. Sci., 1 (2006), pp. 117–124.

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