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Original Article

Co-even domination number of a modified graph by operations on a vertex or an edge

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ABSTRACT: Let $G = (V, E)$ be a simple graph. A dominating set of G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The cardinality of a smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. A dominating set D is called co-even dominating set if the degree of vertex v is even number for all $v \in V \setminus D$. The cardinality of a smallest co-even dominating set of G, denoted by $\gamma_{coe}(G)$, is the co-even domination number of G. In this paper, we study the co-even domination number of graphs which constructed by some operations on a vertex or an edge of a graph.

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Let $G = (V, E)$ be a graph with vertex set V and edge set E. Throughout this paper, we consider graphs without loops and directed edges. For each vertex $v \in V$, the set $N_G(v) = \{u \in V | uv \in E\}$ refers to the open neighbourhood of v and the set $N_G[v] = N_G(v) \cup \{v\}$ refers to the closed neighbourhood of v in G. The degree of v, denoted by $\deg(v)$, is the cardinality of $N_G(v)$. A set $D \subseteq V$ is a dominating set if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. There are various domination numbers in the literature. For a detailed treatment of domination theory, the reader is referred to [6].

A dominating set D is called a co-even dominating set, if the degree of each vertex in $V \setminus D$ is even ([7]). The cardinality of a smallest co-even dominating set of G, denoted by $\gamma_{coe}(G)$, is the co-even domination number of G. Ghanbari in [4], considered binary operations of graphs and presented some bounds for the co-even domination number of join, corona, neighbourhood corona, and Hajós sum of two graphs. In this paper, we present more results for the co-even domination number.

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In the next section, first we mention the definition of vertex and edge removal of a graph and then study the co-even domination number of a graph which constructed by vertex and edge removal and find some bounds for them. In Section 3, we mention the definition of vertex and edge contraction and study the co-even domination number of vertex and edge contraction of a graph. We find some bounds for them, and finally, we present the upper and the lower bounds for the co-even domination number of a graph regarding the vertex (edge) removal and contraction.

1. Vertex and edge removal of a graph

The graph $G-v$ is a graph that is made by deleting the vertex v and all edges connected to v from the graph G and the graph $G-e$ is a graph that obtained from G by simply removing the edge e. Our main results in this section are obtaining some bounds for the co-even domination number of the vertex and the edge removal of a graph. First, we state some known results.

Proposition 1.1 ([7]). *Let $G=(V,E)$ be a graph and D is a co-even dominating set of G. Then,*

(i) All vertices of odd or zero degrees belong to every co-even dominating set.

(ii) $\deg(v)\geq 2$, *for all* $v\in V-D$.

(iii) $\gamma(G)\leq\gamma_{coe}(G)$.

By the definition of co-even domination number, we have the following easy result:

Proposition 1.2. *Let G be a disconnected graph with components G_1 and G_2. Then*

$$\gamma_{coe}(G)=\gamma_{coe}(G_1)+\gamma_{coe}(G_2).$$

Now, we consider the vertex removal of a graph and find an upper bound and a lower one for the co-even domination number of the constructed graph.

Theorem 1.3. *Let $G=(V,E)$ be a graph and $v\in V$. Then,*

$$\gamma_{coe}(G)-\deg(v)-1\leq\gamma_{coe}(G-v)\leq\gamma_{coe}(G)+\deg(v)-1.$$

Proof. Suppose that $v\in V$ and $D_{coe}(G)$ is a co-even dominating set of G. First, we find the upper bound for $\gamma_{coe}(G-v)$. We consider the following cases:

(i) $\deg(v)$ is even and $v\notin D_{coe}(G)$. Then at least one of the neighbours of v should be in $D_{coe}(G)$. Now by adding all other neighbours of v in $D_{coe}(G)$, we have a co-even dominating set for $G-v$. The size of this set is at most $\gamma_{coe}(G)-1+\deg(v)$.

(ii) $\deg(v)$ is even and $v\in D_{coe}(G)$. By removing v and all edges related to it, some of the vertices in its neighbour, may not dominate with any other vertex. So by adding all of the neighbours of v in $D_{coe}(G)-\{v\}$, we have a co-even dominating set for $G-v$. So $\gamma_{coe}(G-v)\leq\gamma_{coe}(G)+\deg(v)-1$.

(iii) $\deg(v)$ is odd. Then by Proposition 1.1, $v\in D_{coe}(G)$. Now by the same argument as (ii), we have $\gamma_{coe}(G-v)\leq\gamma_{coe}(G)+\deg(v)-1$.

Therefore $\gamma_{coe}(G-v)\leq\gamma_{coe}(G)+\deg(v)-1$. Now, we find the lower bound for $\gamma_{coe}(G-v)$. First, we remove v and all the corresponding edges to it. Now we find a co-even dominating set for $G-v$. Suppose that this set is $D_{coe}(G-v)$. One can easily check that $D_{coe}(G-v)\cup N_G[v]$ is a co-even dominating set of G. So

$$\gamma_{coe}(G)\leq\gamma_{coe}(G-v)+\deg(v)+1,$$

and therefore we have the result. □

Remark 1.4. *The bounds in Theorem 1.3 are sharp. For the upper bound, it suffices to consider G as shown in Figure 1. The set of black vertices in G is a co-even dominating set of G. Now, by removing vertex v with degree 4, the set of black vertices is a co-even dominating set of $G-v$, and $\gamma_{coe}(G-v)=\gamma_{coe}(G)+\deg(v)-1$. For the lower bound, it suffices to consider H as shown in Figure 2. The set of black vertices in H and $H-v$ are co-even dominating sets of H and $H-v$, respectively. So $\gamma_{coe}(H-v)=\gamma_{coe}(H)-\deg(v)-1$.*

The following example shows that $\gamma_{coe}(G)$ and $\gamma_{coe}(G-v)$ can be equal.

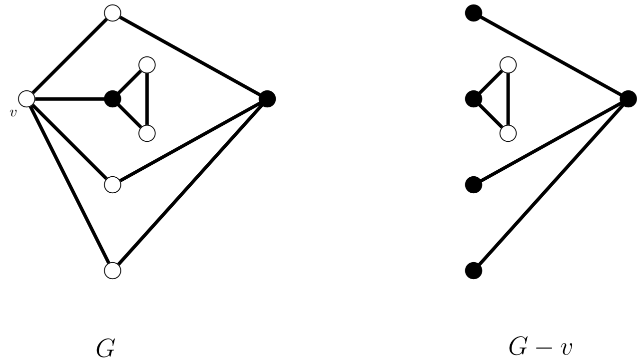

Figure 1: Graphs G and $G - v$

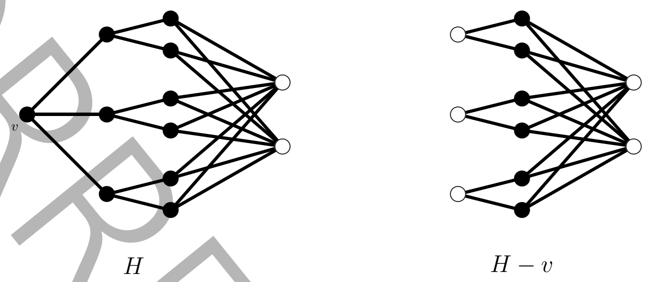

Figure 2: Graphs H and $H - v$

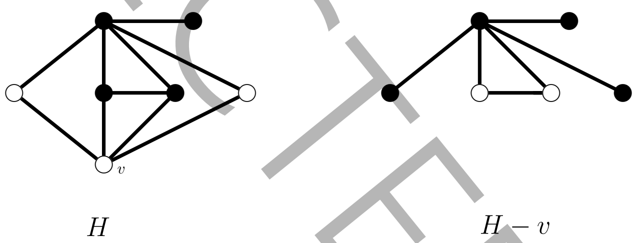

Figure 3: Graphs H and $H - v$ with same co-even domination numbers

Example 1.1. *Consider the graphs H and $H - v$ as shown in Figure 3. Obviously, the set of black vertices of each graph is co-even dominating set with smallest size. Therefore, there are some graphs G such that $\gamma_{coe}(G) = \gamma_{coe}(G - v)$.*

Now, we consider the edge removing of a graph and present the upper and the lower bound for the constructed graph.

Theorem 1.5. *Let $G = (V, E)$ be a graph and $e \in E$. Then,*

$$\gamma_{coe}(G) - 2 \leq \gamma_{coe}(G - e) \leq \gamma_{coe}(G) + 2.$$

Proof. Suppose that $e = uv \in E$ and $D_{coe}(G)$ is a co-even dominating set of G. First we find the upper bound for $\gamma_{coe}(G - e)$. We consider the following cases:

(i) $u, v \notin D_{coe}(G)$. In this case, the degree of these vertices should be even. Now by removing e, the degree of these vertices are odd and they should be in co-even dominating set of $G - e$. Now by considering $D_{coe}(G) \cup \{u, v\}$ as a dominating set of $G - e$, we have a co-even dominating set for that with size $\gamma_{coe}(G) + 2$. So $\gamma_{coe}(G - e) \leq \gamma_{coe}(G) + 2$.

(ii) $u \in D_{coe}(G)$ and $v \notin D_{coe}(G)$. In this case, the degree of v should be even. Now by removing e and the same argument as previous case, $D_{coe}(G) \cup \{v\}$ is a co-even dominating set of $G - e$ and $\gamma_{coe}(G - e) \leq \gamma_{coe}(G) + 1$.

(iii) $u, v \in D_{coe}(G)$. By removing edge e and considering $D_{coe}(G)$ as a domination set of $G - e$, we have a co-even dominating set for $G - e$ too. So $\gamma_{coe}(G - e) \leq \gamma_{coe}(G)$.

Now, we find the lower bound for $\gamma_{coe}(G - e)$. First, we remove e. At this step, we find a co-even dominating set for $G - e$. It is easy to see that $D_{coe}(G - e) \cup \{u, v\}$ is a co-even dominating set of G. So

$$\gamma_{coe}(G) \leq \gamma_{coe}(G - e) + 2,$$

and therefore we have the result. □

We end this section by showing that the bounds are sharp in the Theorem 1.5.

Remark 1.6. *The bounds in Theorem 1.5 are sharp. For the upper bound, it suffices to consider G as shown in Figure 4. The set of black vertices in G is a co-even dominating set of G. Now, by removing edge e, the set of black vertices is a co-even dominating set of $G - e$, and $\gamma_{coe}(G - e) = \gamma_{coe}(G) + 2$. For the lower bound, it suffices to consider H as shown in Figure 5. The set of black vertices in H and $H - e$ are co-even dominating sets of H and $H - e$, respectively. So $\gamma_{coe}(H - e) = \gamma_{coe}(H) - 2$.*

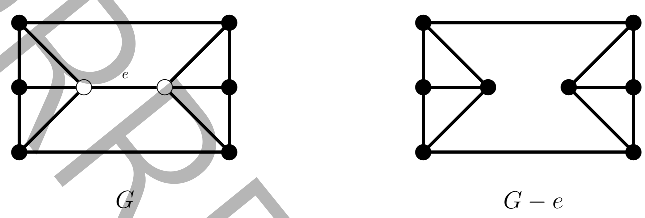

Figure 4: Graphs G and $G - e$

Figure 5: Graphs H and $H - e$

2. Vertex and edge contraction of a graph

Let v be a vertex in graph G. The contraction of v in G denoted by G/v is the graph obtained by deleting v and putting a clique on the open neighbourhood of v. Note that this operation does not create parallel edges; if two neighbours of v are already adjacent, then they remain simply adjacent (see [1, 8]). In a graph G, contraction of an edge e with endpoints u, v is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v. The resulting graph G/e has one less edge than G ([3]). We denote this graph by G/e. In this section, we examine the effects on $\gamma_{coe}(G)$ when G is modified by an edge contraction and vertex contraction. First, we consider the vertex contraction of a graph and find the upper and the lower bound for its co-even domination number.

Theorem 2.1. *Let $G = (V, E)$ be a graph and $v \in V$. Then,*

$$\gamma_{coe}(G) - \deg(v) - 1 \leq \gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1.$$

Proof. Suppose that $v \in V$ and $D_{coe}(G)$ is co-even dominating set of G. First we find the upper bound for $\gamma_{coe}(G/v)$. We consider the following cases:

(i) $\deg(v)$ is odd. So $v \in D_{coe}(G)$. Now, by deleting v and putting a clique on the open neighbourhood of v, we have G/v. One can easily check that

$$(D_{coe}(G) - \{v\}) \cup N_G(v)$$

is a co-even dominating set for G/v with size at most $\gamma_{coe}(G) - 1 + \deg(v)$. So in this case, $\gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1$.

(ii) $\deg(v)$ is even and $v \in D_{coe}(G)$. By the same argument as previous case, we conclude that $\gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1$.

(iii) $\deg(v)$ is even and $v \notin D_{coe}(G)$. Then at least one vertex in $N_G(v)$ should be in $D_{coe}(G)$. Now $D_{coe}(G) \cup N_G(v)$ is a co-even dominating set for G/v with size at most $\gamma_{coe}(G) - 1 + \deg(v)$, and $\gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1$.

Therefore $\gamma_{coe}(G/v) \leq \gamma_{coe}(G) + \deg(v) - 1$. Now we find the lower bound for $\gamma_{coe}(G/v)$. First we remove v and put a clique in open neighbourhood of that. Now we find a co-even dominating set for G/v. It is possible that we have t vertices from $N_G(v)$ in $D_{coe}(G/v)$, where $0 \leq t \leq \deg(v)$. Now we keep our dominating set for G/v and remove all the edges we added before and add vertex v and all corresponding edges to that. In any case, $D_{coe}(G/v) \cup N_G[v]$ is a co-even dominating set for G. So

$$\gamma_{coe}(G) \leq \gamma_{coe}(G/v) + \deg(v) + 1,$$

and therefore we have the result. □

Remark 2.2. *The bounds in Theorem 2.1 are sharp. For the upper bound, it suffices to consider G as shown in Figure 6. The set of black vertices in G and G/v are co-even dominating sets of G and G/v, respectively. Hence $\gamma_{coe}(G/v) = \gamma_{coe}(G) + \deg(v) - 1$. For the lower bound, it suffices to consider H as shown in Figure 7. The set of black vertices in H and H/v are co-even dominating sets of them, respectively. So $\gamma_{coe}(H/v) = \gamma_{coe}(H) - \deg(v) - 1$.*

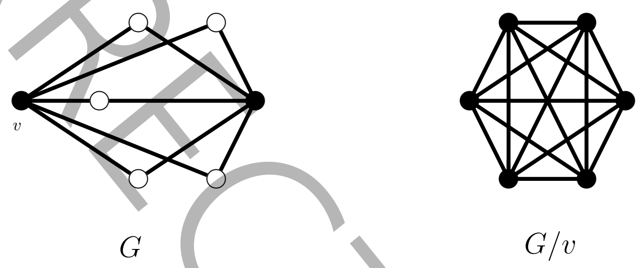

Figure 6: Graphs G and G/v

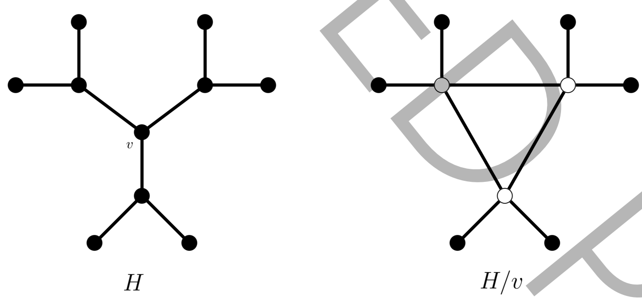

Figure 7: Graphs H and H/v

As an immediate result of Theorems 1.3 and 2.1, we have:

Corollary 2.3. *Let $G = (V, E)$ be a graph and $v \in V$. Then,*

$$\frac{\gamma_{coe}(G - v) + \gamma_{coe}(G/v)}{2} - \deg(v) + 1 \leq \gamma_{coe}(G) \leq \frac{\gamma_{coe}(G - v) + \gamma_{coe}(G/v)}{2} + \deg(v) + 1.$$

Now we consider the edge contraction of a graph and find upper and lower bound for co-even domination number of that.

Theorem 2.4. *Let $G = (V, E)$ be a graph and $e \in E$. Then,*

$$\gamma_{coe}(G) - 2 \leq \gamma_{coe}(G/e) \leq \gamma_{coe}(G).$$

Proof. Suppose that $e = uv \in E$ and $D_{coe}(G)$ is co-even dominating set of G. Also let w be the vertex which is replacement of u and v in G/e. First, we find the upper bound for $\gamma_{coe}(G/e)$. We consider the following cases:

(i) $u, v \notin D_{coe}(G)$. In this case, the degree of these vertices should be even. Now by removing e, the degree of these vertices are odd and therefore the degree of w is even in G/e. Now by considering $D_{coe}(G)$ as a dominating set of G/e too, we have a co-even dominating set for that with size $\gamma_{coe}(G)$. So $\gamma_{coe}(G/e) \leq \gamma_{coe}(G)$.

(ii) $u \in D_{coe}(G)$ and $v \notin D_{coe}(G)$. In this case, $(D_{coe}(G) - \{u\}) \cup \{w\}$ is a co-even dominating set of G/e and $\gamma_{coe}(G/e) \leq \gamma_{coe}(G)$.

(iii) $u, v \in D_{coe}(G)$. In this case, $(D_{coe}(G) - \{u, v\}) \cup \{w\}$ is a co-even dominating set of G/e and $\gamma_{coe}(G/e) \leq \gamma_{coe}(G) - 1$.

So $\gamma_{coe}(G/e) \leq \gamma_{coe}(G)$. Now we find the lower bound for $\gamma_{coe}(G/e)$. First we consider G/e and find a co-even dominating set for that. In the worst case, $w \notin D_{coe}(G/e)$ and the degree of u and v are odd in G. So in any case, $(D_{coe}(G) - \{w\}) \cup \{u, v\}$ is a co-even dominating set for G. Hence $\gamma_{coe}(G) \leq \gamma_{coe}(G/e) + 2$, and therefore we have the result. □

Remark 2.5. *The bounds in Theorem 2.4 are sharp. For the upper bound, it suffices to consider G as shown in Figure 8. The set of black vertices in G is a co-even dominating set of G. Also, the set of black vertices in G/e is a co-even dominating set of G/e, and $\gamma_{coe}(G/e) = \gamma_{coe}(G)$. For the lower bound, it suffices to consider H as shown in Figure 9. The set of black vertices in H and H/e are co-even dominating sets of them, respectively. So $\gamma_{coe}(H/e) = \gamma_{coe}(H) - 2$.*

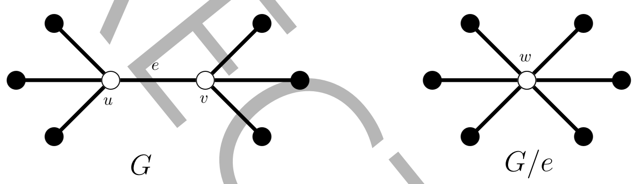

Figure 8: Graphs G and G/e

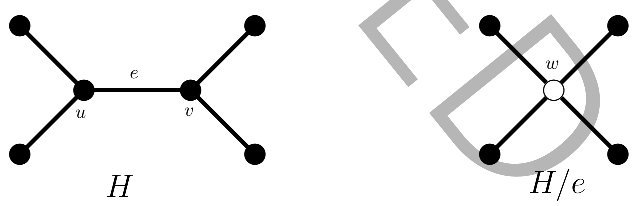

Figure 9: Graphs H and H/e

In Theorem 2.4, we showed that $\gamma_{coe}(G) - 2 \leq \gamma_{coe}(G/e) \leq \gamma_{coe}(G)$, for every $e \in E$. Also in Remark 2.5, we concluded that these bounds are sharp. In the following example, we show that $\gamma_{coe}(G/e)$ can be $\gamma_{coe}(G) - 1$ too:

Example 2.1. *Consider G as shown in Figure 10. The set of black vertices in G is a co-even dominating set of G. Also, the set of black vertices in G/e is a co-even dominating set of G/e, and $\gamma_{coe}(G/e) = \gamma_{coe}(G) - 1$.*

We end this section by an immediate result of Theorems 1.5 and 2.4:

Corollary 2.6. *Let $G = (V, E)$ be a graph and $e \in E$. Then,*

$$\frac{\gamma_{coe}(G - e) + \gamma_{coe}(G/e)}{2} - 1 \leq \gamma_{coe}(G) \leq \frac{\gamma_{coe}(G - e) + \gamma_{coe}(G/e)}{2} + 2.$$

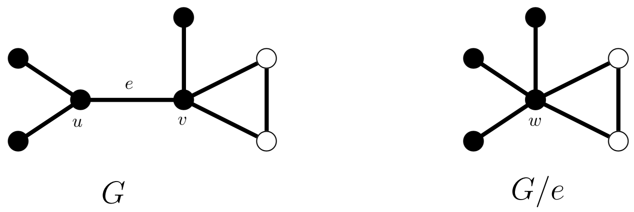

Figure 10: Graphs G and G/e

3. Conclusions

In this paper, we obtained some lower bounds and upper bounds for the co-even domination number of graphs which constructed by the vertex removing, the edge removing, and also vertex and edge contraction regarding the co-even domination number of the main graph. Also we have shown that these bounds are sharp. After that, we presented the upper bounds and the lower bounds for the co-even domination number of a graph regarding vertex (edge) removal and contraction of that as immediate result of our previous results. Future topics of interest for future research include the following suggestions:

(i) Finding the co-even domination number of other unary operations of graphs such as subdivision of a graph (see [2, 5] for some results in this topic), power of a graph, etc.

(ii) Finding the co-even domination number of other operations of graphs such as graph rewriting, line graph, Mycielskian, etc.

(iii) Finding the co-even domination number of interval graphs, intersection graphs, word-representable graphs, etc.

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