AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 6(4) (2025) 361-370
https://doi.org/10.22060/AJMC.2024.23142.1234

Original Article

Geometry of Ricci solitons admitting a new geometric vector field

Farzaneh Shamkhali[a], Ghodratallah Fasihi-Ramandi*[a], Shahroud Azami[a]

[a] *Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran*

ABSTRACT: In the present paper, we introduce a new geometric vector field (it will be called semi-Killing field) on semi-Riemannaian manifolds. A complete classification of semi-Killing vector fields on 3-dimensional Walker manifolds will be derived. Then, we study Ricci solitons admitting this new vector field (called semi-Killing vector field) as their potential. In Riemannain setting, we prove that Ricci solitons with semi-Killing potential vector field are Einstein. Our results show that such Lorentzian solitons have constant scalar curvature. Finally, application of this new structure in theoretical physics has been investigated.

Review History:

Received:28 April 2024
Revised:05 July 2024
Accepted:22 July 2024
Available Online:01 October 2025

Keywords:

Warped product
Geometric vector field
Riemannian geometry

MSC (2020):

53C21; 53C44

1. Introduction

Ricci solitons are the natural generalization of Einstein metrics. A (semi-)Riemannian manifold (M, g) is said to be a Ricci soliton if there exists a vector field $X \in \mathcal{X}(M)$ and a real scalar λ, such that

$$\frac{1}{2}\mathcal{L}_X g + \mathrm{Ric} = \lambda g,$$

where $\mathcal{L}_X$ and Ric denote the Lie derivative in the direction of X, and the Ricci tensor, respectively.

It is called shrinking when $\lambda > 0$, steady when $\lambda = 0$, and expanding when $\lambda < 0$. If $X = \nabla f$ the equation can also be written as

$$\mathrm{Ric} + \mathrm{Hess} f = \lambda g,$$

and is called a gradient (Ricci) soliton. See [1, 4, 5, 7] for background on Ricci solitons and their connection to the Ricci flow. Ricci solitons on closed Riemannian manifolds are gradient and steady or expanding Ricci solitons on close Riemannain manifolds are trivial [12]. Also, every non-compact shrinking soliton is a gradient soliton [11].

*Corresponding author.
E-mail addresses: f.shamkhali56@gmail.com, fasihi@sci.ikiu.ac.ir, azami@sci.ikiu.ac.ir

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many researchers. There are two aspects of the study of Ricci solitons, one looking at the influence on the topology by Ricci soliton (see e.g. [6, 10]) and the other looking at its influence on its geometry (see e.g. [3, 8, 9]). In this paper we are interested in the geometry of Ricci solitons arise frome a new geometric vector fields (called semi-Killing field) on semi-Riemannain manifolds.

This paper is organized as follows: In Section 2, we define a new geometric vector field called semi-Killing vector field and find such fields in the framework of Lorentzian Walker manifolds. Then, in Section 3 we consider Ricci solitons with semi-Killing potential vector fields. As simplest example, we try to examine this new structure on gradient surface solitons. But all we obtain is only trivial solitons. In the next Section, we prove our main theorem that shows why we could not construct any non-trivial example of surface Riemannain solitons. Finally, physical application of this structure is considered.

2. A new geometric vector field

The investigation of geometric vector fields on manifolds provide us some good information about the geometry of underlying manifolds. The study of affine vector fields on a manifold M with linear connection ∇ gives information on the affine transformation group of (M, ∇). Similarly, studying of Killing vector fields on a Riemannian manifold tells us about the isometry group of the manifold. Also, the geometry of harmonic, conformal and projective vector fields has been considered in (semi-)Riemannain manifolds and has led to many useful results. In this section, we define a new geometric vector field on (semi-)Riemannain manifold and consider its geometry in the next sections.

Definition 2.1. *A vector field X in a semi-Riemannian manifold (M, g) is said to be a semi-Killing vector field, if $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$ for some constant α where, $X^\flat$ is dual 1-form of X.*

Clearly, the zero vector field $X = 0$ is a semi-Killing vector field and every Killing vector field X is semi-Killing with $\alpha = 0$. Also, we can construct a non-trivial semi-Killing vector field. Let $M = (a, b) \subset \mathbb{R}$ be an open interval and consider $g = ds^2$. Suppose that X is a nowhere zero semi-Killing vector field on M with some non-zero α. If $X^\flat = h(s)ds$, then the condition $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$ lead us to the following ordinary differential equation

$$-2h'(s) = 2\alpha h^2(s),$$

and solving this equation gives $h(s) = \dfrac{1}{\alpha s + \beta}$ for some constant β.

Theorem 2.2. *Let X be a non-zero semi-Killing vector field on a closed (compact without boundary) Riemannain manifold (M, g). Then X is a Killing vector field.*

Proof. It is well-known that for any vector field X in a Riemannian manifold we have

$$tr_g(\mathcal{L}_X g) = 2\mathrm{div}(X).$$

Now, let X satisfies $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$, and taking the following equality into account

$$tr_g\big(2\alpha X^\flat \otimes X^\flat\big) = 2\alpha |X|_g^2,$$

we obtain

$$\mathrm{div}(X) = \alpha |X|_g^2.$$

But by the divergence theorem we can write

$$\int_M \mathrm{div}(X)\mathrm{dV}_g = 0,$$

which is to say

$$\alpha \int_M |X|_g^2 \mathrm{dV}_g = 0.$$

Since X is non-zero, hence $\alpha = 0$. □

The above theorem shows that the set of semi-Killing vector fields on closed manifolds coincides with the set of all Killing vector fields. Hence, existence of semi-Killing vector fields not only depends on the geometry of underlying manifold but also requires some topological constraints on the manifold.

Homogeneous spaces are among the nicest examples of Riemannian manifolds. In the following, we show that there is no non-trivial left-invariant semi-Killing vector field on a homogeneous manifold M with left-invariant metric g. Let X be a non-zero semi-Killing left invariant vector field on a homogeneous manifold (M, g). Then, by tracing both sides of $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$, we obtain $\mathrm{div}(X) = \alpha |X|_g^2$. Since $\mathrm{div}(X) = 0$ and $|X|_g^2 \neq 0$, we must have $\alpha = 0$.

Theorem 2.3. *Left-invariant semi-Killing vector fields on homogeneous spaces are Killing vector fields.*

For knowing the local generality (up to diffeomorphism) of the set of pair (g, X) satisfying $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$, where X is a non-vanishing, choose 'flow box' coordinates, in which $X = \dfrac{\partial}{\partial x^1}$. The equation for the unknown $g = g_{ij}dx^i dx^j$ then becomes a first order, determined system that has a unique local solutions when one specifies

$$a_{ij}(x^2, \dots, x^n) = g_{ij}(0, x^2, \dots, x^n).$$

Thus, the local solutions appear to depend on $\dfrac{1}{2}n(n+1)$ arbitrary of $n-1$ variables. However, we must bear in mind that flow box coordinate for X depend on n function of $n-1$ variables. Taking this flexibility in mind, we find that the solutions pairs (g, X) up to diffeomorphism depends on $\dfrac{1}{2}n(n-1)$ functions of $n-1$ variables. Since the general metric in n dimensions up to diffeomorphism depends on $\dfrac{1}{2}n(n+1)$ functions of n variables. It follows that we always can find solution pairs locally.

2.1. Semi-Killing vector fields on Walker 3-manifolds

Let (M^3, g) be a three-dimensional Lorentzian Walker manifold (for more details on Walker manifolds see [13]). There exists a system of local coordinates $\{t, x, y\}$ for which the vector field $\frac{\partial}{\partial t}$ spans the parallel null distribution $\mathcal{D}_1$ and the metric g takes the following form:

$$g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t,x,y) \end{bmatrix}, \tag{1}$$

where $\varepsilon = \pm 1$ and $f(t,x,y)$ is a real smooth function. Using the Koszul's formula, one can find [2]

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial t} = \frac{1}{2} f_t \frac{\partial}{\partial t}, \qquad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \frac{1}{2} f_x \frac{\partial}{\partial t},$$

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{1}{2}(f f_t + f_y) \frac{\partial}{\partial t} - \frac{1}{2\varepsilon} f_x \frac{\partial}{\partial x} - \frac{1}{2} f_t \frac{\partial}{\partial y},$$

in which ∇ denotes the Levi-Civita connection of g. The above equations indicate that the vector field $\frac{\partial}{\partial t}$ is parallel ($\nabla \frac{\partial}{\partial t} = 0$) if and only if $f(t,x,y) \equiv f(x,y)$. In this case, (M^3, g) is called a strictly Walker manifold.

Setting $X = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + C\frac{\partial}{\partial Y}$. A straightforward computation shows that

$$L_X g = \begin{bmatrix} 2C_t & \varepsilon B_t + C_x & A_t + fC_t + C_y \\ \varepsilon B_t + C_x & 2\varepsilon B_x & A_x + fC_x + \varepsilon B_y \\ A_t + fC_t + C_y & A_x + fC_x + \varepsilon B_y & X(f) + 2(A_y + fC_y) \end{bmatrix}. \tag{2}$$

By definition of $X^\flat$, we compute

$$X^\flat \otimes X^\flat = \begin{bmatrix} C^2 & \varepsilon BC & C(A + Cf) \\ \varepsilon BC & (\varepsilon B)^2 & \varepsilon B(A + Cf) \\ C(A + Cf) & \varepsilon B(A + Cf) & (A + Cf)^2 \end{bmatrix}. \tag{3}$$

Assume that $X = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + C\frac{\partial}{\partial y}$, then routine computations show that X is a semi-Killing vector field with $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$ if and only if the following system of partial differential equations holds:

$$\begin{cases} C_t - \alpha C^2 = 0, \\ (\varepsilon B_t + C_x) - 2\alpha\varepsilon BC = 0, \\ (A_t + fC_t + C_y) - 2\alpha C(A + Cf) = 0, \\ \varepsilon B_x - \alpha(\varepsilon B)^2 = 0, \\ (A_x + fC_x + \varepsilon B_y) - 2\alpha B\varepsilon(A + Cf) = 0, \\ X(f) + 2(A_y + fC_y) - 2\alpha(A + Cf)^2 = 0. \end{cases} \tag{4}$$

Case 1: Let $C \neq 0$, then first equation of (4) yields

$$C = \frac{1}{a(x,y) - \alpha t}, \tag{5}$$

where $a(x,y)$ is a smooth function. Replacing C in the second equation of the system (4), we compute

$$B_t - \frac{2\alpha}{a(x,y) - \alpha t} B = \frac{\frac{1}{\varepsilon} a_x(x,y)}{a(x,y) - \alpha t}, \tag{6}$$

which asserts that

$$B = \frac{t a_x(x,y) + b(x,y)}{\varepsilon (a(x,y) - \alpha t)^2}, \tag{7}$$

in which $b(x,y)$ is a smooth function. If $B = 0$, then we have

$$a_x(x,y) = b(x,y) = 0, \tag{8}$$

hence, $a(x,y) = a(y)$ and, $C = \dfrac{1}{a(y) - \alpha t}$. Now, we can write

$$\begin{cases} (A_t + fC_t + C_y) - 2\alpha C(A + fC) = 0, \\ A_x = 0, \\ X(f) + 2(A_y + fC_y) - 2\alpha (A + fC)^2 = 0. \end{cases} \tag{9}$$

The second equation of (9) shows that $A(t,x,y) = A(t,y)$. Since $C_t - 2\alpha C^2 \neq 0$, from the first equation of (9), we obtain

$$f = \frac{2\alpha AC - A_t - C_y}{C_t - 2\alpha C^2}. \tag{10}$$

But the left side of above equation is a function on x, y, while the right hand side is a function of t, y. So, there is a smooth function μ on y such that

$$f = \frac{2\alpha AC - A_t - C_y}{C_t - 2\alpha C^2} = \mu(y),$$

which leads us to the following equation

$$A_t - \frac{2\alpha}{a(y) - \alpha t} A = \frac{\alpha(\mu - 1)}{\left(a(y) - \alpha t\right)^2}. \tag{11}$$

Solving this equation, we obtain

$$A = \frac{\alpha t(\mu(y) - 1)}{(a(y) - \alpha t)^2},$$

Therefore, in this case we have

$$A = \frac{\alpha t(\mu(y) - 1)}{(a(y) - \alpha t)^2}, \quad B = 0, \quad C = \frac{1}{a(x,y) - \alpha t},$$

and A, B, C and f have to satisfy the third equation of (9).

On the other hand, the fourth equation of (4), when $B \neq 0$ gives

$$B = \frac{1}{d(t,y) - \alpha \varepsilon x}, \tag{12}$$

where $d(t,y)$ is a smooth function. Comparing this equality with (7) yields

$$\begin{cases} a_x(x,y) = -\varepsilon \alpha, \\ b(x,y) = \varepsilon a(x,y), \\ d(t,y) = -\alpha t + m(y), \end{cases} \tag{13}$$

for some smooth function $m(y)$. Also, we have $a(x,y) = n(y) - \alpha \varepsilon x$ for smooth function $n(y)$ Now, the third equation of (4) gives

$$A_t - \frac{2\alpha}{n(y) - \varepsilon \alpha x - \alpha t} A = \frac{\alpha f - n'(y)}{(n(y) - \varepsilon \alpha x - \alpha t)^2},$$

which shows

$$A = (n(y) - \varepsilon\alpha x - \alpha t)^2\big[p(y) - tn'(y) + \alpha t f\big],$$

for some smooth function $p(y)$. Hence, we have

$$\begin{aligned} A &= (n(y) - \varepsilon\alpha x - \alpha t)^2\big[p(y) - tn'(y) + \alpha t f\big],\\ B &= \frac{1}{m(y) - \alpha(t) - \alpha\varepsilon x},\\ C &= \frac{1}{n(y) - \varepsilon\alpha x - \alpha t}. \end{aligned}$$

In this case, A, B, C and f must satisfy the fifth and sixth equations in (4).

Case 2: If $C = 0$, then

$$\begin{cases} \varepsilon B_t = 0,\\ A_t = 0,\\ \varepsilon B_x - \alpha(\varepsilon B)^2 = 0,\\ (A_x + \varepsilon B_y) - 2\alpha\varepsilon AB = 0,\\ X(f) + 2A_y - 2\alpha A^2 = 0. \end{cases} \tag{14}$$

If $B = 0$, then $A = A(y)$ and A must satisfy

$$A_y = \alpha A^2,$$

since we want X to be non-zero, so for some constant β, we have $A(y) = \dfrac{1}{\beta - \alpha y}$.

If $B \neq 0$, then $A = A(x, y)$ and $B = B(x, y)$ and from the third equation of (14), for some smooth function $p(y)$ we have

$$B = \frac{1}{p(y) - \alpha\varepsilon x}.$$

Taking this equality into account in the fourth equation of (14), we obtain

$$A_x - \frac{2\alpha\varepsilon}{p(y) - \alpha\varepsilon x}A = \frac{\varepsilon p'(y)}{(p(y) - \alpha\varepsilon x)^2},$$

which asserts

$$A = (p(y) - \alpha\varepsilon x)^2\big[\varepsilon p'(y)x + q(y)\big],$$

where $q(y)$ is a smooth function. Hence, we have

$$\begin{aligned} A &= (p(y) - \alpha\varepsilon x)^2\big[\varepsilon p'(y)x + q(y)\big],\\ B &= \frac{1}{p(y) - \alpha\varepsilon x}, \quad C = 0. \end{aligned}$$

In this case, A and f must satisfy the fourth equation of (14).

3. Gradient Ricci soliton on surfaces

In this section, we study the gradient Ricci solitons with semi-Killing vector field as their potential on Riemannain surfaces. If (M^2, g, X, λ) is a soliton on a Riemannian surface M^2, then X is a conformal vector field. This is simply because in two dimensions the Ricci tensor can be written in terms of scalar curvature as $\mathrm{Ric} = \frac{1}{2}Rg$. If $(g, \nabla f, \lambda)$ is a gradient soliton then $J(\nabla f)$ is a Killing vector field where, $J : TM \to TM$ is the complex structure. Also, it has been proven that a surface with a Killing vector field is locally warped product [5]. In particular, a gradient soliton on a surface is locally warped product. Hence, we need to summary some geometric quantities on warped product manifolds.

3.1. Geometry of warped product manifolds

In this subsection, we consider a metric on the product manifold $M^{n+1} = \mathrm{I} \times N^n$ of the form

$$g = ds^2 + f^2(s)g_N,$$

where s is the standard coordinate on an open interval $\mathrm{I} \subset \mathbb{R}$, g_N is a given metric on the manifold N, and $f(s) > 0$ is the warping function, which scales distances along the $N-$factors in the product.

In the rest of this section, we determine geometric quantities of (M, g) in terms of $f(s)$ and geometric quantities of g_N. Smooth vector fields of N, will be denoted by $X, Y, Z, \ldots$, and every smooth vector field on I can be denoted by $\mu(s)\dfrac{\partial}{\partial s}$, for $\mu \in C^\infty(\mathrm{I})$. For the sake of simplicity, we will denote $\dfrac{\partial}{\partial s}$ by ∂_s.

Theorem 3.1. *Let ∇ deonted the Levi-Civita connection of g_N. The Levi-Civita connection of (M, g) denoted by $\overline{\nabla}$ satisfies the following relations.*

$$\overline{\nabla}_{\partial_s}\partial_s = 0, \tag{15}$$

$$\overline{\nabla}_{\partial_s}X = \frac{f'(s)}{f(s)}X, \tag{16}$$

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{f'(s)}{f(s)}g(X, Y)\partial_s. \tag{17}$$

Proof. Straightforward computations using Koszula's formula show the above relations. We prove the equation (16). We have

$$2g(\overline{\nabla}_X\partial_s, \partial_s) = \partial_s g(X, \partial_s) + Xg(\partial_s, \partial_s) - \partial_s g(X, \partial_s) + g([X, \partial_s], \partial_s) + g([\partial_s, X], \partial_s) - g([\partial_s, \partial_s], X) = 0,$$

and

$$\begin{aligned} 2g(\overline{\nabla}_X\partial_s, Y) &= \partial_s g(X, Y) + Xg(Y, \partial_s) - Yg(X, \partial_s) + g([X, \partial_s], Y) - g([\partial_s, Y], X) + g([Y, X], \partial_s) \\ &= \partial_s(f^2(s))g_N(X, Y) = 2\frac{f'(s)}{f(s)}g(X, Y), \end{aligned}$$

Hence, we obtain $\overline{\nabla}_{\partial_s}X = \dfrac{f'(s)}{f(s)}X$. □

In the rest of this section, the curvature and Ricci curvature tensors of g and g_N are denoted respectively by $\overline{R}, \overline{\mathrm{Ric}}, R, \mathrm{Ric}$.

Theorem 3.2. *Reimannain curvature tensor of (M, g) denoted by $\overline{R}$ satisfies the following relations.*

$$\overline{R}(\partial_s, X)(\partial_s) = \frac{f''(s)}{f(s)}X, \tag{18}$$

$$\overline{R}(\partial_s, X)(Y) = -\frac{f''(s)}{f(s)}g(X, Y)\partial_s, \tag{19}$$

$$\overline{R}(X, Y)(\partial_s) = 0, \tag{20}$$

$$\overline{R}(X, Y)(Z) = R(X, Y)(Z) + (\frac{f'(s)}{f(s)})^2\big(g(X, Z)Y - g(Y, Z)X\big). \tag{21}$$

Proof. Straightforward computations demonstrate the above equations. For instance we compute the equality (18).

$$\begin{aligned} \overline{R}(\partial, X)(\partial_s) &= \overline{\nabla}_{\partial_s}\overline{\nabla}_X\partial_s - \overline{\nabla}_X\overline{\nabla}_{\partial_s}\partial_s - \overline{\nabla}_{[\partial_s, X]}\partial_s \\ &= \overline{\nabla}_{\partial_s}\frac{f'(s)}{f(s)}X = \partial_s(\frac{f'(s)}{f(s)})X + \frac{f'(s)}{f(s)}\nabla_{\partial_s}X \\ &= \frac{f(s)f''(s) - (f'(s))^2}{f^2(s)}X + (\frac{f'(s)}{f(s)})^2X = \frac{f''(s)}{f(s)}X. \end{aligned}$$

□

The sectional curvature of (M, g) satisfies the following relation

$$\overline{K}(X, \frac{\partial}{\partial s}) = -\frac{f''(s)}{f(s)},$$

for every unite vector field $X \in \mathcal{X}(N)$.

Let $\{E_i\}_{i=1}^n$ be an orthonormal basis of (N, g_N) with reciprocal basis $\{E^i\}_{i=1}^n$, then $\{\partial_s, f^{-2}(s)E_i\}_{i=1}^n$ is an orthonormal basis for (M, g) with $\{\partial_s, f^{-2}(s)E^i\}_{i=1}^n$ as its reciprocal basis.

Theorem 3.3. *Ricci curvature tensor of (M, g) denoted by $\overline{\text{Ric}}$ satisfies the following relations.*

$$\overline{\text{Ric}}(\partial_s, \partial_s) = -n\frac{f''(s)}{f(s)}, \tag{22}$$

$$\overline{\text{Ric}}(\partial_s, X) = 0, \tag{23}$$

$$\overline{\text{Ric}}(X, Y) = \text{Ric}(X, Y) - [\frac{f''(s)}{f(s)} + (n-1)(\frac{f'(s)}{f(s)})^2]g(X, Y). \tag{24}$$

Proof. Directly from the definition of the Ricci tensor, we can

$$\begin{aligned}\overline{\text{Ric}}(\partial_s, \partial_s) &= g\big(\overline{R}(\partial_s, \partial_s)(\partial_s), \partial_s\big) + \sum_{i=1}^n g\big(\overline{R}(\partial_s, f^{-2}(s)E_i)(f^{-2}(s)E^i), \partial_s\big)\\ &= \sum_{i=1}^n f^{-4}(s)g\big(\overline{R}(\partial_s, E_i)(E^i), \partial_s\big)\\ &= \sum_{i=1}^n -\frac{f''(s)}{f(s)} g_N(E_i, E^i)\\ &= -n\frac{f''(s)}{f(s)}.\end{aligned}$$

□

In this setting for any function θ of the radial coordinate s, the Hessian of θ with respect to g is given by

$$\nabla^2\theta = \theta''(s)ds^2 + f(s)f'(s)\theta'(s)g_N$$

3.2. Constructing 2-dimensional steady soliton

Next, we aim to construct a complete, steady, rotationally symmetric gradient soliton metric on $\mathbb{R}^2$ with a semi-Killing vector field. Such a metric will be a warped product $\mathrm{I} \times_f S^1$ and it is to natural to assume that $\text{Ric} + \nabla^2\theta = 0$, and $\mathcal{L}_{\nabla\theta}g = 2\nabla^2\theta = 2\alpha d\theta \otimes d\theta = 2\alpha(\theta')^2 ds^2$. Then, f and θ have to satisfy

$$\begin{aligned}&f\theta'' - f'' = 0,\\ &ff'\theta' = 0,\\ &\theta'' - \alpha(\theta')^2 = 0,\\ &f(f'\theta' - f'') = 0.\end{aligned}$$

So, we must have $f' = 0$ or $\theta' = 0$. If $\theta' = 0$, then $\theta = c$ for a real constant c, and $f = as^2 + bs + c$ where $a, b, c \in \mathbb{R}$. Considering the case $f' = 0$, then by suitable translation and time dilation in s we get $f(s) = s$, (which giving the flat metric) and we have $\alpha(\theta')^2 = \theta'' = 0$. Therefore, $\theta = c$ for a real constant c.

Theorem 3.4. *The complete, steady, rotationally symmetric gradient soliton metrics on $\mathbb{R}^2$ with semi-Killing potential vector field are the flat metrics.*

3.3. Constructing 2-dimensional non-steady soliton

Now, we wish to construct a complete, expanding, rotationally symmetric gradient soliton metric on $\mathbb{R}^2$ with semi-Killing vector field. As mentioned above, such a metric will be a warped product $\mathrm{I} \times_f S^1$ and it is natural to assume that $\mathrm{Ric} + \nabla^2\theta = \lambda g$ ($\lambda \neq 0$), and $\mathcal{L}_{\nabla\theta} g = 2\nabla^2\theta = 2\alpha d\theta \otimes d\theta = 2\alpha(\theta')^2 ds^2$. Then, f and θ have to satisfy

$$\begin{aligned}
&\theta'' - \lambda = \frac{f''}{f},\\
&ff'\theta' - \lambda f^2 = ff'',\\
&ff'\theta' = 0,\\
&\theta'' - \alpha(\theta')^2 = 0.
\end{aligned}$$

The third equation shows that $\theta' = 0$ or $f' = 0$. Let us assume $\theta' = 0$, then we have $f'' + \lambda f = 0$. So,

- for $\lambda = a^2$, where $a > 0$ we have $f(s) = c_1 \cos as + c_2 \sin as$, for constant c_1, c_2,
- for $\lambda = -a^2$, where $a > 0$ we have $f(s) = c_1 \cosh as + c_2 \sinh as$, for constant c_1, c_2.

If $f' = 0$, then the second equation becomes $-\lambda f^2 = 0$, which is impossible.

Theorem 3.5. *There is no non-trivial complete, non-steady, rotationally symmetric gradient soliton metric on $\mathbb{R}^2$ with semi-Killing potential vector field.*

4. Ricci soliton with semi-Killing vector fields

In the previous section, we showed that there is no non-trivial complete, rotationally symmetric gradient soliton metric on $\mathbb{R}^2$ with semi-Killing potential vector field. In this section, we show that this result can be generalized for all Riemannian manifolds. In fact, if (M^n, g, X, λ) be a Ricci soliton with potential semi-Killing field, then X has to be a Killing vector field and (M, g) reduces to be an Einstein manifold.

Let (M, g, X, λ) be a Riemannain Ricci soliton with $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$. Then, we have

$$\mathrm{Ric} = -2\alpha X^\flat \otimes X^\flat + \lambda g.$$

By computation traces of two sides of the above equation, we find $R = -2\alpha|X|^2 + n\lambda$, so by addition suitable expression to each side of the equation, we obtain

$$\mathrm{Ric} - \frac{1}{2}Rg + (\frac{n-2}{2})\lambda g = \alpha(|X|^2 g - 2X^\flat \otimes X^\flat).$$

As Einstein tensor $\mathrm{Ric} - \frac{1}{2}Rg$ is divergence free, so the right hand side of above equation must be divergence free.

Lemma 4.1. *Let X be a non-zero vector field on a Riemannian manifold (M, g). If divergence of symmetric tensor $T := |X|^2 g - 2X^\flat \otimes X^\flat$ vanishes, then* $\mathrm{div}(X) = 0$.

Proof. Let $\{e_i\}_{i=1}^n$ is an orthonormal base on M and denote its reciprocal base by $\{e^i\}_{i=1}^n$. So, we can write

$$\begin{aligned}
\mathrm{div}(X^\flat \otimes X^\flat)(Y) &= \sum_{i=1}^4 (\nabla_{e_i} X^\flat \otimes X^\flat)(e^i, Y)\\
&= \sum_{i=1}^4 \big((\nabla_{e_i} X^\flat) \otimes X^\flat + X^\flat \otimes (\nabla_{e_i} X^\flat)\big)(e^i, Y)\\
&= \sum_{i=1}^4 \big((\nabla_{e_i} X^\flat)(e^i) X^\flat(Y) + X^\flat(e^i)(\nabla_{e_i} X^\flat)(Y)\\
&= \mathrm{div}(X^\flat)\langle X, Y\rangle + (\nabla_X X^\flat)(Y),\\
&= \mathrm{div}(X^\flat)\langle X, Y\rangle + \langle \nabla_X X, Y\rangle.
\end{aligned}$$

Also,

$$\mathrm{div}(|X|^2 g)(Y) = d(|X|^2)(Y) = Y\langle X, X\rangle = 2\langle \nabla_Y X, X\rangle.$$

The above computations show that $\mathrm{div}(T) = 0$, if and only if for all vector field Y, we have

$$\mathrm{div}(X^\flat)\langle X, Y\rangle + \langle \nabla_X X, Y\rangle - \langle \nabla_Y X, X\rangle = 0.$$

Setting $Y = X$, we obtain

$$\mathrm{div}(X^\flat)\langle X, X\rangle = 0 \implies \mathrm{div}(X) = 0,$$

as required. □

Now, we can prove the following Theorem.

Theorem 4.2. *Riemannain Ricci solitons* (M^n, g, X, λ) *with* $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$*, are Einstein manifold.*

Proof. Since (M^n, g, X, λ) is a Ricci soliton, we have

$$\mathrm{Ric} + 2\alpha X^\flat \otimes X^\flat = \lambda g.$$

If X is identically zero, then we have nothing to prove. Let X be a non-zero vector field, so Lemma 4.1 indicates that $\mathrm{div}(X) = 0$. On the other hand, we have

$$\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat.$$

Tracing both side of the above formula gives

$$\mathrm{div}(X) = \alpha |X|^2,$$

consequently, $\alpha = 0$, and we have completed the proof. □

The above theorem shows that there is no not-trivial Riemannian Ricci soliton with semi-Killing potential vector fields. Hence, we have to look for such structure in Lorentzian or other semi-Riemannain settings.

Theorem 4.3. *If* (M^n, g, X, λ) *is a Lorentz Ricci soliton with* $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$*, then* M *has constant scalar curvature* R*.*

Proof. By arguments similar to those used in the proof of previous Lemma and Theorem 4.2 we deduce that in Lorentzain case always $\mathrm{div}(X) = 0$. Now, taking trace of both sides of the following equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

gives $R = n\lambda$. □

The above theorem shows that in Lorentzian setting, type of soliton (to be shrinking, steady or expanding) is directly related to the sign of scalar curvature.

5. Application to physics

In this section, let (M^4, g, X, λ) is a Lorentz Ricci soliton which we regard it as a space-time manifold. Then, the Ricci soliton equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

become a generalization of Einstein field equation. In fact, tracing the both side of the above equation yields $R + \mathrm{div}(X) = 4\lambda$. The above equation can be rewritten as

$$\mathrm{Ric} - \frac{1}{2}Rg + \lambda g = \frac{1}{2}(\mathrm{div}(X)g - \mathcal{L}_X g).$$

In general theory of relativity, the scalar curvature R is related to distribution of mass in points of space-time, so regardless of λ which can be interpreted as cosmological constant, we may deduce that $\mathrm{div}(X)$ is related to notion of matter in space-time and $\frac{1}{2}(\mathrm{div}(X)g - \mathcal{L}_X g)$ is the momentum-energy tensor of this matter. Therefore, a Ricci soliton is a geometric structure which capable of describing matter and gravity, simultaneously.

However, the Ricci flow can be a framework for geometrization of matter in general relativity, it gives no more information about g as a potential for gravity and X as a potential for matter. Hence, it is natural to posing any other relation on X and g. If X be a Killing vector field, then the Ricci soliton equations coincides to Einstein equation in vacuum, and X gives the symmetries of this space-time. In this paper, we suggest X to satisfy the equation $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$ for a non-zero constant α. Under this assumption, the Ricci soliton equation as a generalization of Einstein field equation, becomes

$$\mathrm{Ric} - \frac{1}{2}Rg + \lambda g = \alpha(\frac{|X|^2}{2}g - X^\flat \otimes X^\flat).$$

This equation shows that symmetric 2-tensor $T = |X|^2 g - X^\flat \otimes X^\flat$ must be divergence free. Applying this fact, a similar argument with Theorem 4.2 shows that X has to be a light-like vector field.

As we mentioned before, such structure in Riemannain settings lead to $X = 0$, and the structure reduces to Einstein manifold. But, as soon as we consider this structure in Lorentzian setting, we derive new field equation, with an internal relation between X and g. So, in our theory, X can be in related to the notion of dark matter in general relativity. Because, when in small scales we consider Riemannain geometry, we do not contact to any dark matter or dark energy, but in large scales and the framework of Lorentzian manifold this notion to be appeared. These interpretations are logical facts which bear out form our theory, but physical experiments can only verify how close this theory is to reality.

References

[1] H.-D. Cao, *Geometry of Ricci solitons*, Chinese Ann. Math. Ser. B, 27 (2006), pp. 121–142.

[2] M. Chaichi, E. García-Río, and M. E. Vázquez-Abal, *Three-dimensional Lorentz manifolds admitting a parallel null vector field*, J. Phys. A, 38 (2005), pp. 841–850.

[3] B.-Y. Chen and S. Deshmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl., 19 (2014), pp. 13–21.

[4] B. Chow and D. Knopf, *The Ricci flow: an introduction*, vol. 110 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2004.

[5] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci flow*, vol. 77 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006.

[6] A. Derdzinski, *A Myers-type theorem and compact Ricci solitons*, Proc. Amer. Math. Soc., 134 (2006), pp. 3645–3648.

[7] A. Derdziński, *Ricci solitons*, Wiad. Mat., 48 (2012), pp. 1–32.

[8] S. Deshmukh, *Jacobi-type vector fields on Ricci solitons*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 55(103) (2012), pp. 41–50.

[9] S. Deshmukh, H. Alodan, and H. Al-Sodais, *A note on Ricci solitons*, Balkan J. Geom. Appl., 16 (2011), pp. 48–55.

[10] M. Fernández-López and E. García-Río, *A remark on compact Ricci solitons*, Math. Ann., 340 (2008), pp. 893–896.

[11] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math., 645 (2010), pp. 125–153.

[12] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, 2002. arXiv.

[13] A. G. Walker, *Canonical form for a Riemannian space with a parallel field of null planes*, Quart. J. Math. Oxford Ser. (2), 1 (1950), pp. 69–79.

Please cite this article using:

Farzaneh Shamkhali, Ghodratallah Fasihi-Ramandi, Shahroud Azami, Geometry of Ricci solitons admitting a new geometric vector field, AUT J. Math. Comput., 6(4) (2025) 361-370

https://doi.org/10.22060/AJMC.2024.23142.1234