AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 6(4) (2025) 341-351
https://doi.org/10.22060/AJMC.2024.22491.1161

Original Article

Weighted composition, Stević-Sharma, Volterra integral and integral type operators between Dirichlet-Zygmund spaces

Sepideh Nasresfahani*[a], Alireza Parvizi[b]

[a] *Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, Iran*
[b] *Department of Architecture, Sepehr Daneshe Moaser Institute of Higher Education, Isfahan, Iran*

ABSTRACT: In this paper, we study the boundedness and compactness of weighted composition operators between Dirichlet-Zygmund spaces. We also briefly investigate boundedness and compactness of the Stević-Sharma, Volterra-integral and integral-type operators between Dirichlet-Zygmund spaces.

Review History:

Received:19 June 2023
Revised:21 June 2024
Accepted:23 September 2024
Available Online:01 October 2025

Keywords:

Weighted composition operator
Dirichlet-Zygmund spaces
Boundedness
Compactness

MSC (2020):

47B33; 46E15; 47B38

1. Introduction

Let $\mathcal{S} = \mathcal{S}(\mathbb{D})$ be the class of all holomorphic self-maps of the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$. For $\psi \in \mathcal{S}$ and $u \in H(\mathbb{D})$, the weighted composition operator, induced by ψ and u is given by

$$W_{u,\psi}(f) := u \cdot f \circ \psi, \quad f \in H(\mathbb{D}).$$

We can regard this operator as a generalization for a multiplication operator M_u and a composition operator C_ψ induced by ψ, where $M_u f = u \cdot f$ and $C_\psi f = f \circ \psi$. An extensive study concerning the theory of (weighted) composition operators has been established during the past four decades on various settings. We refer to standard references [6, 14, 22] and [10] for various aspects about the theory of composition operators acting on holomorphic function spaces, especially the problems of relating operator-theoretic properties of C_ψ to function theoretic properties of ψ. The differentiation operator D is defined by $Df = f'$, for $f \in H(\mathbb{D})$. Note that D is typically unbounded on many

*Corresponding author.
E-mail addresses: sepide.nasr@gmail.com, alirezaa.parvizi@gmail.com

familiar spaces of holomorphic functions. The differential operator plays an important role in various fields such as dynamical system theory and operator theory.

For $\psi \in \mathcal{S}$, $u_1, u_2 \in H(\mathbb{D})$, the following sum of operators, has been introduced by Stević et al. [16] and is called as Stević-Sharma operator

$$T_{u_1,u_2,\psi}(f) = W_{u_1,\psi}(f) + W_{u_2,\psi}D(f) = u_1 \cdot f \circ \psi + u_2 \cdot f' \circ \psi. \tag{1}$$

One of the reasons why this operator is of some importance, is that all products of multiplication, composition and differentiation operators can be obtained from the operator $T_{u_1,u_2,\psi}$ by choosing appropriate u_1, u_2. For recent studies about the Stević-Sharma operator on various holomorphic function spaces, we refer to [1, 2, 19, 18] and references therein.

For any analytic function $\xi \in H(\mathbb{D})$, the Volterra integral operator V_ξ, may be defined as follows

$$V_\xi(f) = \int_0^z f(w)\xi'(w)dw, \quad f \in H(\mathbb{D}).$$

Moreover, the integral type operator I_ξ is defined by

$$I_\xi(f) = \int_0^z f'(w)\xi(w)dw, \quad f \in H(\mathbb{D}).$$

The importance of these two operators comes from the fact that

$$V_\xi(f) + I_\xi(f) = M_\xi f - f(0)\xi(0),$$

where $M_\xi f(z) = \xi(z)f(z)$ is the multiplication operator. In [13], Pommerenke introduced and studied the Volterra integral operator V_ξ on Hardy spaces. After that many researchers considered Volterra integral and Integral type operators on analytic function spaces. See for example [9, 15, 21].

There are very few investigates about the weighted composition operator, Stević-Sharma, Volterra integral and integral type operators in the setting of spaces of weak vector valued holomorphic functions. The main concern of the present paper is to discuss the boundedness and compactness of these operators on weak vector valued (scalar valued) Dirichlet-Zygmund spaces. To this end, we first recall our function spaces to work on. We denote by $H^\infty(\mathbb{D})$, the space of all analytic functions with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Let dA be the area measure on $\mathbb{D}$ normalized to have the total mass 1. For $1 \le p < \infty$ and $\alpha > -1$, the weighted Bergman space $A_{p,\alpha}(\mathbb{D})$ is the space of all holomorphic functions f on $\mathbb{D}$ for which the norm

$$\|f\|_{A_{p,\alpha}} := \left\{ \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right\}^{\frac{1}{p}}$$

is finite, where $(1 - |z|^2)^\alpha dA(z) = dA_\alpha(z)$. The analytic Dirichlet space $\mathcal{B}_{p,\alpha}(\mathbb{D})$, is the space of all functions $f \in H(\mathbb{D})$, for which

$$\|f\|^p_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = \int_{\mathbb{D}} |f'(z)|^p(1 - |z|^2)^\alpha dA(z) + |f(0)| < \infty.$$

The analytic function $f \in H(\mathbb{D})$, is considered to be in Dirichlet-Zygmund space $\mathcal{Z}_{p,p-1}(\mathbb{D})$, if the following norm

$$\|f\|^p_{\mathcal{Z}_{p,p-1}(\mathbb{D})} = \|f'\|^p_{\mathcal{B}_{p,p-1}} + |f(0)| = \|f''\|^p_{A_{p,p-1}} + |f'(0)| + |f(0)|$$

is finite. To the best of our knowledge, [23], is the only work to study Dirichlet-Zygmund spaces, where X. Zhu considered the boundedness and compactness of weighted composition operators from Dirichlet-Zygmund spaces into Zygmund type and Bloch type spaces.

Let X be a complex Banach space. The corresponding weak version vector-valued Dirichlet-Zygmund space $w\mathcal{Z}_{p,p-1}(X)$ consists of the analytic functions $f : \mathbb{D} \to X$ for which $x^* \circ f \in \mathcal{Z}_{p,p-1}(\mathbb{D})$ for every $x^* \in X^*$, equipted with the following norm

$$\|f\|_{w\mathcal{Z}_{p,p-1}(X)} = \sup_{x^*\in B_{X^*}} \|x^* \circ f\|_{\mathcal{Z}_{p,p-1}(\mathbb{D})} < \infty.$$

Here and in the sequel, X^* is the dual space of X and $B_{X^*} = \{x^* \in X^* : \|x^*\|_{X^*} \le 1\}$ is the closed unit ball of X^*. In fact, such weak version spaces $wE(X)$ can be introduced under more general conditions on any Banach spaces E consisting of holomorphic functions, see [3, 4, 7, 8, 12] and references therein.

In this paper, our aim is to characterize the boundedness and compactness of the weighted composition, Stević-Sharma, Volterra-integral and Integral type operator between weak vector valued (scalar valued) Dirichlet-Zygmund spaces.

Throughout this paper, constants are denoted by C, they are positive and not necessary the same as each occurrence. Also we use $A \preceq B$ if there exists a constant $C > 0$, such that $A \le CB$.

2. Boundedness

The following key lemma, which gives us a characterization on Dirichlet-Zygmund spaces, will help us to prove our main results.

Lemma 2.1. *Let $1 < p < \infty$. Then*

(i) for any $f \in \mathcal{B}_{p,p-1}(\mathbb{D})$,

$$|f(z)| \preceq \|f\|_{\mathcal{B}_{p,p-1}}.$$

(ii) for any $f \in \mathcal{Z}_{p,p-1}(\mathbb{D})$,

$$|f(z)| \preceq \|f\|_{\mathcal{Z}_{p,p-1}}, \quad |f'(z)|(1-|z|^2)^{1/p} \preceq \|f\|_{\mathcal{Z}_{p,p-1}}$$

Proof. We refere part (i) to [11] and part (ii) to [23]. □

Cuckovic and Zhao in [17], characterized the boundedness of operator $W_{u,\psi}$ between weighted Bergman spaces, in terms of an integral operator, as follows.

Lemma 2.2. *Let $-1 < \alpha, \beta < \infty$, ψ be an analytic self map on $\mathbb{D}$ and $u \in H(\mathbb{D})$. If $0 < p < \infty$, then the weighted composition operator $W_{u,\psi} : A_{p,\alpha}(\mathbb{D}) \to A_{p,\beta}$ is bounded if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2})^{2+\alpha} |u(w)|^p dA_\beta(w) < \infty.$$

Now we provide a characterization for the boundedness of operator $W_{u,\psi}$ between Dirichlet-Zygmund spaces.

Theorem 2.3. *Let $1 < p < \infty$, ψ be an analytic self map on $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements are equivalent:*
(a) $W_{u,\psi}$ is bounded on $w\mathcal{Z}_{p,p-1}(X)$.
(b) $W_{u,\psi}$ is bounded on $\mathcal{Z}_{p,p-1}(\mathbb{D})$.
(c) $u, u\psi \in \mathcal{Z}_{p,p-1}$ and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2})^{p+1} |u(w)\psi'^2(w)|^p dA_{p-1}(w) < \infty.$$

Proof. (a)⇒(b). For any analytic function $f \in \mathcal{Z}_{p,p-1}(\mathbb{D})$ and $\nu \in X$ with $||\nu|| = 1$, define $h : \mathbb{D} \to X$, such that $h(z) = \nu f(z)$, for any $z \in \mathbb{D}$. Then

$$\begin{aligned}(\nu^* \circ h)'(z) &= \lim_{w\to z} \frac{\nu^*(\nu f(w)) - \nu^*(\nu f(z))}{w-z} = \lim_{w\to z} \frac{f(w)\nu^*(\nu) - f(z)\nu^*(\nu)}{w-z} \\ &= \nu^*(\nu) f'(z).\end{aligned}$$

Hence $(\nu^* \circ h)''(z) = (nu^* \nu f'(z))' = \nu^*(\nu) f''(z)$ and we obtain

$$\begin{aligned}||h||^p_{w\mathcal{Z}_{p,p-1}(X)} &= \sup_{||\nu^*||_{X^*}\le 1} \left(\int_{\mathbb{D}} |(\nu^* \circ h)''(z)|^p dA_{p-1}(z) + |(\nu^* \circ h)'(0)| + |(\nu^* \circ h)(0)| \right) \\ &= \sup_{||\nu^*||_{X^*}\le 1} \left(\int_{\mathbb{D}} |\nu^*(\nu) f''(z)|^p dA_{p-1}(z) + |\nu^*(\nu) f'(0)| + |\nu^*(\nu) f(0)| \right) \\ &= \int_{\mathbb{D}} |f''(z)|^p dA_{p-1}(z) + |f'(0)| + |f(0)| = \|f\|^p_{\mathcal{Z}_{p,p-1}(\mathbb{D})}. \end{aligned} \tag{2}$$

In a similar way, we get

$$\begin{aligned}\|W_{u,\psi} h\|^p_{w\mathcal{Z}_{p,p-1}(X)} &= \sup_{\|\nu^*\|_{X^*}\le 1} \left(\int_{\mathbb{D}} |(\nu^*(uC_\psi h))''(z)|^p dA_{p-1}(z) + \left|\nu^*(uC_\psi h))'(0)\right| + \left|(\nu^*(uC_\psi h))(0)\right| \right) \\ &= \sup_{\|\nu^*\|_{X^*}\le 1} \left(\int_{\mathbb{D}} |(\nu^* uC_\psi(\nu f))''(z)|^p dA_{p-1}(z) + \left|\nu^* uC_\psi(\nu f))'(0)\right| + \left|(\nu^* uC_\psi(\nu f))(0)\right| \right) \\ &= \sup_{\|\nu^*\|_{X^*}\le 1} \left(\int_{\mathbb{D}} |\nu^*(\nu)(uC_\psi f)''(z)|^p dA_{p-1}(z) + \left|\nu^*(\nu)(uC_\psi f)'(0)\right| + \left|\nu^*(\nu)(uC_\psi f)(0)\right| \right) \\ &= \int_{\mathbb{D}} |(uC_\psi f)''(z)|^p dA_{p-1}(z) + |(uC_\psi f)'(0)| + |(uC_\psi)(0)| = \|W_{u,\psi} f\|^p_{\mathcal{Z}_{p,p-1}(\mathbb{D})}. \end{aligned} \tag{3}$$

Notice that, in the details of equations (2) and (3), equality happens becouse all the functions get their suprimum at the same point. By applying equations (2) and (3) it's easy to see that, the boundedness of $W_{u,\psi}$ on $w\mathcal{Z}_{p,p-1}(X)$ gives us the boundedness of $W_{u,\psi}$ on $\mathcal{Z}_{p,p-1}(\mathbb{D})$.
(b)⇒(a). For any $h \in w\mathcal{Z}_{p,p-1}$ and $v^* \in X^*$, according to the definition of $w\mathcal{Z}_{p,p-1}$, we have that $v^* \circ h \in \mathcal{Z}_{p,p-1}$. But we have supposed that $W_{u,\psi}$ is bounded on $\mathcal{Z}_{p,p-1}$, hence

$$\begin{aligned}\|W_{u,\psi}(h)\|_{w\mathcal{Z}_{p,p-1}} &= \sup_{\|v^*\|_{X^*}\leq 1} \|v^* \circ W_{u,\psi}h\|_{\mathcal{Z}_{p,p-1}} = \sup_{\|v^*\|_{X^*}\leq 1} \|W_{u,\psi}(v^* \circ h)\|_{\mathcal{Z}_{p,p-1}} \\ &\leq \sup_{\|v^*\|_{X^*}\leq 1} \|v^* \circ h\|_{\mathcal{Z}_{p,p-1}} = \|h\|_{w\mathcal{Z}_{p,p-1}},\end{aligned}$$

which gives us the desired result.

(b)⇒(c). Let $W_{u,\psi}$ is bounded on $\mathcal{Z}_{p,p-1}$, then $\|W_{u,\psi}(f)\|_{\mathcal{Z}_{p,p-1}} \preceq \|f\|_{\mathcal{Z}_{p,p-1}}$, for all $f \in \mathcal{Z}_{p,p-1}$. Now for any $f \in A_{p,p-1}$, assume that $h(z) = \int_0^z f(w)dw$ and $g(z) = \int_o^z h(w)dw$, then we have that $g'(z) = h(z) \in \mathcal{B}_{p,p-1}$, $g'(0) = g(0) = 0$. So $g''(z) = f(z) \in A_{p,p-1}$ and $g(z) \in \mathcal{Z}_{p,p-1}$. On the other hand

$$\begin{aligned}(u.g(\psi(z)))'' &= u''(z)g(\psi(z)) + 2u'(z)\psi'(z)g'(\psi(z)) \\ &\quad + u(z)\psi''(z)g'(\psi(z)) + u(z)\psi'^2(z)g''(\psi(z)).\end{aligned}$$

Applying the boundedness of $W_{u,\psi}$ on $\mathcal{Z}_{p,p-1}$, for functions $f_1(z) = 1$ and $f_2(z) = Id$, give us

$$\|u''\psi + \psi''u + 2\psi'u'\|_{A_{p,p-1}} = \|(u\psi)''\|_{A_{p,p-1}} \leq \|W_{u,\psi}(f_2)\|_{\mathcal{Z}_{p,p-1}} < \infty, \tag{4}$$

$$\|u''\|_{A_{p,p-1}} = \|W_{u,\psi}(f_1)\|_{\mathcal{Z}_{p,p-1}} < \infty. \tag{5}$$

Then by the boundedness of ψ on $\mathbb{D}$ and the equations (4) and (5), we obtain $\|\psi''u + 2u'\psi'\|_{A_{p,p-1}} < \infty$. But

$$\begin{aligned}\|W_{u\psi'^2,\psi}(f)\|_{A_{p,p-1}} &= \|W_{u\psi'^2,\psi}(g'')\|_{A_{p,p-1}} = \|u\psi'^2 g'' \circ \psi\|_{A_{p,p-1}} \\ &= \|(ug \circ \psi)'' - (u\psi'' + 2u'\psi')g' \circ \psi - u''g \circ \psi\|_{A_{p,p-1}} \\ &\leq \|W_{u,\psi}(g)\|_{\mathcal{Z}_{p,p-1}} + \|g'\|_\infty \|u\psi'' + 2u'\psi'\|_{A_{p,p-1}} + \|g\|_\infty \|u''\|_{A_{p,p-1}}.\end{aligned}$$

By applying Lemma 2.1 and the boundedness of $W_{u,\psi}$ on $\mathcal{Z}_{p,p-1}$, we get that

$$\begin{aligned}\|W_{u\psi'^2,\psi}(f)\|_{A_{p,p-1}} &\preceq (\|u\psi'' + 2u'\psi'\|_{A_{p,p-1}} + \|u\|_{\mathcal{Z}_{p,p-1}} + 1)\|g\|_{\mathcal{Z}_{p,p-1}} \\ &\preceq \|g\|_{\mathcal{Z}_{p,p-1}} = \|g''\|_{A_{p,p-1}} = \|f\|_{A_{p,p-1}}.\end{aligned}$$

Which implies the boundedness of operator $W_{u\psi'^2,\psi} : A_{p,p-1}(\mathbb{D}) \to A_{p,p-1}(\mathbb{D})$. Then equations (4) and (5) along with Lemma 2.2, give us the desired result.
(c)⇒(b). With the assumptions in (c) and Lemma 2.2, we get that $u, u.\psi \in \mathcal{Z}_{p,p-1}(\mathbb{D})$ and operator $W_{u\psi'^2,\psi}$ is bounded between weighted Bergman spaces $A_{p,p-1}(\mathbb{D})$. Also

$$\|u\psi'' + 2u'\psi'\|_{A_{p,p-1}} \leq \|(u.\psi)''\|_{A_{p,p-1}} + \|u''\psi\|_{A_{p,p-1}} < \infty, \tag{6}$$

and since for any $g \in \mathcal{Z}_{p,p-1}(\mathbb{D})$, $g'' \in A_{p,p-1}(\mathbb{D})$, we have that

$$\|u\psi'^2 g''\|_{A_{p,p-1}} \leq \|g''\|_{A_{p,p-1}}.$$

Therefore,

$$\begin{aligned}\|(ug \circ \psi)''\|_{A_{p,p-1}} &\leq \|u''\|_{A_{p,p-1}}\|g\|_\infty + \|u\psi'' + 2u'\psi'\|_{A_{p,p-1}}\|g'\|_\infty + |g''\|_{A_{p,p-1}} \\ &\leq \|g\|_{\mathcal{Z}_{p,p-1}}.\end{aligned} \tag{7}$$

On the other hand, by applying Lemma 2.1, we have that

$$\begin{aligned}|(ug \circ \psi)(0)| &\preceq \|u(0)\|\|g\|_{\mathcal{Z}_{p,p-1}}, \\ |(ug \circ \psi)'(0)| &\preceq \Big(|u'(0)| + \frac{(u\psi')(0)}{(1 - |\psi(0)|^{\frac{1}{p}}}\Big)\|g\|_{\mathcal{Z}_{p,p-1}}.\end{aligned} \tag{8}$$

Hence, (7) and (8), give us the boundedness of operator $W_{u,\psi}$ between Dirichlet-Zygmund spaces $\mathcal{Z}_{p,p-1}(\mathbb{D})$, which completes the proof. □

By applying Lemma 2.2 and Theorem 2.3, we get the following corollary.

Corollary 2.4. *Let $1 < p < \infty$, ψ be an analytic self-map on $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then $W_{u,\psi}$ is bounded on $\mathcal{Z}_{p,p-1}(\mathbb{D})$ if and only if $W_{u\psi'^2,\psi}$ is bounded on $A_{p,p-1}(\mathbb{D})$ and u, $(u\psi) \in \mathcal{Z}_{p,p-1}(\mathbb{D})$.*

Lemma 2.5. *Let $1 < p < \infty$, ψ be an analytic self-map on $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then $W_{u,\psi}$ is bounded from $\mathcal{Z}_{p,p-1}(\mathbb{D})\big(\mathcal{B}_{p,p-1}(\mathbb{D})$ or $A_{p,p-1}(\mathbb{D})\big)$ into $A_{p,p-1}(\mathbb{D})$, if and only if $u \in A_{p,p-1}(\mathbb{D})$.*

Proof. The proof is easy by using lemma 2.1, so we skip the details. □

Definition of norm in Dirichlet-Zygmund spaces, gives us that for any $g \in \mathcal{Z}_{p,p-1}(\mathbb{D})$, $u \in H(\mathbb{D})$ and $u \in \mathcal{S}$,

$$\|ug \circ \psi\|_{\mathcal{Z}_{p,p-1}(\mathbb{D})} = \|u''g \circ \psi + (2u'\psi' + u\psi'')g' + u\psi'^2 g''\|_{A_{p,p-1}(\mathbb{D})} + M,$$

for a constant M. Hence, for studing operator $W_{u,\psi}$ on Dirichlet Zygmund spaces, we may deal with the three operators $M_{u''}C_\psi$, $M_{2u'\phi'+u\psi''}C_\psi D$ and $M_{u\psi'^2}C_\psi D^2$. As an interesting result, by using the details in the proof of Theorems 2.3 and Lemma 2.5, the following Corollary obtains.

Corollary 2.6. *Let $1 < p < \infty$ $\psi \in \mathcal{S}$ and $u \in H(\mathbb{D})$. Then $W_{u,\psi} : \mathcal{Z}_{p,p-1}(\mathbb{D}) \to \mathcal{Z}_{p,p-1}(\mathbb{D})$ is bounded if and only if, all the three following operators are bounded*

$$\begin{aligned} &W_{u'',\psi} : \mathcal{Z}_{p,p-1}(\mathbb{D}) \to A_{p,p-1}(\mathbb{D}),\\ &W_{(2u'\psi'+u\psi''),\psi} : \mathcal{B}_{p,p-1}(\mathbb{D}) \to A_{p,p-1}(\mathbb{D}),\\ &W_{u\psi'^2,\psi} : A_{p,p-1}(\mathbb{D}) \to A_{p,p-1}(\mathbb{D}). \end{aligned}$$

In [20] Lemma 5, the authors provide another characterization for Bergman spaces which asserts that for any $\alpha > -1$ and $p > 0$, there exists $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \le C[|f(0)|^p + \int_{\mathbb{D}} |g(z)|^p dA_\alpha(z)], \tag{9}$$

$$|f(0)|^p + \int_{\mathbb{D}} |g(z)|^p dA_\alpha(z) \le C \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \tag{10}$$

for all analytic functions f in $\mathbb{D}$, where $g(z) = (1 - |z|^2)f'(z)$ for $z \in \mathbb{D}$. Hence $f \in A_{p,\alpha}(\mathbb{D})$ if and only if $f' \in A_{p,\alpha+1}(\mathbb{D})$. This characterization will help us in the proof of the next Theorem, which gives us a characterization for the boundedness of Stević-Sharma operator $T_{u_1,u_2,\psi}$, between some Dirichlet-Zygmund spaces.

Theorem 2.7. *Let $1 < p < \infty$, $u_1 \in \mathcal{Z}_{p,p-1}(\mathbb{D})$, $u_2 \in H(\mathbb{D})$ and $\psi \in \mathcal{S}$. Then the following statements are equivalent:*

(a) $T_{u_1,u_2,\psi} : w\mathcal{Z}_{p,3p-1}(X) \to w\mathcal{Z}_{p,3p-1}(X)$ is bounded.

(b) $T_{u_1,u_2,\psi} : \mathcal{Z}_{p,3p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}(\mathbb{D})$ is bounded.

(c) $u_2, (u_2\psi) \in \mathcal{Z}_{p,3p-1}(\mathbb{D})$ and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2})^{3p+1} |u_1(w)\psi'^2(w)|^p dA_{3p-1}(w) < \infty, \tag{11}$$

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2})^{4p+1} |u_2(w)\psi'^2(w)|^p dA_{3p-1}(w) < \infty. \tag{12}$$

Proof. (a) ⇒ (b). Let $T_{u_1,u_2,\psi} : w\mathcal{Z}_{p,3p-1}(X) \to w\mathcal{Z}_{p,3p-1}(X)$ be bounded. Then similar to the proof of part (a) to (b) of Theorem 2.3, for any $f \in \mathcal{Z}_{p,3p-1}$ and $\nu \in X$ with $\|\nu\| = 1$, if we define $h = \nu f(z)$ for any $z \in \mathbb{D}$, we get that $\|h\|^p_{w\mathcal{Z}_{p,3p-1}} = \|f\|_{\mathcal{Z}_{p,3p-1}}$ and also $\|T_{u_1,u_2,\psi}h\|^p_{w\mathcal{Z}_{p,3p-1}} = \|T_{u_1,u_2,\psi}f\|_{\mathcal{Z}_{p,3p-1}}$ which gives us the desired result.

(b)⇒(a). It is similar to the proof of part (b)⇒(a) of Theorem 2.3, just we have operator $T_{u_1,u_2,\psi}$ instead of $W_{u,\psi}$, so we skip the details.
(b) ⇒ (c). Let $u_1 \in \mathcal{Z}_{p,p-1}(\mathbb{D}) \subset A_{p,p-1}(\mathbb{D})$. But according to [20] we know that $f \in A_{p,\alpha}$ if and only if $f^{(n)} \in A_{p,\alpha+np}$, therefore

$$\begin{aligned} ||u_1\psi||_{\mathcal{Z}_{p,3p-1}} &= ||(u_1\psi)''||_{A_{p,3p-1}} + |u_1(0)\psi(0)| + |(u_1\psi)'(0)|\\ &\le ||(u_1\psi)'||_{A_{p,2p-1}} + C\\ &\le ||u_1\psi||_{A_{p,p-1(\mathbb{D})}} + C\\ &\le ||\psi||_\infty ||u_1||_{A_{p,p-1}(\mathbb{D})} + C < \infty. \end{aligned} \tag{13}$$

Since $T_{u_1,u_2,\psi}$ is bounded on $\mathcal{Z}_{p,3p-1}(\mathbb{D})$, by setting $f = z$, we obtain

$$||u_1C_\psi f + u_2C_\psi Df||_{\mathcal{Z}_{p,3p-1}(\mathbb{D})} = ||u_1\psi + u_2||_{\mathcal{Z}_{p,3p-1}(\mathbb{D})} < \infty. \tag{14}$$

Then the triangle inequality, (14) and (13) yield that $u_2 \in \mathcal{Z}_{p,3p-1}(\mathbb{D})$. In addition, by applying $||(u_1\psi)''||_{A_{p,3p-1}(\mathbb{D})} < \infty$ and $u_1'' \in A_{p,p-1}(\mathbb{D}) \subset A_{p,3p-1}(\mathbb{D})$, similar to the details of (4), we get that

$$||u_1\psi'' + 2u_1'\psi'||_{A_{p,3p-1}} < \infty. \tag{15}$$

On the other hand, for any $f \in A_{p,3p-1}(\mathbb{D})$ such that $f(0) = 0$, by putting $h(z) = \int_0^z f(w)dw$ and $g(z) = \int_0^z h(w)dw$, we have that $g''(z) = f(z) \in A_{p,3p-1}(\mathbb{D})$ and $g(0) = g'(0) = 0$. Hence by applying Lemma 2.1 and (15), we obtain

$$\begin{aligned}
\|u_1\psi'^2 f\circ\psi(z)\|_{A_{p,3p-1}(\mathbb{D})} &= \|u_1\psi'^2 g''\circ\psi(z)\|_{A_{p,3p-1}(\mathbb{D})}\\
&\le \|(u_1 g\circ\psi)''\|_{A_{p,3p-1}} + \|(u_1\psi'' + 2u_1'\psi')g'\circ\psi\|_{A_{p,3p-1}} + \|u_1'' g\circ\psi\|_{A_{p,3p-1}}\\
&\le \|u_1 g\circ\psi\|_{A_{p,p-1}} + \|g'\|_\infty\|u_1\psi'' + 2u_1'\psi'\|_{A_{p,3p-1}} + \|g\|_\infty\|u_1''\|_{A_{p,3p-1}}\\
&\le \|g\|_\infty\|u_1\|_{A_{p,p-1}} + \|g'\|_\infty\|u_1\psi'' + 2u_1'\psi'\|_{A_{p,3p-1}} + \|g\|_\infty\|u_1''\|_{A_{p,3p-1}}\\
&\preceq \|g\|_{\mathcal{Z}_{p,3p-1}} \preceq \|g''\|_{A_{p,3p-1}} = \|f\|_{A_{p,3p-1}}.
\end{aligned} \tag{16}$$

Thus, lemma 2.2, gives us

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2}\right)^{3p+1}|u_1(w)\psi'^2(w)|^p dA_{3p-1}(w) < \infty. \tag{17}$$

Also by (16), (13), $u_1 \in \mathcal{Z}_{p,p-1}$ and Theorem 2.3 we have the boundedness of $W_{u_1,\psi} : \mathcal{Z}_{p,3p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}(\mathbb{D})$. Now with the triangle inequality, for any $f \in \mathcal{Z}_{p,3p-1}(\mathbb{D})$,

$$\begin{aligned}
\|W_{u_2,\psi}Df\|_{\mathcal{Z}_{p,3p-1}(\mathbb{D})} &\le \|T_{u_1,u_2,\psi}(f)\|_{\mathcal{Z}_{p,3p-1}(\mathbb{D})} + \|W_{u_1,\psi}(f)\|_{\mathcal{Z}_{p,3p-1}(\mathbb{D})}\\
&\preceq \|f\|_{\mathcal{Z}_{p,3p-1}(\mathbb{D})}.
\end{aligned}$$

On the other hand, $W_{u_2,\psi}D : \mathcal{Z}_{p,3p-1} \to \mathcal{Z}_{p,3p-1}$ is bounded if and only if $W_{u_2,\psi} : \mathcal{Z}_{p,4p-1} \to \mathcal{Z}_{p,3p-1}$ is bounded. Therefore, applying theorem 2.3 and lemma 2.2 complete the proof.

(c)⇒(b). Assume that (c) holds. Since the boundedness of $W_{u_2,\psi} : \mathcal{Z}_{p,4p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}$ is equivalent to the boundedness of $W_{u_2,\psi}D$ between Dirichlet-Zygmund spaces $\mathcal{Z}_{p,3p-1}$, Then theorem 2.3 and lemma 2.2, along with the assumption $u_1 \in \mathcal{Z}_{p,p-1} \subset \mathcal{Z}_{p,3p-1}$ and equation (13), give us the boundedness of $W_{u_2,\psi}D$ and $W_{u_1,\psi}$ between $\mathcal{Z}_{p,3p-1}(\mathbb{D}$. Therefore by using the triangle inequality, $T_{u_1,u_2,\psi} = W_{u_1,\psi} + W_{u_2,\psi}D$ is bounded on $\mathcal{Z}_{p,3p-1}(\mathbb{D})$. ☐

We can easily check that $M_uDC_\psi = T_{0,u\psi',\psi}$. Therefore, applying theorem 2.7, gives us the following corollary.

Corollary 2.8. *Let $1 < p < \infty$, $u \in H(\mathbb{D})$ and $\psi \in \mathcal{S}$. Then M_uDC_ψ is bounded on $\mathcal{Z}_{p,3p-1}(\mathbb{D})$, if and only if $u\psi', (u\psi'\psi) \in \mathcal{Z}_{p,3p-1}(\mathbb{D})$ and*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2}\right)^{4p+1}|u(w)\psi'^3(w)|^p dA_{3p-1}(w) < \infty.$$

Theorem 2.9. *Let $1 < p < \infty$ and $\xi \in H(\mathbb{D})$. Then for any $f \in \mathcal{Z}_{p,p-1}(\mathbb{D})$, the following statements are equivalent*

(a) The Volterra-integral operator $V_\xi(f) = \int_0^z f(w)\xi'(w)dw$ is bounded on Dirichlet-Zygmund spaces.

(b) $\xi \in \mathcal{Z}_{p,p-1}(\mathbb{D})$.

(c) The integral-type operator $I_\xi(f) = \int_0^z f'(w)\xi(w)dw$ is bounded on Dirichlet-Zygmund spaces.

Proof. (a) ⇒(b). Suppose that operatpr V_ξ is bounded on $\mathcal{Z}_{p,p-1}(\mathbb{D})$, then for $f = 1$, we get that

$$|\xi(0)| + \|\xi\|_{\mathcal{Z}_{p,p-1}} \le \|V_\xi f\|_{\mathcal{Z}_{p,p-1}} \le \|V_\xi\| < \infty. \tag{18}$$

Therefore, $\xi \in \mathcal{Z}_{p,p-1}(\mathbb{D})$.
(b)⇒(c). Let $\xi \in \mathcal{Z}_{p,p-1}(\mathbb{D})$. Then $\xi \in \mathcal{B}_{p,p-1}(\mathbb{D})$ and by applying lemma 2.1, $\|\xi\|_\infty \le \|\xi\|_{\mathcal{Z}_{p,p-1}} < \infty$. Then for any $h \in \mathcal{Z}_{p,p-1}(\mathbb{D})$,

$$\begin{aligned}
\|I_\xi(h)\|_{\mathcal{Z}_{p,p-1}} &= \|h''\xi\|_{A_{p,p-1}} + \|h'\xi'\|_{A_{p,p-1}}\\
&\le \|h\|_{\mathcal{Z}_{p,p-1}}\|\xi\|_\infty + \|h'\|_\infty\|\xi'\|_{A_{p,p-1}}\\
&\le \|h\|_{\mathcal{Z}_{p,p-1}}\|\xi\|_{\mathcal{Z}_{p,p-1}} + \|h\|_{\mathcal{Z}_{p,p-1}}\|\xi\|_{\mathcal{B}_{p,p-1}}\\
&\preceq \|h\|_{\mathcal{Z}_{p,p-1}}.
\end{aligned}$$

(c)⇒(b). By putting function $f(z) = z$, and using the boundedness property of opertors I_ξ on $\mathcal{Z}_{p,p-1}(\mathbb{D})$, we get the disired result.
(b)⇒(a). Let $\xi \in \mathcal{Z}_{p,p-1}(\mathbb{D}) \subset \mathcal{B}_{p,p-1}(\mathbb{D})$.Then for any $h \in \mathcal{Z}_{p,p-1}(\mathbb{D})$, using Lemma 2.1, give us

$$\begin{aligned}\|V_\xi(h)\|_{\mathcal{Z}_{p,p-1}} &= \|h'\xi'\|_{A_{p,p-1}} + \|h\xi''\|_{A_{p,p-1}}\\ &\le \|h'\|_\infty\|\xi'\|_{A_{p,p-1}} + \|h\|_\infty\|\xi''\|_{A_{p,p-1}}\\ &\le \|h'\|_{\mathcal{B}_{p,p-1}}\|\xi\|_{\mathcal{B}_{p,p-1}} + \|h\|_\infty\|\xi\|_{\mathcal{Z}_{p,p-1}}\\ &\le \|h\|_{\mathcal{Z}_{p,p-1}}\|\xi\|_{\mathcal{B}_{p,p-1}} + \|h\|_{\mathcal{Z}_{p,p-1}}\|\xi\|_{\mathcal{Z}_{p,p-1}}\\ &\preceq \|h\|_{\mathcal{Z}_{p,p-1}},\end{aligned}$$

which completes the proof. □

3. Compactness

In this section we aim to consider the compactness of weighted composition and Stevi'c sharma operatros, we also briefly investigate the compactness of Voltera integral and integral type operators between Dirichtet Zygmund spaces.

Lemma 3.1. *Let $1 < p < \infty$, ψ be an analytic self map on $\mathbb{D}$ and $u \in H(\mathbb{D})$. Suppose that $W_{u,\psi}$ is a bounded weighted composition operator on $A_{p,p-1}, (\mathcal{B}_{p,p-1} or \mathcal{Z}_{p,p-1})$. Then $W_{u,\psi}$ is compact on $A_{p,p-1}, (\mathcal{B}_{p,p-1}$ or $\mathcal{Z}_{p,p-1}$) if and only if for any bounded sequence $\{f_n\}_0^\infty$ in $A_{p,p-1}, (\mathcal{B}_{p,p-1},$ or $\mathcal{Z}_{p,p-1})$ such that $\{f_n\}_0^\infty \to 0$ uniformly on compact subsets on $\mathbb{D}$ as $n \to \infty$, we have $\|W_{u,\psi}(f_n)\|_{A_{p,p-1}} \to 0$, $(\|W_{u,\psi}(f_n)\|_{\mathcal{B}_{p,p-1}} \to 0$, or $\|W_{u,\psi}\|_{\mathcal{Z}_{p,p-1}} \to 0$).*

Proof. The proof is similar to Proposition 3.11 [6] or Lemma 2.1 [5], so we skip the details. □

Note that, if X be a reflexive Banach space, then Montel's theorem is valid for $H(\mathcal{D}, X)$ endowded with compact-open(co) topology. Hence in this case, Lemma 3.1 is valid for $w\mathcal{Z}_{p,p-1}(X)$.
The following Lemma from [17], gives us a characterization for compactness of operator $W_{u,\psi}$ Between weighted Bergman spaces and will help us to get our desired results.

Lemma 3.2. *Let ψ be an analytic self map on $\mathbb{D}$ and $u \in H(\mathbb{D})$. If $0 < p < \infty$ and $-1 < \alpha, \beta$, then the weighted composition operator $W_{u,\psi} : A_{p,\alpha}(\mathbb{D}) \to A_{p,\beta}(\mathbb{D})$ is compact if and only if*

$$\limsup_{|a|\to 1^-} \int_{\mathbb{D}} \left(\frac{(1-|a|)^2}{|1-\overline{a}\psi(w)|^2}\right)^{2+\alpha} |u(w)|^p dA_\beta(w) = 0.$$

Theorem 3.3. *Let $1 < p$, X is a reflexive Banach space and operator $W_{u,\psi} : w\mathcal{Z}_{p,p-1}(X) \to w\mathcal{Z}_{p,p-1}(X)$ is bounded. Then the following statements are equivalent:*

(a) $W_{u,\psi} : w\mathcal{Z}_{p,p-1}(X) \to w\mathcal{Z}_{p,p-1}(X)$ is weakly compact.

(b) $W_{u,\psi} : \mathcal{Z}_{p,p-1}(\mathbb{D}) \to \mathcal{Z}_{p,p-1}(\mathbb{D})$ is compact.

(c) $u, u.\psi \in \mathcal{Z}_{p,p-1}(\mathbb{D})$ and

$$\limsup_{|a|\to 1^-} \int_{\mathbb{D}} \left(\frac{(1-|a|)^2}{|1-\overline{a}\psi(w)|^2}\right)^{p+1} |u(w)\psi'^2(w)|^p dA_{p-1}(w) = 0.$$

Proof. (a) ⇒ (b). Assume that $W_{u,\psi} : w\mathcal{Z}_{p,p-1}(X) \to w\mathcal{Z}_{p,p-1}(X)$ is bounded, then by using Theorem 2.3, we get the boundedness of $W_{u,\psi} : \mathcal{Z}_{p,p-1}(\mathbb{D}) \to \mathcal{Z}_{p,p-1}(\mathbb{D})$. Now let the bounded sequence $(f_n)_{n=1}^\infty \subset \mathcal{Z}_{p,p-1}$ converges uniformly to zero on compact subsets of $\mathbb{D}$, and $\nu \in X$ with $\|\nu\| = 1$, define the sequence $(h_n)_{n=1}^\infty$ such that $h_n := \nu f_n(z)$ for any $z \in \mathbb{D}$. Then similar to the proof of part (a)⇒(b) of Theorem 2.3 we get that $\|h_n\|^p_{w\mathcal{Z}_{p,p-1}} = \|f_n\|^p_{\mathcal{Z}_{p,p-1}} < \infty$, so $(h_n) \subset w\mathcal{Z}_{p,p-1}$ is a bounded sequence which converges uniformly to zero on compact subsets of $\mathbb{D}$. Then by using Lemma 3.1, we get that $\|W_{u,\psi}h_n\|_{w\mathcal{Z}_{p,p-1}}$ converges to zero as $n \to \infty$. Also, similar to equation (3) of Theorem 2.3, we have that $\|W_{u,\psi}f_n\|^p_{\mathcal{Z}_{p,p-1}} = \|W_{u,\psi}h_n\|^p_{w\mathcal{Z}_{p,p-1}}$. Therefore $\|W_{u,\psi}f_n\|_{\mathcal{Z}_{p,p-1}}$ converges to zero as $n \to \infty$. Therefore, Lemma 3.1 completes the proof.

(b)⇒(a). Let $(h_n)_{n=1}^\infty \subset w\mathcal{Z}_{p,p-1}$ be a bounded sequence which converges uniformly to zero on compact subsets of $\mathbb{D}$. Then (h_n) is bounded and converges pointwise to zero. Hence for $v^* \in X^*$ such that $\|v^*\|_{X^*} \le 1$, if we consider sequence $(v^* \circ h_n) \subset \mathcal{Z}_{p,p-1}$ we have that $(v^* \circ h_n)$ is bounded and converges pointwise to zero. Then by applying Corollary 1.3 of [6], we get that $(v^* \circ h_n)$ converges weakly to zero. But we assumed that $W_{u,\psi} : \mathcal{Z}_{p,p-1}(\mathbb{D}) \to \mathcal{Z}_{p,p-1}(\mathbb{D})$ is compact, and hence it is completely continuous. Therefore, $\|W_{u,\psi}(v^*\circ h_n)\|_{\mathcal{Z}_{p,p-1}} \to \|W_{u,\psi}(0)\|_{\mathcal{Z}_{p,p-1}} = 0$. On the other hand, $\|W_{u,\psi}h_n\|_{w\mathcal{Z}_{p,p-1}} = \sup_{\|w^*\|_{X^*}\leq 1} \|W_{u,\psi}(w^*\circ h_n)\|_{\mathcal{Z}_{p,p-1}} = \|W_{u,\psi}(v^*\circ h_n)\|_{\mathcal{Z}_{p,p-1}}$, for $v^* \in X^*$. Therefore $\|W_{u,\psi}h_n\|_{w\mathcal{Z}_{p,p-1}} \to 0$ and applying Lemma 3.1, gives us the desired result.

(b)⇒ (c). Since we supposed that $W_{u,\psi}$ is bounded on $\mathcal{Z}_{p,p-1}$, then Theorem 2.3 gives us $u, u.\psi \in \mathcal{Z}_{p,p-1}$. Now let $w, z \in \mathbb{D}$ and $\xi_{\psi(w)}(z) = \frac{\psi(w)-z}{1-\overline{\psi(w)}z} \in Aut(\mathbb{D})$. Also suppose that $(a_n) \subset \mathbb{D}$ be a sequence with $\lim_{n\to\infty} |\psi(a_n)| = 1$, then define the test function

$$B_{n,w}(z) := \big(\xi_{\psi(a_n)}(z)\big)^2\big(1 - |\psi(a_n)|^2\big)\frac{1}{1-\overline{\psi(a_n)}z}.$$

Since $\|\xi_{\psi(a_n)}\|_\infty < \infty$, so $\big(B_{n,w}\big) \in \mathcal{Z}_{p,p-1}$ converges uniformly to zero on compact subsets of $\mathbb{D}$, as $n \to \infty$. On the other hand, $B_{n,w}(\psi(a_n)) = 0$ and $B'_{n,w}(\psi(a_n)) = 2\xi'_{\psi(a_n)}(\psi(a_n))\xi_{\psi(a_n)}(\psi(a_n)) = 0$ also $B''_{n,w}(\psi(a_n)) = \frac{2}{(1-|\psi(a_n)|^2)^2}$. Therefore, we get that

$$\begin{aligned}\|W_{u,\psi}(B_{n,w})\|_{\mathcal{Z}_{p,p-1}} &\geq \|\big(u(w)B_{n,w}(\psi(a_n))\big)''\|_{A_{p,p-1}}\\ &= \|u''(w)B_{n,w}(\psi(a_n)) + (2u'\psi' + u\psi'')(w)B'_{n,w}(\psi(a_n)) + u\psi'^2(w)B''_{n,w}(\psi(a_n))\|_{A_{p,p-1}}\\ &= \|u\psi'^2(w)B''_{n,w}(\psi(a_n))\|_{A_{p,p-1}} = \int_{\mathbb{D}} |u(w)\psi'^2(w)|^p\Big|\frac{2}{(1-|\psi(a_n)|^2)^2}\Big|^p dA_{p-1}(w)\end{aligned}$$

But we assumed that $W_{u,\psi}$ is compact on $\mathcal{Z}_{p,p-1}$. Then by applying Lemma 3.1, we get the desired result.

(c) ⇒ (b). Let $\psi, u.\psi \in \mathcal{Z}_{p,p-1}$. Then $\|u''\|_\infty \preceq \|u''\|_{A_{p,p-1}} \leq \|u\|_{\mathcal{Z}_{p,p-1}} < \infty$ and by Lemma 2.1, we have that $\|2u'\psi' + u\psi''\|_\infty \leq \|u\psi\|_\infty + \|u''\psi\|_\infty \preceq \|u\psi\|_{\mathcal{Z}_{p,p-1}} + \|u\|_{\mathcal{Z}_{p,p-q}} < \infty$. Now let $\{f_n\}_{n=0}^\infty \subset \mathcal{Z}_{p,p-1}(\mathbb{D}) \subset \mathcal{B}_{p,p-1} \subset A_{p,p-1}$ be a sequence such that f_n converges to zero uniformly, on compact subsets of $\mathbb{D}$. Then, whenever $|\psi(z)| > s$ for $s \in (0,1)$, since $|u(w)\psi'^2(w)| \neq 0$ and so $1 \preceq |u(w)\psi'^2(w)|$, then we get that

$$\limsup_{|\psi(w)|\to 1}\int_{\mathbb{D}}\Big(\frac{1}{|1-|\psi(w)|^2}\Big)^{p+1}dA_{p-1}(w) \preceq \limsup_{|\psi(w)|\to 1}\int_{\mathbb{D}}\Big(\frac{1}{|1-|\psi(w)|^2}\Big)^{p+1}|u(w)\psi'^2(w)|^p dA_{p-1}(w).$$

Hence in this case, our assumption in (c) along with Lemma 3.2, give us the compactness of operator $C_\psi : A_{p,p-1} \to A_{p,p-1}$ and so by applying Lemma 3.1, we have $\|f_n \circ \psi\|_{A_{p,p-1}} \to 0$.
On the other hand, if $|\psi(z)| \leq s$, since $f_n \to 0$ uniformly on $\{|w| \leq t\}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|f_n(w)| \leq \epsilon$ for an arbitrary $\epsilon > 0$. Therefore, by assuming $|\psi(z)| \leq s$, we get $\|f_n\|_\infty \to 0$. But $H^\infty \subset A_{p,p-1}$ hence, $\|f_n \circ \psi\|_{A_{p,p-1}} \leq \|f_n\|_\infty \to 0$. So in the both cases we obtain $\|f_n \circ \psi\|_{A_{p,p-1}} \to 0$. With a similar argument, since we have supposed that $\{f_n\} \in A_{p,p-1}$ and then $\{f'_n\} \in \mathcal{B}_{p,p-1} \subset A_{p,p-1}$, we get that $\|f'_n \circ \psi\|_{A_{p,p-1}} \to 0$.
Also we have supposed that $\{f_n\}_{n=0}^\infty \subset \mathcal{Z}_{p,p-1}(\mathbb{D}) \subset \mathcal{B}_{p,p-1} \subset A_{p,p-1}$ converges to zero uniformly, on compact subsets of $\mathbb{D}$. Therefore f_n converges poinwise on $\mathbb{D}$ and hence $|f_n \circ \psi(0)| \to 0$. With a similar argument we see that $|(f_n \circ \psi)'(0)| \to 0$. Indeed, by using Lemma 3.2 and our assumptions, $W_{u\psi'^2,\psi} : A_{p,p-1}(\mathbb{D}) \longmapsto A_{p,p-1}(\mathbb{D})$ is compact. So Lemma 3.1, gives us

$$A_n := \|u\psi'^2 f''_n \circ \psi\|_{A_{p,p-1}(\mathbb{D})} \longrightarrow 0,\ as\ n \to \infty.$$

Hence, we obtain

$$\begin{aligned}&\|W_{u,\psi}(f_n)\|_{\mathcal{Z}_{p,p-1}(\mathbb{D})} = \|(u.f_n \circ \psi)''\|_{A_{p,p-1}} + |(u.f_n \circ \psi)'(0)| + |u.f_n \circ \psi(0)|\\ &\leq \|u\psi'^2 f''_n \circ \psi\|_{A_{p,p-1}\mathbb{D})} + \|u'' f_n \circ \psi\|_{A_{p,p-1}} + \|(2u'\psi' + u\psi'')f'_n \circ \psi\|_{A_{p,p-1}}\\ &\quad + |(f_n \circ \psi)'(0)| + |f_n \circ \psi(0)|\\ &\leq \Big(A_n + \|u''\|_\infty\|f_n \circ \psi\|_{A_{p,p-1}} + \|2u'\psi' + u\psi''\|_\infty\|f'_n \circ \psi\|_{A_{p,p-1}} + |(f_n \circ \psi)'(0)| + |f_n \circ \psi(0)|\Big) \to 0,\end{aligned}$$

as $n \to \infty$. Therefore, Lemma 3.1 completes the proof. □

As a result of Theorems 2.3, 3.3 and Lemma 3.2, we have the following corollary.

Corollary 3.4. *Let $1 < p < \infty$, ψ be an analytic self map on $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then $W_{u,\psi}$ is bounded on $\mathcal{Z}_{p,p-1}(\mathbb{D})$ if and only if $W_{u\psi'^2,\psi}$ is bounded on $A_{p,p-1}(\mathbb{D})$ and u, $(u\psi) \in \mathcal{Z}_{p,p-1}(\mathbb{D})$.*

Moreover, by aplying Theorem 2.3, Lemma 2.1, Corollary 2.6 and the details of the proof of Theorem 3.3, we get the following result.

Corollary 3.5. *Let $1 < p < \infty$, $\psi \in \mathcal{S}$, $u \in H(\mathbb{D})$ and $W_{u,\psi} : \mathcal{Z}_{p,p-1}(\mathbb{D}) \to \mathcal{Z}_{p,p-1}(\mathbb{D})$ is bounded. Then $W_{u,\psi} : \mathcal{Z}_{p,p-1}(\mathbb{D}) \to \mathcal{Z}_{p,p-1}(\mathbb{D})$ is compact if and only if all the three following operators are compact*

$$W_{u'',\psi} : \mathcal{Z}_{p,p-1}(\mathbb{D}) \to A_{p,p-1}(\mathbb{D}),$$
$$W_{2u'\psi'+u\psi'',\psi} : \mathcal{B}_{p,p-1}(\mathbb{D}) \to A_{p,p-1}(\mathbb{D}),$$
$$W_{u\psi'^2,\psi} : A_{p,p-1}(\mathbb{D}) \to A_{p,p-1}(\mathbb{D}).$$

Theorem 3.6. *Let $1 < p < \infty$, X be a reflexive Banach space, $u_1 \in \mathcal{Z}_{p,p-1}(\mathbb{D})$, $u_2 \in H(\mathbb{D})$, $\psi \in \mathcal{S}$ and $T_{u_1,u_2,\psi} : \mathcal{Z}_{p,3p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}(\mathbb{D})$ is bounded. Then the following statements are equivalent:*

(a) $T_{u_1,u_2,\psi} : w\mathcal{Z}_{p,3p-1}(X) \to w\mathcal{Z}_{p,3p-1}(X)$ is weakly compact.

(b) $T_{u_1,u_2,\psi} : \mathcal{Z}_{p,3p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}(\mathbb{D})$ is compact.

(c) $u_2, (u_2\psi) \in \mathcal{Z}_{p,3p-1}(\mathbb{D})$ and

$$\limsup_{|a|\to 1^-} \int_{\mathbb{D}} \Big(\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2}\Big)^{3p+1} |u_1(w)\psi'^2(w)|^p dA_{3p-1}(w) = 0,$$
$$\limsup_{|a|\to 1^-} \int_{\mathbb{D}} \Big(\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2}\Big)^{4p+1} |u_2(w)\psi'^2(w)|^p dA_{3p-1}(w) = 0.$$

Proof. (a) $\Leftrightarrow$ (b). It is similar to the proof of Theorem 3.3.

(b) $\Rightarrow$ (c). Let $T_{u_1,u_2,\psi} : \mathcal{Z}_{p,3p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}(\mathbb{D})$ is bounded, then according to Theorem 2.7,

$$u_1\psi, u_2, (u_2\psi) \in \mathcal{Z}_{p,3p-1}(\mathbb{D}), \quad u_1'' \in A_{p,3p-1}(\mathbb{D}), \; u_1\psi'' + 2u_1'\psi' \in A_{p,3p-1}. \tag{19}$$

Now let $w, z \in \mathbb{D}$ and $\xi_{\psi(w)}(z) = \frac{\psi(w)-z}{1-\overline{\psi(w)}z} \in Aut(\mathbb{D})$. Also suppose that $a_n \subset \mathbb{D}$ be a sequence with $\lim_{n\to\infty} |\psi(a_n)| = 1$, then define the test function

$$B_{n,w}(z) := \big(\xi_{\psi(a_n)}(z)\big)^3 \big(1-|\psi(a_n)|^2\big) \frac{1}{1-\overline{\psi(a_n)}z}$$

Since $\|\xi_{\psi(a_n)}\|_\infty < \infty$, so $B_{n,w} \in \mathcal{Z}_{p,3p-1}$ converges uniformly to zero on compact subsets of $\mathbb{D}$, as $n \to \infty$. On the other hand, $B_{n,w}(\psi(a_n)) = 0$, $B'_{n,w}(\psi(a_n)) = 0$, $B''_{n,w}(\psi(a_n)) = 0$ and $B'''_{n,w}(\psi(a_n)) = \frac{6}{(1-|\psi(a_n)|^2)^3}$. Therefore, by Lemma 3.1, we get that $\|T_{u,\psi}(B_{n,w})\|_{\mathcal{Z}_{p,3p-1}} \to 0$. Also

$$\begin{aligned}\|T_{u_1,u_2,\psi}(B_{n,w})\|_{\mathcal{Z}_{p,3p-1}} &\geq \big\|\big(u_1(w)B_{n,w}(\psi(a_n) + u_2(w)B'_{n,w}(\psi(a_n))\big)''\big\|_{A_{p,3p-1}} \\ &= \|u_2\psi'^2(a_n)B'''_{n,w}(\psi(a_n))\|_{A_{p,3p-1}} = \int_{\mathbb{D}} |u_2(a_n)\psi'^2(a_n)|^p \Big|\frac{2}{(1-|\psi(a_n)|^2)^3}\Big|^p dA_{3p-1}(a_n).\end{aligned}$$

Then applying Lemma 3.2 and the fact that $0 < |1-|\psi(z)|^2| < 1$, give us

$$\limsup_{|a|\to 1^-} \int_{\mathbb{D}} \Big(\frac{(1-|a|^2)}{|1-\bar{a}\psi(w)|^2}\Big)^{4p+1} |u_2(w)\psi'^2(w)|^p dA_{3p-1}(w) = 0. \tag{20}$$

Therefore, by (19), (20) and Theorem 3.3, we get the compactness of $W_{u_2,\psi} : \mathcal{Z}_{p,4p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}(\mathbb{D})$, which is equivalent to compactness of $W_{u_2,\psi}D : \mathcal{Z}_{p,3p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}(\mathbb{D})$. Now suppose that $\{f_n\} \subset \mathcal{Z}_{p,3p-1}(\mathbb{D})$ be bounded a sequence converges to zero on compact subsets of $\mathbb{D}$. Since we also assumed that $T_{u_1,u_2,\psi}$ is compact on $\mathcal{Z}_{p,3p-1}(\mathbb{D})$, hence by using the triangle inequality,

$$\|W_{u_1,\psi}f_n\|_{\mathcal{Z}_{p,3p-1}(\mathbb{D})} \leq \|T_{u_1,u_2,\psi}f_n\|_{\mathcal{Z}_{p,3p-1}(\mathbb{D})} + \|W_{u_2,\psi}f_n\|_{\mathcal{Z}_{p,4p-1}(\mathbb{D})}, \tag{21}$$

so by using Lemma 3.1, we obtain the compactness of $W_{u_1,\psi} : \mathcal{Z}_{p,3p-1} \to \mathcal{Z}_{p,3p-1}$. Then Theorem 3.3 completes the proof.

(c)⇒(b). Assume that (c) holds. Then by applying Theorem 2.3, Theorem 3.3 and (13), we have the compactness of two operators

$$W_{u_2,\psi} : \mathcal{Z}_{p,4p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1},$$
$$W_{u_1,\psi} : \mathcal{Z}_{p,3p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}.$$

But the compactness of $W_{u_2,\psi} : \mathcal{Z}_{p,4p-1}(\mathbb{D}) \to \mathcal{Z}_{p,3p-1}$ is equivalent to the compactness of operator $W_{u_2,\psi}D : \mathcal{Z}_{p,3p-1} \to \mathcal{Z}_{p,3p-1}$. Therefore by using the triangle inequality and lemma 3.1, $T_{u_1,u_2,\psi} = W_{u_1,\psi} + W_{u_2,\psi}D$ is compact on $\mathcal{Z}_{p,3p-1}(\mathbb{D})$. □

Theorem 3.7. *Let $1 < p < \infty$ and $\xi \in H(\mathbb{D})$. Then for any $f \in \mathcal{Z}_{p,p-1}(\mathbb{D})$,*

(a) $V_\xi(f) = \int_0^z f(w)\xi'(w)dw$ is bounded on Dirichlet-Zugmund spaces if and only if it is compact.

(b) $I_\xi(f) = \int_0^z f'(w)\xi(w)dw$ is bounded on Dirichlet-Zugmund spaces if and only if it is compact.

Proof. The proof is clear by applying theorem 2.9, and lemma 3.1, so we skip it. □

Acknowledgement

Very grateful to the reviewer's insightful comments which improved the manuscript.

References

[1] E. Abbasi, S. Nasresfahani, and K. Khalilpour, *Stević-sharma type operator from Besov space into Zygmund space*, J. Adv. Math. Model, 11 (2021), pp. 573–584.

[2] S. Acharyya and T. Ferguson, *Sums of weighted differentiation composition operators*, Complex Anal. Oper. Theory, 13 (2019), pp. 1465–1479.

[3] O. Blasco, *Composition operators fronm weak to strong vector-valued Hardy spaces*, Complex Anal. Oper. Theory, 14 (2020), pp. Paper No. 81, 11.

[4] J. Bonet, P. Domański, and M. Lindström, *Weakly compact composition operators on analytic vector-valued function spaces*, Ann. Acad. Sci. Fenn. Math., 26 (2001), pp. 233–248.

[5] F. Colonna and M. Tjani, *Operator norms and essential norms of weighted composition operators between Banach spaces of analytic functions*, J. Math. Anal. Appl., 434 (2016), pp. 93–124.

[6] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[7] J. Laitila, *Weakly compact composition operators on vector-valued BMOA*, J. Math. Anal. Appl., 308 (2005), pp. 730–745.

[8] J. Laitila, H.-O. Tylli, and M. Wang, *Composition operators from weak to strong spaces of vector-valued analytic functions*, J. Operator Theory, 62 (2009), pp. 281–295.

[9] S. Li and S. Stević, *Volterra-type operators on Zygmund spaces*, J. Inequal. Appl., (2007), pp. Article ID 32124, 10.

[10] J. Liu, S. Ponnusamy, and H. Xie, *Complex symmetric weighted composition-differentiation operators*, Linear Multilinear Algebra, 71 (2023), pp. 737–755.

[11] S. Nasresfahani and E. Abbasi, *Product-type operators on weak vector valued α-Besov spaces*, Turkish J. Math., 46 (2022), pp. 1210–1223.

[12] S. Nasresfahani, M. Hassanlou, and E. Abbasi, *Carleson measure and composition operators on vector valued weighted besov type spaces*, J. Adv. Math. Model, 12 (2022), pp. 1–12.

[13] C. Pommerenke, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation*, Comment. Math. Helv., 52 (1977), pp. 591–602.

[14] J. H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

[15] S. Stević, *On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball*, J. Math. Anal. Appl., 354 (2009), pp. 426–434.

[16] S. Stević, A. K. Sharma, and A. Bhat, *Products of multiplication composition and differentiation operators on weighted Bergman spaces*, Appl. Math. Comput., 217 (2011), pp. 8115–8125.

[17] e. Čučković and R. Zhao, *Weighted composition operators between different weighted Bergman spaces and different Hardy spaces*, Illinois J. Math., 51 (2007), pp. 479–498.

[18] S. Wang, M. Wang, and X. Guo, *Differences of Stević-Sharma operators*, Banach J. Math. Anal., 14 (2020), pp. 1019–1054.

[19] ——, *Products of composition, multiplication and iterated differentiation operators between Banach spaces of holomorphic functions*, Taiwanese J. Math., 24 (2020), pp. 355–376.

[20] H. Wulan and K. Zhu, *Lipschitz type characterizations for Bergman spaces*, Canad. Math. Bull., 52 (2009), pp. 613–626.

[21] H. Xie, J. Liu, and S. Ponnusamy, *Volterra-type operators on the minimal Möbius-invariant space*, Canad. Math. Bull., 66 (2023), pp. 509–524.

[22] K. Zhu, *Operator theory in function spaces*, vol. 138 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, second ed., 2007.

[23] X. Zhu, *Weighted composition operators from Dirichlet-Zygmund spaces into Zygmund-type spaces and Bloch-type spaces*, J. Funct. Spaces, (2022), pp. Article ID 9332406, 8.

Please cite this article using:

Sepideh Nasresfahani, Alireza Parvizi, Weighted composition, Stević-Sharma, Volterra integral and integral type operators between Dirichlet-Zygmund spaces, AUT J. Math. Comput., 6(4) (2025) 341-351
https://doi.org/10.22060/AJMC.2024.22491.1161