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Original Article

Exact double domination in the generalized Sierpiński graphs

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ABSTRACT: A subset D of vertices of a simple graph G is an exact double dominating set if each vertex v of G is dominated by exactly two vertices of D, i.e. $|N_G[v] \cap D| = 2$, in which $N_G[v]$ is the closed neighborhood of v in G. The generalized Sierpiński graph $S(G,t)$ is a fractal-like graph that uses G as a building block and can be constructed recursively in t steps from the base graph G. In this paper we study and determine the existence of exact double dominating sets in generalized Sierpiński graphs $S(P_n,t)$, $S(C_n,t)$, $S(K_{1,n},t)$ and $S(K_n,t)$.

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1. Introduction

Let $G = (V, E)$ be a finite and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The set $N_G(u)$ denotes the (open) *neighborhood* of $u \in V(G)$, which means the set of all adjacent vertices to u in G, the *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$ and the *degree* of u is $\deg_G(u) = |N_G(u)|$. A *matching* in G is a set of pairwise non-incident edges of E and for a subset of vertices $X \subseteq V$, $G[X]$ denotes the subgraph *induced* by X. In recent years much attention drawn to the domination theory which is an interesting branch in graph theory. Each vertex of a graph is said to *dominate* every vertex in its closed neighborhood. A subset S of $V(G)$ is a *dominating set* for G if each vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G. In [6] Fink and Jacobson generalized the concept of dominating sets. Let k be a positive integer. A subset D of vertices in G is a *k-dominating set* if each vertex in $V \setminus D$ is adjacent to at least k vertices in D. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. Hence, for $k = 1$, 1-dominating sets are the classical dominating sets. A vertex subset

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D is a *perfect k-dominating set* if each vertex v of G, not in D, is adjacent to exactly k vertices of D. The perfect k-domination problem is NP-complete for general graphs, see [3]. Note that every nontrivial graph has a perfect k-dominating set, since the entire vertex set is such a set and there are graphs whose only perfect k-dominating set is the entire vertex set (consider the stars $K_{1,t}$ for $1 < k < t$). A set $S \subseteq V$ is a *double dominating set* for G if each vertex in V is dominated by at least two vertices in S, note that this concept is different from 2-domination. Note that sets that are both perfect dominating and independent are called *perfect codes* by Biggs in [2] or *efficient dominating sets* by Bange, Barkauskas and Slater in [1]. Analogously to perfect or efficient domination, Harary and Haynes in [10] defined an *efficient doubly dominating set* as a subset S of vertices such that each vertex of G is dominated by exactly two vertices of S, i.e $|N_G[v] \cap S| = 2$ for each $v \in V$, see also [9]. Chellali, Khelladi and Maffray in [4] prefer to use the phrase *exact* double (doubly) dominating set for this concept and they show that the complexity of the problem of deciding whether a graph admits an exact double dominating set is NP-Complete. Note that not all graphs admit an exact double dominating set (for example consider the 4-cycle C_4 or the stars). In [4] a constructive characterization of those trees that admit an exact double dominating set is provided, and they establish a necessary and sufficient condition for the existence of an exact double dominating set in a connected cubic graph. Also, in [4] it is shown that if there exists an exact double dominating set, then all such sets have the same size. Hence, the concept *exact double domination number* is redundant.

Theorem 1.1 ([4]). *The vertex set of every exact double dominating set induces a matching.*

Theorem 1.2 ([4]). *If G has an exact double dominating set, then all such sets have the same size.*

Motivated by topological studies of the Lipscomb's space, Klavžar et al. introduced the *Sierpiński graph* $S(K_n, t)$ for the first time (in which the base graph is the complete graph K_n) and they show that $S(K_3, t)$ is isomorphic to the Tower of Hanoi game graph with t disks, see [13]. More generally, see [8], the t-th *generalized Sierpiński* of an arbitrary graph $G = (V, E)$, denoted by $S(G, t)$, is the graph with vertex set V^t (the set of all words of length t on the alphabet V) and two vertices $\mathbf{u} = u_1u_2 \dots u_t$ and $\mathbf{v} = v_1v_2 \dots v_t$ are adjacent in it if and only if there exist $i \in \{1, \dots, t\}$ such that

(i) $u_j = v_j$ if $j < i$,

(ii) $u_i \neq v_i$ and $u_iv_i \in E(G)$,

(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

In this case, $\mathbf{uv}$ is considered as a linking edge appeared at step i.

For convenient, we usually let $V = \{1, 2, \dots, n\}$, and some times to avoid confusing we may use the notation "$u_1u_2 \dots u_t$" instated of $u_1u_2 \dots u_t$. For example, see Figure 1, and Figure 2 for $S(C_4, 3)$ in which C_4 is assumed to be a cycle with the vertex set $\{1, 2, 3, 4\}$ and the edge set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$.

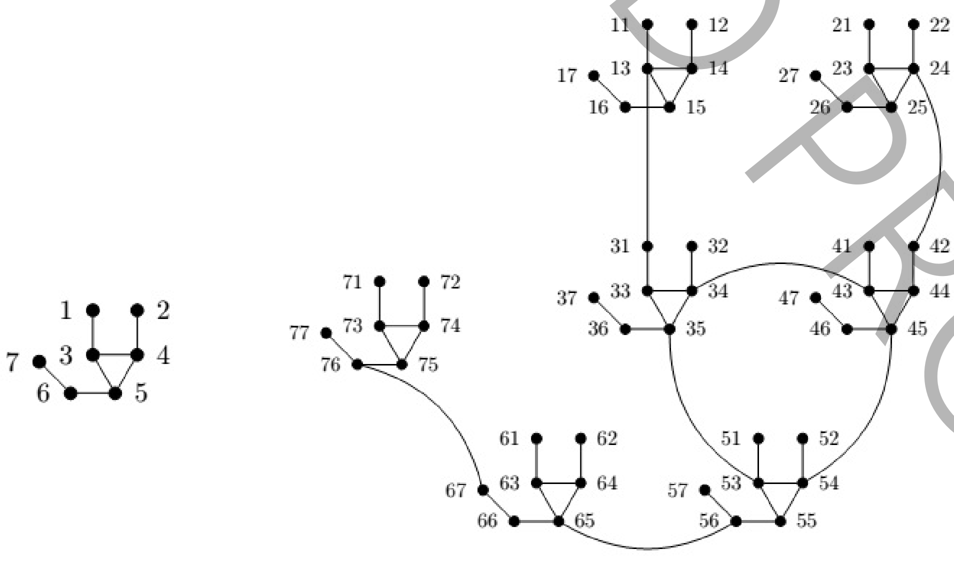

Figure 1: A graph G and its generalized Sierpiński $S(G, 2)$.

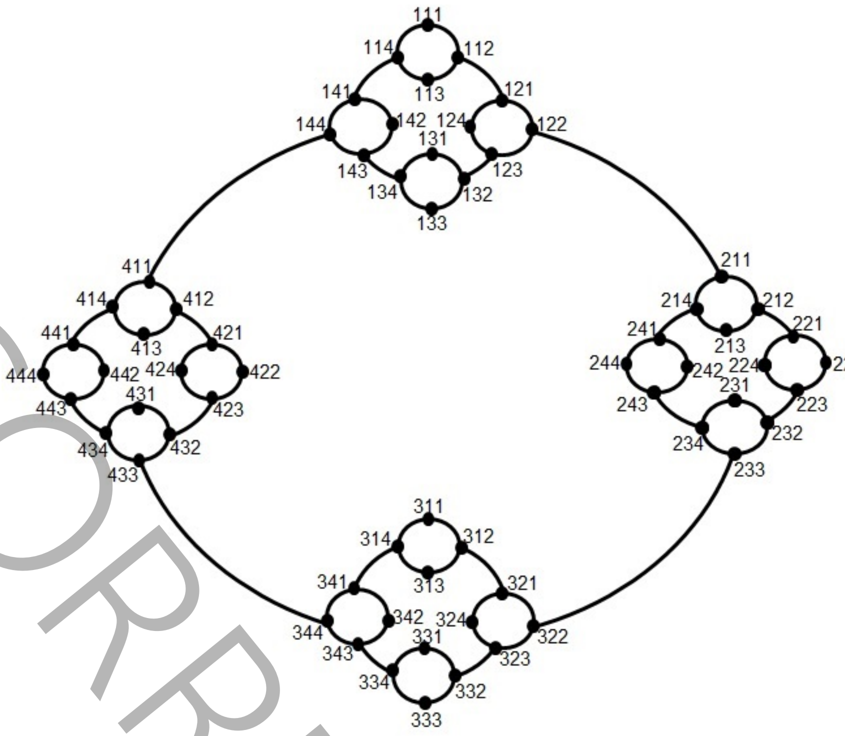

Figure 2: The generalized Sierpiński graph $S(C_4, 3)$.

In fact $S(G,t)$ is a fractal-like graph that uses G as a building block and in general, $S(G,t)$ can be constructed recursively from the base graph G with the following process: $S(G,1) = G$ and, for $t \geq 2$, we copy n times $S(G,t-1)$ and add the letter i at the beginning of each label of the vertices belonging to the copy of $S(G,t-1)$ corresponding to vertex i. Then for each edge $ij \in E(G)$, we add an edge between two vertices "$ijj\ldots j$" and "$jii\ldots i$" (a linking edge). Vertices of the form "$ii\ldots i$" (where $1 \leq i \leq n$) are called extreme vertices. For each $i \in \{1,2,\ldots,n\}$ let $S_i(G,t)$ be the subgraph of $S(G,t)$ induced by the vertices of the form "$i\ldots$" (i.e., vertices that they first entry is " i "). Note that $S_i(G,t)$ is isomorphic to $S(G,t-1)$ and consequently, $S(G,t)$ contains n^{t-1} copies of the graph $S(G,1) = G$. Also, if $ij \in E(G)$, then the vertex "$ijj\ldots j$' in the copy $S_i(G,t)$ is adjacent to the vertex "$jii\ldots i$' in copy $S_j(G,t)$ and this is the unique edge between these two copies. It is shown in [15] that the order of $S(G,t)$ is n^t and its size is $|E(G)|\frac{n^t-1}{n-1}$. When G is the complete graph K_n, the (ordinary) Sierpiński graph $S_n^t = S(K_n,t)$ is obtained. Sierpiński graphs are studied from numerous points of view and it is well known that $S(K_n,t)$ contains n (extreme) vertices of degree $n-1$ and all the other vertices are of degree n, see [11]. Hence, Sierpiński graphs $S(K_n,t)$ are almost regular. Polymer networks and WK-recursive networks can be modeled by generalized Sierpiński graphs, see [16]. Rodríguez-Velázquez et al. in [15] obtained closed formulae for several graphical parameters of the generalized Sierpiński graph $S(G,t)$ (including chromatic, vertex cover, clique, independence and domination number) in terms of the similar parameters of the base graph G. In [12] the degree sequence of $S(G,t)$ is completely determined in terms of the degree sequence of G. In [8] some interesting results about the generalized Sierpiński graphs (concerning their automorphism groups, perfect codes and distinguishing numbers) are obtained. The total chromatic number for some families of these graphs is determined by Geetha and Somasundaram in [7]. In [5] the distance between vertices of $S(G,t)$ is expressed in terms of the distance between vertices of the base graph G. The Roman domination number of $S(G,t)$ is investigated in [14].

2. Generalized Sierpiński

In this section, we first state a proposition for the generalized Sierpiński graph $S(G,t)$ when G has a pendant vertex. Then, we determine the existence of an exact double dominating set for $S(K_{1,n-1},t)$, $S(P_n,t)$ and $S(C_n,t)$. A necessary condition is stablished in Theorem 2.6 for the existence of an exact double dominating set and for the cases $n \in \{2,3\}$ it is completely determined. Finally, we establish a conjecture about the existence of an exact double dominating set for the Sierpiński graph $S(K_n,t)$. For paths and cycles, we have the following theorem.

Theorem 2.1 ([4]).
a) A path P_n has an exact double dominating set if and only if $n \equiv 2 \pmod 3$. If this holds, then the size of any

such set is $\frac{2(n+1)}{3}$.
b) A cycle C_n *has an exact double dominating set if and only if* $n \equiv 0 \pmod 3$. *If this holds, then the size of any such set is* $\frac{2n}{3}$.

Recall that a pendant vertex is a vertex of degree one.

Proposition 2.2. *If* G *is a graph with at least one pendant vertex which has not any exact double dominating set, then for each positive integer* t *the generalized Sierpiński graph* $S(G,t)$ *has not any exact double dominating set.*

Proof. Assume that $V(G) = \{1, 2, \dots, n\}$ and G does not have any exact double dominating set. Also, let i be a pendant vertex in G and its unique neighbor be j.
In $S(G,t)$ the vertex "$ii \dots iij$" is adjacent to the vertex "$ii \dots iji$". Since "$ii \dots ii$" is a pendant vertex in $S(G,t)$ and $N_{S(G,t)}[ii \dots i] = \{ii \dots ii, ii \dots ij\}$, if there exists an exact double dominating set D for $S(G,t)$, then we must have $\{ii \dots ii, ii \dots ij\} \subseteq D$. Hence "$ii \dots iji$" $\notin D$. Then, since no vertex in

$$D \setminus \{ii \dots ik : \ 1 \le k \le n\}$$

can dominate a vertex in $\{ii \dots ik : \ 1 \le k \le n\}$, the set $D \cap \{ii \dots ik : \ 1 \le k \le n\}$ must be an exact double dominating set for the induced subgraph of $S(G,t)$ on the set $\{ii \dots ik : \ 1 \le k \le n\}$. This is a contradiction because this induced subgraph and G are isomorphic and G does not have any exact double dominating set. The proof is complete. □

Corollary 2.3. *If* G *has two pendant vertices with the same neighbor in* G*, then* $S(G,t)$ *has not any exact double dominating set for each* $t \ge 1$. *Specially, for each* $n \ge 3$ *and each* $t \ge 1$, $S(K_{1,n-1}, t)$ *has not any exact double dominating set.*

Theorem 2.4. *Let* $t \ne 1$ *be a positive integer. Then there exists an exact double dominating set for* $S(P_n, t)$ *if and only if* t *is an odd integer and* $n = 2$.

Proof. If $n \equiv 0, 1 \pmod 3$, then Theorem 2.1 and Proposition 2.2 imply that $S(P_n,t)$ has not any exact double dominating set. Suppose that $n \equiv 2 \pmod 3$. If $n = 2$, then $S(P_2,t)$ is a path with 2^t vertices. Note that $2^t \equiv (-1)^t \pmod 3$ and Theorem 2.1 implies that $S(P_2,t)$ has an exact double dominating set if and only if t is an odd integer. Hereafter, let $n \ge 3$ and assume on the contrary that there exists an exact double dominating set D for $S(P_n,t)$. Now consider the below $2n$ vertices in $S(P_n,t)$.

$$\mathbf{x_1} = 11 \dots 1(n-1)1, \ \mathbf{x_2} = 11 \dots 1(n-1)2, \ \mathbf{x_3} = 11 \dots 1(n-1)3, \ \dots, \ \mathbf{x_n} = 11 \dots 1(n-1)n,$$
$$\mathbf{y_1} = 11 \dots 1n1, \ \mathbf{y_2} = 11 \dots 1n2, \ \mathbf{y_3} = 11 \dots 1n3, \ \dots, \ \mathbf{y_n} = 11 \dots 1nn.$$

Note that we have

$$\deg_{S(P_n,t)}(\mathbf{x_1}) = \deg_{S(P_n,t)}(\mathbf{y_1}) = \deg_{S(P_n,t)}(\mathbf{y_n}) = 1,$$

and among these $2n$ vertices only $\mathbf{x_{n-2}}$ has a neighbor not in $\{\mathbf{x_1}, \dots, \mathbf{x_n}, \mathbf{y_1}, \dots, \mathbf{y_n}\}$ (for $n = 8$ the induced subgraph of $S(P_n,t)$ on these $2n$ vertices is shown in Figure 3). Since $\deg_{S(P_n,t)}(\mathbf{y_1}) = 1$, we have $\{\mathbf{y_1}, \mathbf{y_2}\} \subseteq D$ and hence,

$$\mathbf{y_3} \notin D, \ \{\mathbf{y_4}, \mathbf{y_5}\} \subseteq D, \ \mathbf{y_6} \notin D, \ \dots, \ \{\mathbf{y_{n-1}}, \mathbf{y_n}\} \subseteq D,$$

because $n \equiv 2 \pmod 3$. Since $\{\mathbf{y_{n-1}}, \mathbf{y_n}\} \subseteq D$, we have $\mathbf{x_n} \notin D$ and hence, $\{\mathbf{x_{n-2}}, \mathbf{x_{n-1}}\} \subseteq D$. Also, since $\deg_{S(P_n,t)}(\mathbf{x_1}) = 1$, we have $\{\mathbf{x_1}, \mathbf{x_2}\} \subseteq D$, and hence

$$\mathbf{x_3} \notin D, \ \{\mathbf{x_4}, \mathbf{x_5}\} \subseteq D, \ \mathbf{x_6} \notin D, \ \dots, \ \{\mathbf{x_{n-4}}, \mathbf{x_{n-3}}\} \subseteq D.$$

Now we get

$$\left|D \cap N_{S(P_n,t)}[x_{n-2}]\right| = \left|\{\mathbf{x_{n-3}}, \mathbf{x_{n-2}}, \mathbf{x_{n-1}}\}\right| = 3 \ne 2,$$

which is a contradiction. Therefore, $S(P_n,t)$ has not any double dominating set and the proof is complete. □

Now we want to investigate the existence of exact double dominating sets in the generalized sierpiński graph of cycles. Note that the cycle C_3 can be considered as the complete graph K_3 and Theorem 9 completely determines whether there exists an exact double dominating set for the sierpiński graph $S(K_3,t)$.

Theorem 2.5. *Let* $t \ge 2$ *and* $n \ge 4$ *be two positive integers. Then, there exists an exact double dominating set* D *for* $S(C_n,t)$ *if and only if* $n \equiv 1 \pmod 3$ *and in this case we have* $|D| = 2n^{t-1}\lfloor \frac{n}{3} \rfloor$.

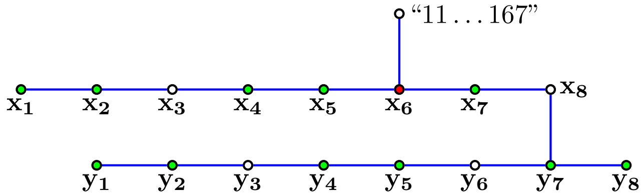

Figure 3: An induced part of $S(P_8, t)$.

Proof. Assume that $V(C_n) = \{1, 2, 3, \ldots, n\}$ and $E(C_n) = \{12, 23, 34, \ldots, (n-1)n, n1\}$. We consider the following three cases.

Case 1. $n \equiv 0 \pmod 3$:

In this case, the condition $n \geq 4$ implies that $n \in \{6, 9, 12, \ldots\}$. Assume on the contrary that $S(C_n, t)$ has an exact double dominating set D. Let $\mathbf{w} =$ "$11\ldots1$" which is a string of length $t-3$ (if $t = 3$, then $\mathbf{w}$ is an empty string!). Also, let

$$\mathbf{x_1} = \mathbf{w}411 = \text{"}11\ldots1411\text{"},\ \mathbf{x_2} = \mathbf{w}412,\ \mathbf{x_3} = \mathbf{w}413,\ \ldots,\ \mathbf{x_n} = \mathbf{w}41n.$$

Note that $14 \notin E(C_n)$, $N_{S(C_n,t)}(\mathbf{x_1}) = \{\mathbf{x_2}, \mathbf{x_n}\}$ and the induced subgraph of $S(C_n, t)$ on the set $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$ is isomorphic to C_n , see Figure 4 for $n = 9$.

If $\mathbf{x_1} \in D$, then since $|N_{S(C_n,t)}[\mathbf{x_1}] \cap D| = 2$ we must have $\mathbf{x_2} \in D$ or $\mathbf{x_n} \in D$. Without loss of generality, assume that $\mathbf{x_2} \in D$. Let

$$\mathbf{y_1} = \mathbf{w}421 = \text{"}11\ldots1421\text{"},\ \mathbf{y_2} = \mathbf{w}422,\ \mathbf{y_3} = \mathbf{w}423,\ \ldots,\ \mathbf{y_n} = \mathbf{w}42n.$$

Note that the induced subgraph of $S(C_n, t)$ on the set $\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_n}\}$ is isomorphic to C_n, and $\mathbf{x_2}\mathbf{y_1} \in E(S(C_n, t))$ because $12 \in E(C_n)$. Since $\{\mathbf{x_1}, \mathbf{x_2}\} \subseteq D$, we have $\mathbf{y_1} \notin D$. Since $14 \notin E(C_n)$, we have $N_{S(C_n,t)}(\mathbf{y_2}) = \{\mathbf{y_1}, \mathbf{y_3}\}$, and since $|N_{S(C_n,t)}[\mathbf{y_2}] \cap D| = 2$, we must have $\{\mathbf{y_2}, \mathbf{y_3}\} \subseteq D$. Thus, $\mathbf{y_4} \notin D$. Note that for each $4 \leq i \leq n$ we have $N_{S(C_n,t)}(\mathbf{y_i}) = \{\mathbf{y_{i-1}}, \mathbf{y_{i+1}}\}$. Thus, by repeating this argument we must have

$$\{\mathbf{y_5}, \mathbf{y_6}\} \subseteq D,\ \mathbf{y_7} \notin D, \ldots,\ \{\mathbf{y_{n-1}}, \mathbf{y_n}\} \subseteq D,$$

because $n \equiv 0 \pmod 3$, see Figure 4 (a). Now we obtain

$$|N_{S(C_n,t)}[\mathbf{y_1}] \cap D| = |\{\mathbf{x_2}, \mathbf{y_2}, \mathbf{y_n}\}| = 3 \neq 2,$$

a contradiction.

Thus, $\mathbf{x_1} \notin D$. This implies that $\{\mathbf{x_n}, \mathbf{x_2}\} \subseteq D$. Since $|N_{S(C_n,t)}[\mathbf{x_2}] \cap D| = 2$ and $N_{S(C_n,t)}(\mathbf{x_2}) = \{\mathbf{x_1}, \mathbf{x_3}, \mathbf{y_1}\}$ we must have $\mathbf{x_3} \in D$ or $\mathbf{y_1} \in D$. If $\mathbf{y_1} \in D$, then $\mathbf{x_3} \notin D$ and we have

$$\{\mathbf{x_4}, \mathbf{x_5}\} \subseteq D,\ \mathbf{x_6} \notin D, \ldots,\ \{\mathbf{x_{n-2}}, \mathbf{x_{n-1}}\} \subseteq D,$$

because $n \equiv 0 \pmod 3$, see Figure 4 (b). Now we get

$$|N_{S(C_n,t)}[\mathbf{x_{n-1}}] \cap D| = |\{\mathbf{x_n}, \mathbf{x_{n-1}}, \mathbf{x_{n-2}}\}| = 3 \neq 2,$$

a contradiction. Hence, $\mathbf{y_1} \notin D$ and $\mathbf{x_3} \in D$. This implies that $\mathbf{y_2} \in D$ and $\mathbf{y_3} \in D$, see Figure 5. Therefore,

$$\mathbf{y_4} \notin D,\ \{\mathbf{y_5}, \mathbf{y_6}\} \subseteq D,\ \mathbf{y_7} \notin D, \ldots, \{\mathbf{y_{n-1}}, \mathbf{y_n}\} \subseteq D.$$

This means that $|N_{S(C_n,t)}[\mathbf{y_1}] \cap D| = 3$, which is a contradiction. Thus, Case 1 leads to a contradiction.

Case 2. $n \equiv 2 \pmod 3$:

In this case, $n \in \{5, 8, 11, \ldots\}$. Assume on the contrary that $S(C_n, t)$ has an exact double dominating set D, and let $\mathbf{w}, \mathbf{x_1}, \ldots, \mathbf{x_n}, \mathbf{y_1}, \ldots, \mathbf{y_n}$ be as defined in Case 1, see Figure 6 for $n = 8$. If $\mathbf{x_1} \in D$, then $|D \cap N_{S(C_n,t)}[\mathbf{x_1}]| = 2$ and hence, $\mathbf{x_2} \in D$ or $\mathbf{x_n} \in D$. Without loss of generality, assume that $\mathbf{x_2} \in D$. Hence, $\mathbf{x_3} \notin D$ and by using the fact $n \equiv 2 \pmod 3$ we get (see Figure 6 (a) which is depicted for $n = 8$)

$$\{\mathbf{x_4}, \mathbf{x_5}\} \subseteq D,\ \mathbf{x_6} \notin D,\ \ldots,\ \{\mathbf{x_{n-1}}, \mathbf{x_n}\} \subseteq D.$$

Thus,

$$|N_{S(C_n,t)}[\mathbf{x_n}] \cap D| = |\{\mathbf{x_{n-1}}, \mathbf{x_n}, \mathbf{x_1}\}| = 3 \neq 2,$$

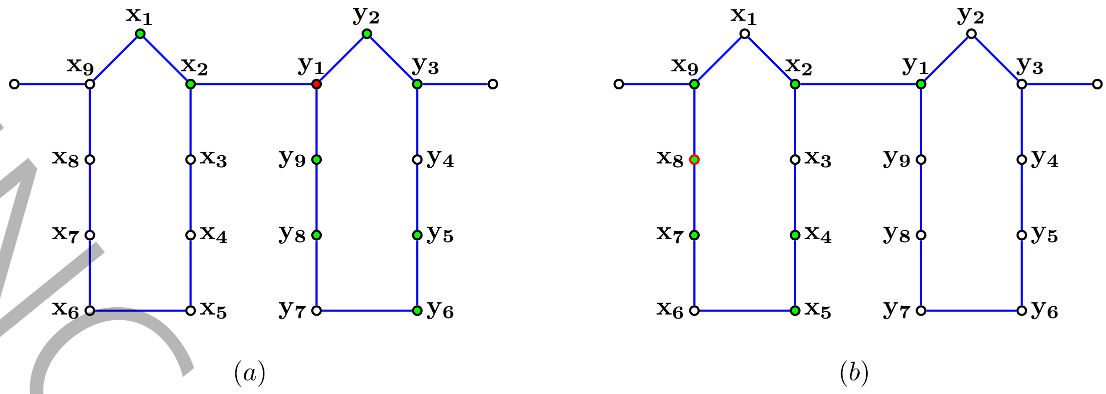

Figure 4: An induced part of $S(C_9, t)$.

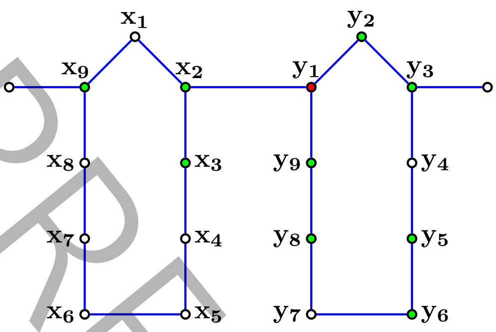

Figure 5:

which is a contradiction. Therefore, $\mathbf{x_1} \notin D$.
Now the condition $|N_{S(C_n,t)}[\mathbf{x_1}] \cap D| = 2$ implies that $\{\mathbf{x_n}, \mathbf{x_2}\} \subseteq D$, see Figure 6 (b) for $n = 8$. Since $|N_{S(C_n,t)}[\mathbf{x_2}] \cap D| = 2$, we have $|\{\mathbf{x_3}, \mathbf{y_1}\} \cap D| = 1$. If $\mathbf{x_3} \in D$, then $\mathbf{y_1} \notin D$. Since $N_{S(C_n,t)}[\mathbf{y_2}] = \{\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}\}$, we have $\mathbf{y_2} \in D$ and $\mathbf{y_3} \in D$, see Figure 6 (b). Therefore,

$$\mathbf{y_4} \notin D,\ \mathbf{y_5} \in D,\ \mathbf{y_6} \in D,\ \mathbf{y_7} \notin D,\ \ldots, \mathbf{y_n} \in D,$$

because $n \equiv 2 \pmod 3$. Now we see that $|N_{S(C_n,t)}[\mathbf{y_1}] \cap D| = 3$, a contradiction. Thus $\mathbf{x_3} \notin D$ and $\mathbf{y_1} \in D$. Also, since D is an exact double dominating set we see that

$$\mathbf{y_n} \notin D,\ \mathbf{y_{n-1}} \in D,\ \mathbf{y_{n-2}} \in D,\ \mathbf{y_{n-3}} \notin D,\ \ldots,\ \mathbf{y_4} \in D,\ \mathbf{y_3} \in D,$$

because $n \equiv 2 \pmod 3$. Let

$$\mathbf{z_1} = \mathbf{w}431 = \text{“}11\ldots1431\text{”},\ \mathbf{z_2} = \mathbf{w}432,\ \mathbf{z_3} = \mathbf{w}433,\ \ldots,\ \mathbf{z_n} = \mathbf{w}43n.$$

Since $|N_{S(C_n,t)}[\mathbf{y_3}] \cap D| = 2$, we have $\mathbf{z_2} \notin D$ (see figure 7). Since $|N_{S(C_n,t)}[\mathbf{z_1}] \cap D| = 2$, $\mathbf{z_1} \in D$ and $\mathbf{z_n} \in D$. Since $|N_{S(C_n,t)}[\mathbf{z_2}] \cap D| = 2$, $\mathbf{z_3} \notin D$. Since D is an exact double dominating set, if we proceed (counterclockwise) on the cycle induced by $\mathbf{z}_i's$, we see that

$$\mathbf{z_{n-1}} \notin D,\ \mathbf{z_{n-2}} \in D,\ \mathbf{z_{n-3}} \in D,\ \mathbf{z_{n-4}} \notin D,\ \ldots,\ \mathbf{z_4} \notin D,$$

because $n \equiv 2 \pmod 3$. These facts imply that $N_{S(C_n,t)}[\mathbf{z_3}] \cap D \subseteq \{\mathbf{w}344\}$ and hence, $|N_{S(C_n,t)}[\mathbf{z_3}] \cap D| \leq 1$ which is a contradiction. This contradiction completes the proof in this case.
Case 3. $n \equiv 1 \pmod 3$:
Consider $S(C_n, 2)$ and for each $i \in \{1, 2, \ldots, n\}$ let $D'_{1,i} \subseteq V(S(C_n, 2))$ be defined as

$$D'_{1,i} = \{ii\} \bigcup \left(\bigcup_{j=1}^{\frac{n-1}{3}} \{i(i-1+3j)\} \right),$$

in which the numbers are considered in module n. Now define $D_2 \subseteq V(S(C_n, 2))$ as

$$D_2 = \bigcup_{i=1}^{n} \left(\{i1, i2, \ldots, in\} \setminus D'_{1,i} \right).$$

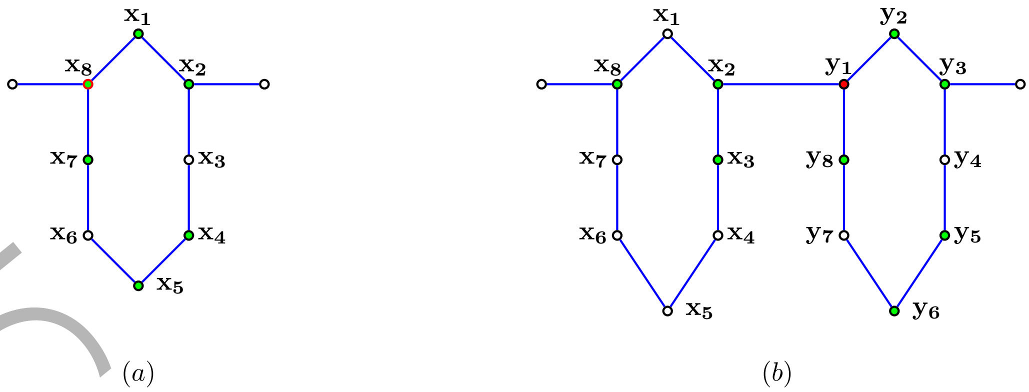

Figure 6: An induced part of $S(C_8, t)$.

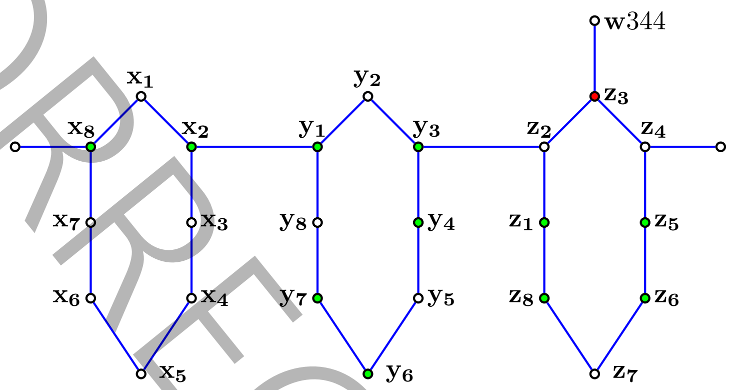

Figure 7:

It is straightforward to check that D is an exact double dominating set for $S(C_n, 2)$, see Figure 8 for $n = 7$. Also, note that $|D_2| = n\left(\frac{n-1}{3} \times 2\right)$ and $ii \notin D_2$ for each $i \in \{1, 2, \ldots, n\}$.

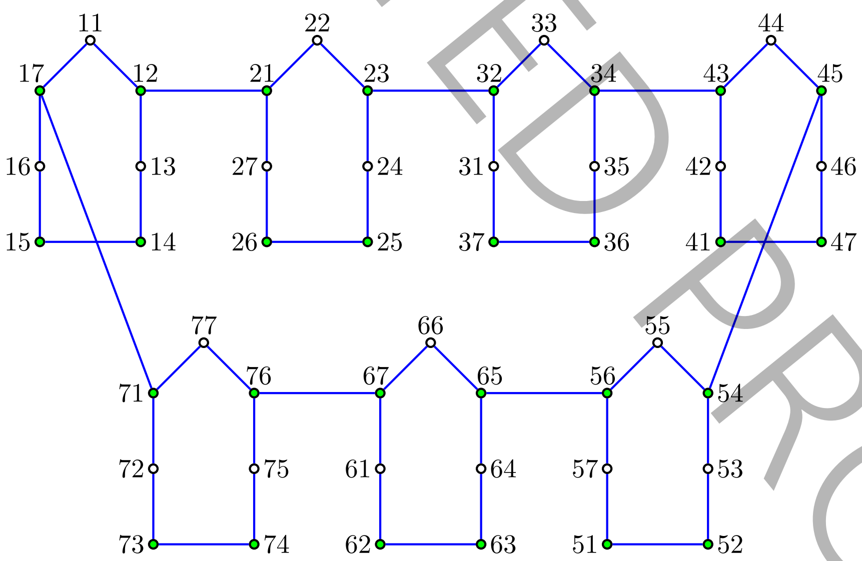

Figure 8: An exact double dominating set for $S(C_7, 2)$.

For $t = 3$ define the set $D_3 \subseteq V(S(C_n, 3))$ as

$$D_3 = \bigcup_{i=1}^{n} \left(\{i\} \times D_2\right),$$

in which $\{i\} \times \{j_1 j_2, j_1' j_2'\}$ is considered as $\{ij_1 j_2, ij_1' j_2'\}$. Note That $S(C_n, 3)$ consists of n copies of $S(C_n, 2)$ and for each $i \in \{1,2,\ldots,n\}$, $\{i\} \times D_2$ is an exact double dominating set for the i-th copy of $S(C_n,2)$ in $S(C_n,3)$ i.e., for $S_i(C_n,3)$. For each $i \in \{1,2,\ldots,n\}$, the vertex "$i(i+1)(i+1)$" in $S_i(C_n,3)$ is adjacent to the vertex "$(i+1)ii$" in $S_{i+1}(C_n,3)$ and

$$D_3 \cap \{i(i+1)(i+1),(i+1)ii\} = \emptyset.$$

If $|j-i| \geq 2$, then no vertex in $S_i(C_n,3)$ has a neighbor in $S_j(C_n,3)$. Thus, when $j \neq i$ no vertex in $D_3 \cap V(S_i(C_n,3))$ has a neighbor in $S_j(C_n,3)$. This means that D_3 is an exact double dominating set for $S(C_n,3)$. In a similar way and recursively, for each $t \geq 3$ define $D_t \subseteq V(S(C_n,t))$ as

$$D_t = \bigcup_{i=1}^{n} \left(\{i\} \times D_{t-1}\right),$$

in which $\{i\} \times \{j_1 j_2 \ldots j_{t-1}, j'_1 j'_2 \ldots j'_{t-1}\}$ is again considered as $\{ij_1 j_2 \ldots j_{t-1}, ij'_1 j'_2 \ldots j'_{t-1}\}$. By a similar argument (and in an inductive way), we can see that D_t is an exact double dominating set for $S(C_n,t)$. Also, note that $|D_t| = 2n^{t-1}\left(\frac{n-1}{3}\right)$. Now the proof is complete. □

Let $n \geq 3$ be a positive integer and K_n be the complete graph with the vertex set $\{1,2,\ldots,n\}$. Obviously, the set $D = \{1,2\}$ is an exact double dominating set for $K_n = S(K_n,1)$. When $t \geq 2$ we have

$$|V(S(K_n,t))| = n^t, \qquad |E(S(K_n,t))| = \frac{n^t-1}{n-1}\binom{n}{2},$$

and in $S(K_n,t)$ extreme vertices "$11\ldots1$", "$22\ldots2$", …, "$nn\ldots n$" have degree $n-1$ and other vertices have degree n.

Theorem 2.6. *Let $n \geq 2$ be a positive integer. If there exists an exact double dominating set D for $S(K_n,t)$, then t is an odd integer, $|D| = \frac{2(n^t+1)}{n+1}$ and*

$$\big|D \cap \{11\ldots1, 22\ldots2, \ldots, nn\ldots n\}\big| = 2.$$

Proof. If $n=2$, then $K_2 = P_2$, $S(K_2,t) = P_{2^t}$ and the result directly follows from Theorem 2.4, part (a) in Theorem 2.1 and this fact that two vertices "$11\ldots1$" and "$nn\ldots n$" = "$22\ldots2$" are leaves. Hence, assume that $n \geq 3$. For $t=1$ the proof is obvious. Hereafter, assume that $t \geq 2$ and suppose that $D \subseteq V(S(K_n,t))$ is an exact double dominating set for $S(K_n,t)$. By Theorem 1.1, the induced subgraph of $S(K_n,t)$ on the set D, i.e. $S(K_n,t)[D]$, is a matching and hence

$$\big|E\big(S(K_n,t)[D]\big)\big| = \frac{|D|}{2}.$$

Since D is an exact double dominating set, each vertex in $V(S(K_n,t)) \setminus D$ has exactly two neighbors in D. Hence,

$$\big|\big\{e = \mathbf{uv} \in E(S(K_n,t)) : \ \mathbf{u} \notin D, \ \mathbf{v} \in D\big\}\big| = |V(S(K_n,t)) \setminus D| \times 2 = (n^t - |D|)2.$$

Let

$$\overline{D} = V(S(K_n,t)) \setminus D, \qquad S = \overline{D} \cap \{11\ldots1, 22\ldots2, \ldots, nn\ldots n\}, \qquad s = |S|.$$

Note that for each $\mathbf{x} \in S$ and each $\mathbf{y} \in \overline{D} \setminus S$ we have

$$\deg_{S(K_n,t)}(\mathbf{x}) = n-1, \qquad \deg_{S(K_n,t)}(\mathbf{y}) = n.$$

Each vertex in $\overline{D}$ has exactly two neighbors in D. Thus, for each $\mathbf{x} \in S$ and each $\mathbf{y} \in \overline{D} \setminus S$ we have

$$\deg_{S(K_n,t)[\overline{D}]}(\mathbf{x}) = n-3, \qquad \deg_{S(K_n,t)[\overline{D}]}(\mathbf{y}) = n-2.$$

Now by using the handshaking lemma we see that

$$\begin{aligned}\big|E\big(S(K_n,t)[\overline{D}]\big)\big| &= \frac{1}{2}\big(|S|(n-3) + (|\overline{D}| - |S|)(n-2)\big)\\ &= \frac{1}{2}\big(s(n-3) + (n^t - |D| - s)(n-2)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{n^t-1}{n-1}\binom{n}{2} &= |E(S(K_n,t))| \\ &= \big|E\big(S(K_n,t)[D]\big)\big| + \big|E\big(S(K_n,t)[\overline{D}]\big)\big| + \big|\{\mathbf{uv}\in E(S(K_n,t)):\ \mathbf{u}\in\overline{D},\ \mathbf{v}\in D\}\big| \\ &= \frac{|D|}{2} + \frac{1}{2}\big(s(n-3)+(n^t-|D|-s)(n-2)\big) + (n^t-|D|)2.\end{aligned}$$

This implies that

$$|D| = \frac{2n^t+n-s}{n+1}.$$

Since $|D|$ is an integer, $(n+1)$ must divide $2n^t+n-s$, i.e. $2n^t+n-s \equiv 0 \pmod{n+1}$. Since $n \equiv -1 \pmod{n+1}$, we have

$$0 \overset{n+1}{\equiv} 2n^t+n-s \overset{n+1}{\equiv} 2(-1)^t+(-1)-s.$$

Thus,

$$s \overset{n+1}{\equiv} 2(-1)^t+(-1).$$

This using the fact $s\in\{0,1,2,\dots,n\}$ implies that either t is an even integer and $s=1$, or t is an odd integer and $s=n-2$.
Suppose (on the contrary) that t is an even integer and $s=1$. Hence,

$$|D| = \frac{2n^t+n-s}{n+1} = \frac{2n^t+n-1}{n+1}.$$

Since $|D|$ is an even integer (see Theorem 1.1), $2n^t+n-1$ must be an even integer and hence, n is an odd integer. Thus, $n=2k+1$ for some integer $k\geq 1$. This implies that

$$|D| = \frac{2n^t+n-1}{n+1} = \frac{2(2k+1)^t+(2k+1)-1}{(2k+1)+1} = \frac{(2k+1)^t+k}{k+1}.$$

Again, since $|D|$ is an even integer, $2(k+1)$ must divide $(2k+1)^t+k$. Hence,

$$0 \overset{2k+2}{\equiv} (2k+1)^t+k \overset{2k+2}{\equiv} (-1)^t+k = 1+k,$$

which is a contradiction because $2k+2$ do not divide $k+1$. This contradiction shows that t is an odd integer and $s=n-2$. Specially, we obtain

$$|D| = \frac{2n^t+n-s}{n+1} = \frac{2(n^t+1)}{n+1},$$

and

$$\big|D\cap\{11\dots1, 22\dots2, \dots, nn\dots n\}\big| = n-|S| = n-s = n-(n-2) = 2,$$

which completes the proof. □

Theorem 2.7. *For each odd integer $t\geq 1$, there exists an exact double dominating set for the Sierpiński graph $S(K_3,t)$.*

Proof. For $t=1$, Figure 9 (A_1) provides an exact double dominating set D_1 for $S(K_3,1)=K_3$ in which green vertices are the elements of D_1. Also, for $t=3$, (A_3) in Figure 9 provides an exact double dominating set D_3 for $S(K_3,3)$ in which green vertices are the elements of D_3. In an inductive way, assume that t is an odd integer and the structure A_t provides an exact double dominating set D_t for $S(K_3,t)$, see Figure 10 in which the inner vertices are not depicted and only three extreme vertices are clearly indicated. Let $\overline{D_t} = V\big(S(K_3,t)\big)\setminus D$ be the complement set of D_t with respect to $V\big(S(K_3,t)\big)$, and $\overline{A_t}$ be the structure corresponding to $\overline{D_t}$ (which is obtained from A_t in which D_t is replaced by $\overline{D_t}$). Then let A_{t+1} be the structure which is isomorphic to $S(K_3,t+1)$ in which its upper part is A_t, its below-left part is a clockwise rotation of $\overline{A_t}$, and its below-right part is a counter clockwise rotation of $\overline{A_t}$, see Figures 9 and 10. Similarly, let A_{t+2} be the structure which is isomorphic to $S(K_3,t+2)$ in which its upper part is A_{t+1}, its below-left part is a clockwise rotation of $\overline{A_{t+1}}$, and its below-right part is a counter clockwise rotation of $\overline{A_{t+1}}$. Now let $D_{t+2}\subseteq V\big(S(K_3,t+2)\big)$ be the set corresponding to the structure A_{t+2}. By the induction assumption, D_t is an exact double dominating set for $S(K_3,t)$ and hence, each vertex in A_t has two green vertices in its closed neighborhood. Note that the closed neighborhood of each of three extreme vertices $\mathbf{a},\mathbf{b},\mathbf{c}$ contains 3 members and the closed neighborhood of each non-extreme vertex contains 4 vertices. Hence, in $\overline{A_t}$ each extreme vertex has $3-2=1$ green vertex in its closed neighborhood and each non-extreme vertex has $4-2=2$ green vertex in its closed neighborhood. Thus, in A_{t+1} just two extreme vertices $\mathbf{b}', \mathbf{c}'$ have 1 green vertex in their (own) closed neighborhood and other vertices have 2 green vertices in their (own) closed neighborhood. Hence, in $\overline{A_{t+1}}$ just $\mathbf{a}'$ has 1 green vertex in its closed neighborhood and other vertices have 2 green vertices in their (own) closed neighborhood. Therefore and finally, in A_{t+2} each vertex has 2 green vertices in its closed neighborhood which means D_{t+2} is an exact double dominating set. This completes the proof. □

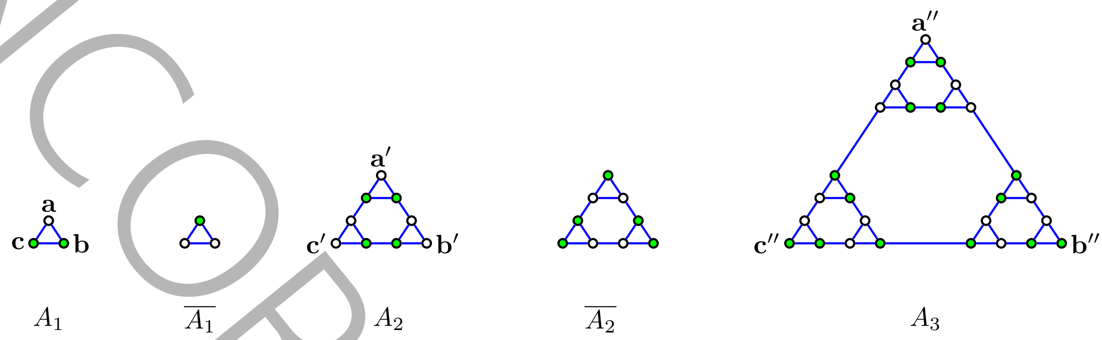

Figure 9:

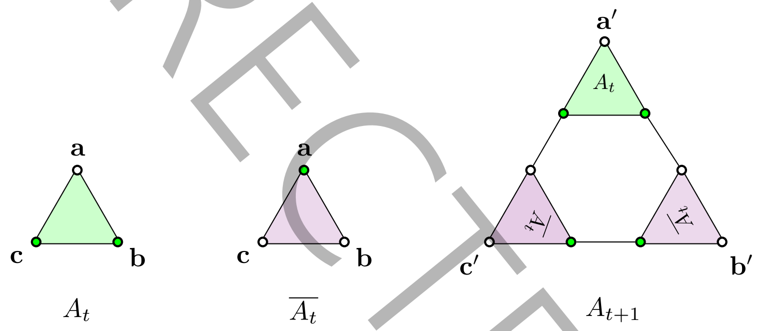

Figure 10:

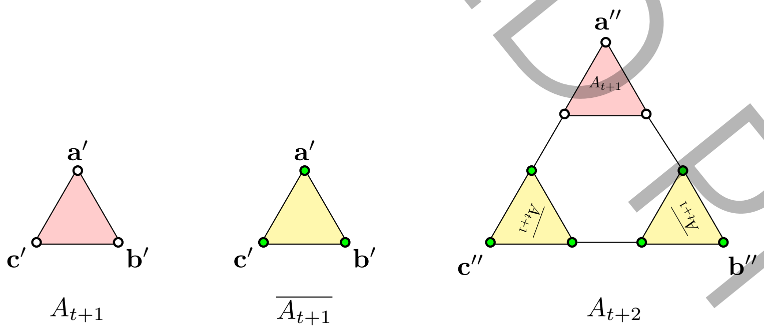

Figure 11:

Investigating some other small values of n and t, motivated us to provide the following conjecture.

Conjecture 2.8. *For each integer $n \geq 4$ and each odd integer $t \geq 1$, there exists an exact double dominating set for the Sierpiński graph $S(K_n, t)$.*

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The authors declare that they have no competing interests.

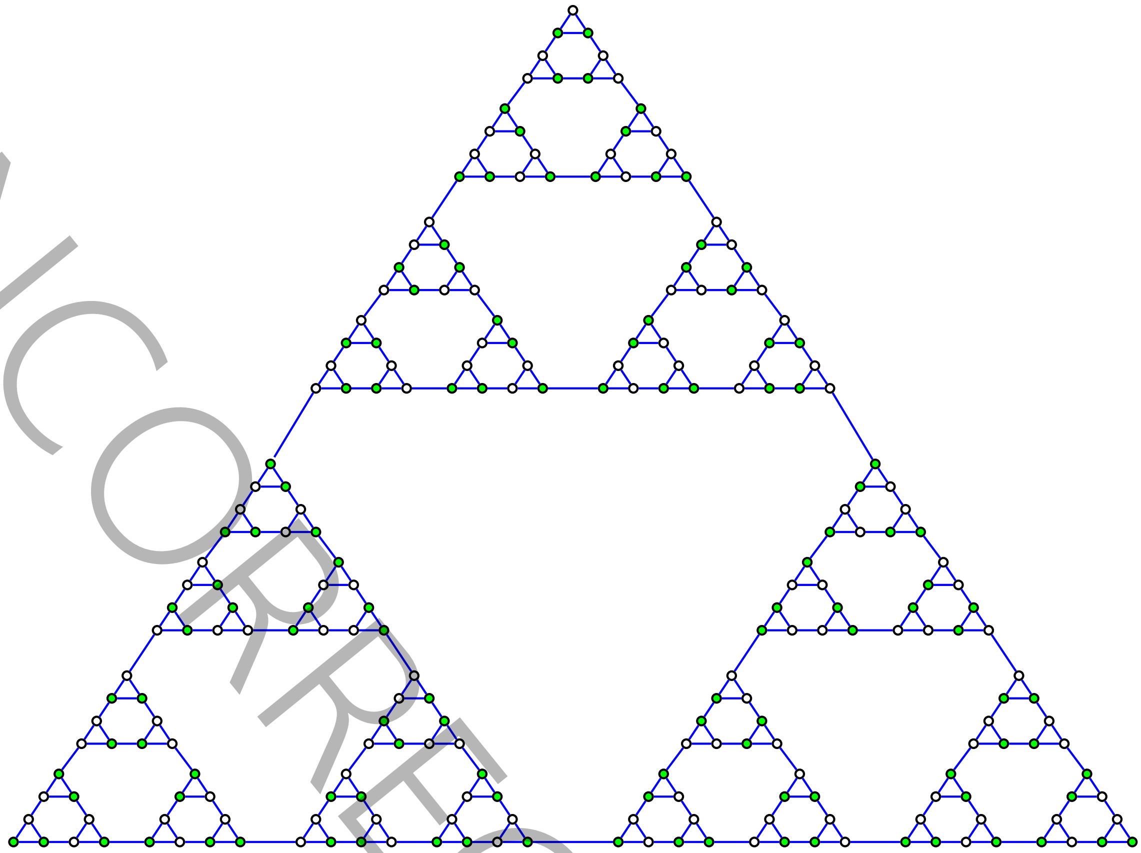

Figure 12: An exact double dominating set for $S(K_3, 5)$.

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