AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 7(2) (2026) 201-203
https://doi.org/10.22060/AJMC.2025.23797.1308

Short Contribution

A proof for a general form of the Serre-Swan theorem

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ABSTRACT: In this brief note, we present a proof for a general form of the Serre-Swan theorem.

Review History:

Received:02 January 2025
Revised:08 January 2025
Accepted:09 January 2025
Available Online:01 May 2026

Keywords:

C^*-algebra
Hilbert C^*-module
Serre-Swan theorem

1. Introduction

For a commutative C^*-algebra $C_0(\Omega)$, where Ω is a locally compact Hausdorff space, a general form of the Serre-Swan theorem [6] asserts that the category of Hilbert $C_0(\Omega)$-modules is equivalent to the category of continuous fields of Hilbert spaces over Ω (see [4, 7]). For more information on the continuous field of Banach spaces, see [1, 3]. This approach can be used to study the structure of Hilbert C^*-modules over a commutative C^*-algebra. For example, in [5] this approach has been used to study the frame existence problem for a unital and commutative C^*-algebra. However, there is no basic and direct proof of the theorem in the literature. In this brief note, we shall state a basic and direct proof of the mentioned theorem.

2. A proof for a general form of the Serre-Swan theorem

Henceforth, we assume that Ω is a locally compact Hausdorff space and $C_0(\Omega)$ is the commutative C*-algebra consisting of all complex-valued functions on Ω vanishing at infinity. In the following, we will recall the definition of continuous field of Hilbert spaces from [1].

Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of Hilbert spaces. Every element of $\prod_{\omega\in\Omega} H_\omega$, that is, every function x defined in Ω such that $x(\omega) \in H_\omega$ for every $\omega \in \Omega$, is called a vector field.

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Definition 2.1. *Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of Hilbert spaces. We denote by $C_0(\Omega) - \prod_{\omega\in\Omega} H_\omega$ the set*

$$\{x \in \prod_{\omega\in\Omega} H_\omega : \omega \mapsto \|x(\omega)\| \in C_0(\Omega)\}.$$

For a subset Γ of $C_0(\Omega) - \prod_{\omega\in\Omega} H_\omega$, we say an element $x \in C_0(\Omega) - \prod_{\omega\in\Omega} H_\omega$ is locally a member of Γ and denote it by $x \in_{loc} \Gamma$, if for every $\omega \in \Omega$ and every $\epsilon > 0$, there is some $x' \in \Gamma$such that $\|x(\omega') - x'(\omega')\| < \epsilon$, for all ω' in some neighborhood of ω.

Definition 2.2. *Let Ω be a locally compact Hausdorff space. A continuous field Γ of Hilbert spaces over Ω is a family $\{H_\omega\}_{\omega\in\Omega}$ of Hilbert spaces, with a linear subspace $\Gamma \subseteq \prod_{\omega\in\Omega} H_\omega$ satisfying the following properties:*

1. *Γ is a subset of $C_0(\Omega) - \prod_{\omega\in\Omega} H_\omega$.*
2. *For every $\omega \in \Omega$, the set $\{x(\omega) : x \in \Gamma\}$ is equal to H_ω.*
3. *If $x \in_{loc} \Gamma$, then $x \in \Gamma$.*

We denote a continuous field of Hilbert spaces over a locally compact Hausdorff space Ω by $(\{H_\omega\}_{\omega\in\Omega}, \Gamma)$. Indeed, $(\{H_\omega\}_{\omega\in\Omega})$ can be considered as a Hilbert $C_0(\Omega)$-module equipped with the point-wise multiplication

$$(f.x)(\omega) = f(\omega)x(\omega),$$

for every $\omega \in \Omega$ and $C_0(\Omega)$-valued inner product

$$\langle x, y\rangle(\omega) = \langle x(\omega), y(\omega)\rangle,$$

for all $f \in C_0(\Omega), x, y \in \Gamma$, and $\omega \in \Omega$ (for more details, we refer to [1]). Moreover, corresponding to every Hilbert $C_0(\Omega)$-module $\mathcal{E}$, there is a unique continuous field of Hilbert spaces isomorphic to $\mathcal{E}$ (see [?, 7]). However, there is no direct proof for the following fact in the literature, so we present its proof.

Theorem 2.3. *Let $\mathcal{E}$ be a Hilbert $C_0(\Omega)$-module. There is a continuous field of Hilbert spaces $(\{H_\omega\}_{\omega\in\Omega}, \Gamma)$ isomorphic to $\mathcal{E}$.*

Proof. For every $\omega \in \Omega$, let $N_\omega = \{x \in \mathcal{E} : \langle x, x\rangle = 0\}$. Since, for every $x \in \mathcal{E}$ there is some $y \in \mathcal{E}$ that $x = y\langle y, y\rangle^{\frac{1}{2}}$, one can easily see that N_ω is a closed right Hilbert submodule of $\mathcal{E}$. Consider the quotient space $H_\omega = \frac{\mathcal{E}}{N_\omega}$, equipped with inner product $[x + N_\omega, y + N_\omega]_\omega := \langle x, y\rangle(\omega)$. Indeed, $(H_\omega, [.,.]_\omega)$ is complete and hence a Hilbert space. Now, for every $x \in \mathcal{E}$, define $\hat{x} \in \prod_{\omega\in\Omega} H_\Omega$ by

$$\hat{x}(\omega) = x + N_\omega.$$

Since $\|\hat{x}(\omega)\|^2 = \langle x, x\rangle(\omega)$, we have $\hat{x} \in C_0(\Omega) - \prod_{\omega\in\Omega} H_\omega$.

Now, we let $\Gamma = \{\hat{x} : x \in E\}$ and show that Γ is a continuous field of Hilbert spaces. Indeed, Γ satisfies Properties (1) and (2) in Definition 2.2. Suppose that $\pi \in C_0 - \prod_{\omega\in\Omega} H_\omega$ and for every $\epsilon > 0$ and every $\omega' \in \Omega$, there is an open subset U of Ω consisting of ω' that for every $\omega \in U$, $\|\pi(\omega) - \hat{x}(\omega)\| < \epsilon$. Let $\epsilon > 0$. Since, $\pi \in C_0 - \prod_{\omega\in\Omega} H_\omega$, there is a compact set K, such that for every $\omega \in K^c$, $\|\pi(\omega)\| < \epsilon$. Now, by the assumption, for every $\omega' \in K$, there exist an open subset $U_{\omega'}$, containing ω' and some $x \in \mathcal{E}$ such that for every $\omega \in \Omega$, $\|\pi(\omega) - \hat{x}(\omega)\| < \epsilon$. Now, since the compact set K is a subset of $\bigcup_{\omega'\in K} U_{\omega'}$, there exist $\omega_1, \omega_2, \ldots, \omega_n \in K$, $x_1, x_2, \ldots, x_n \in \mathcal{E}$ and open subsets $U_{\omega_1}, U_{\omega_2}, \ldots, U_{\omega_n}$ containing $\omega_1, \omega_2, \ldots, \omega_n$ such that for every $\omega \in U_i$, $\|\pi(\omega) - \hat{x}_i(\omega)\| < \epsilon$. Now, the partition of unity theorem yields that for the finite family of open sets $\{U_i : 1 \leq i \leq n\}$ there exist a finite family $\{f_i : \Omega \to [0, 1] | 1 \leq i \leq n\}$ of continuous functions, such that for every $1 \leq i \leq n$, Supp $f_i \subseteq U_i$ and also $\Sigma_{i=1}^n f_i(\omega) = 1$, for all $\omega \in \bigcup_{i=1}^n U_i$. (We refer to [2], for this version of partition of unity theorem).

Now, considering $y = \Sigma_{i=1}^n x_i.f_i$, we have $y \in \mathcal{E}$. Moreover, for every $\omega \in \Omega$,

$$\|\pi(\omega) - \hat{y}(\omega)\| < \epsilon.$$

In order to see that, for an arbitrary $\omega \in \Omega$, if $\omega \in \bigcup_{i=1}^n U_i$, let $I_\omega = \{i \in \{1, \ldots, n\} : \omega \in U_i\}$. The following holds:

$$\begin{aligned} \|\pi(\omega) - \hat{y}(\omega)\| &= \|\Sigma_{i=1}^n f_i(\omega)\pi(\omega) - \Sigma_{i=1}^n \hat{x}_i(\omega) f_i(\omega)\| \\ &\leq \Sigma_{i=1}^n f_i(\omega)\|\pi(\omega) - \hat{x}_i(\omega)\| \\ &= \Sigma_{i\in I_\omega} f_i(\omega)\|\pi(\omega) - \hat{x}_i(\omega)\| + \Sigma_{i\notin I_\omega} f_i(\omega)\|\pi(\omega) - \hat{x}_i(\omega)\| \\ &= \Sigma_{i\in I_\omega} f_i(\omega)\|\pi(\omega) - \hat{x}_i(\omega)\| < \epsilon\Sigma_{i\in I_\omega} f_i(\omega) \leq \epsilon. \end{aligned} \tag{1}$$

If $\omega \in (\bigcup_{i=1}^n U_i)^c$, then for every $i \in \{1, \ldots, n\}$, we have $f_i(\omega) = 0$. Hence, $\hat{y}(\omega) = 0$, which yields that $\|\pi(\omega) - \hat{y}(\omega)\| = \|\pi(\omega)\| < \epsilon$. So, for every $\omega \in \Omega$, $\|\pi(\omega) - \hat{y}(\omega)\| < \epsilon$. In other words, $\|\pi - \hat{y}\| = \sup_{\omega\in\Omega} \|\pi(\omega) - \hat{y}(\omega)\| \leq \epsilon$. Hence, with choosing $\epsilon = \frac{1}{k}$, for every $k \in \mathbb{N}$, there is some $x_k \in \mathcal{E}$ with $\|\pi - \hat{x}_k\| \leq \frac{1}{k}$. Since, for every $x \in \mathcal{E}$, $\|x\| = \|\hat{x}\|$, one can easily see that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}$ and consequently, there is some $x \in \mathcal{E}$ such that $\{x_k\}_{k\in\mathbb{N}}$ converges to it. Moreover, $\pi = \hat{x}$ and so $\pi \in \Gamma$. $\square$

Acknowledgement

The second author is supported by the National Elites Foundation.

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Mohammad Bagher Asadi, Zahra Hassanpour-Yakhdani, A proof for a general form of the Serre-Swan theorem, AUT J. Math. Comput., 7(2) (2026) 201-203
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