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Original Article

Zero product determined of abstract Segal algebras

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ABSTRACT: At the present article, we investigate the notion of zero product determined for category of abstract Segal algebras. Indeed, where $\mathfrak{X}$ is an abstract segal algebra with respect to $\mathfrak{A}$, we prove that under some conditions this notion inherits from $\mathfrak{X}$ to $\mathfrak{A}$. Applying these results, we obtain some sufficient conditions in which the Fourier algebra $A(\mathfrak{G})$ is zero product determined, when $\mathfrak{G}$ is a locally compact group.

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1. Introduction

Studying of disjointness preserving operators on (Banach) function algebras is due to Beckenstein and Narici in [3]. Afterwards, this problem has been studied by many authors with the name Lamperti operators (or separating maps). For more details, see [7, 9, 10]. Moreover, it is worthwhile to mention that as a generalization of Lamperti operators, some authors introduced and considered the form of operators preserving zero products on some classes of Banach algebras. We also mention that zero product preserving maps can be considered as an extension of weighted homomorphisms on Banach algebras.

Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are two Banach algebras and $S : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a linear operator. We say that S is *zero product preserving map* if

$$x_1, x_2 \in \mathfrak{A},\ x_1x_2 = 0 \quad \Longrightarrow \quad S(x_1)S(x_2) = 0.$$

By this hypothesis, it is clear that the induced bilinear operator $\psi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{B}$ defined by $\psi(x_1, x_2) = S(x_1)S(x_2)$ has the following property:

$$x_1, x_2 \in \mathfrak{A},\ x_1x_2 = 0 \quad \Longrightarrow \quad \psi(x_1, x_2) = 0. \tag{1}$$

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In this case, the bilinear operator $\psi : \mathfrak{A} \times \mathfrak{A} \longrightarrow B$ is called *preserving zero products*. These such bilinear operators were introduced and considered by Alaminos et. al. in [2]. Mapping preserving zero products have an important role in studying Lamperti operators, for example see [1, 5, 11, 15].

We also note that if ρ is a bounded linear functional on $\mathfrak{A}$, then the bounded bilinear operator $\psi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{C}$, which is defined by $\psi(x_1, x_2) = \rho(x_1x_2) \quad (x_1, x_2 \in \mathfrak{A})$, satisfies the property (1). Motivated by this fact, a Banach algebra $\mathfrak{A}$ is called *zero product determined* if every bounded preserving zero product bilinear operator $\psi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{C}$, has the form

$$\psi(x_1, x_2) = \rho(x_1x_2) \qquad (x_1, x_2 \in \mathfrak{A}),$$

for some bounded linear functional $\rho : \mathfrak{A} \longrightarrow \mathbb{C}$.

We mention that the notion of zero product determined first has been defined and studied in the algebraic version without the continuity hypothesis. In the theory of Banach algebras this concept plays a key role in the studying local derivations and homomorphisms on Banach algebras. Recently, M. Brešar in [4] has presented a comprehensive information on this subject.

In the case where $\mathfrak{G}$ is a locally compact group, it is proved that the group convolution algebra $L^1(\mathfrak{G})$ is always zero product determined [4, Theorem 5.21]. Motivated by this result, we shall study the notion of zero product determined for some certain Banach algebras associated with locally compact groups.

A Banach space $(S(\mathfrak{G}), \|\cdot\|_{S(\mathfrak{G})})$ is called Segal algebra if it is a dense subspace of the group algebra $L^1(\mathfrak{G})$ with the following conditions:

(i) there is a constant $C > 0$ such that for all $h \in S(\mathfrak{G})$, $\|h\|_1 \leq C\|h\|_{S(\mathfrak{G})}$.

(ii) $S(\mathfrak{G})$ is invariant under left translation and for all $h \in S(\mathfrak{G})$ the map $g \mapsto \delta_g * h$ is continuous.

(iii) $\|\delta_g * h\|_{S(\mathfrak{G})} = \|h\|_{S(\mathfrak{G})}$, for all $g \in \mathfrak{G}$ and $h \in S(\mathfrak{G})$.

Recall that Segal algebras on locally compact groups are a special case of a general well-known Banach algebras which is called abstract Segal algebra (see Definition 2.1).

In this paper, we intend to investigate the concept of zero product determined for category of abstract Segal algebras $\mathfrak{X}$ with respect to $\mathfrak{A}$. To do this, we first study a milder notion which is so called the property $\mathbb{B}$ for abstract Segal algebras. Indeed, we show that under some conditions this notion inherits from $\mathfrak{X}$ to $\mathfrak{A}$. As an application, we obtain some sufficient conditions on the locally compact group $\mathfrak{G}$ in which the Fourier algebra $A(\mathfrak{G})$ is zero product determined.

2. Main results

First, we begin with the definition of abstract Segal algebras.

Definition 2.1. *Suppose that $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$ and $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ are Banach algebras and $\mathfrak{X}$ is a dense left ideal in $\mathfrak{A}$. Then we say that $\mathfrak{X}$ is an abstarct Segal algebra with respect to $\mathfrak{A}$ if*

(i) there is a constant $M_1 > 0$ such that for all $b \in \mathfrak{X}$, $\|b\|_{\mathfrak{A}} \leq M_1\|b\|_{\mathfrak{X}}$,

(ii) there is a constant $M_2 > 0$ such that

$$\|ab\|_{\mathfrak{X}} \leq M_2\|b\|_{\mathfrak{X}}\|a\| \qquad (a \in \mathfrak{A}, b \in \mathfrak{X}).$$

Moreover, if $\mathfrak{X}$ is an (two-sided) ideal in $\mathfrak{A}$ and

$$\|ba\|_{\mathfrak{X}} \leq M_2\|b\|_{\mathfrak{X}}\|a\|_{\mathfrak{A}} \qquad (a \in \mathfrak{A}, b \in \mathfrak{X}),$$

then $\mathfrak{X}$ is called a symmetric abstarct Segal algebra.

Here, we want to study the notion of zero product determined for abstract Segal algebras. We first need to study a milder notion which is so called the property $\mathbb{B}$ for abstract Segal algebras. We note that the property $\mathbb{B}$ originally introduced and studied by Alaminos et al. in [1] for some certain Banach algebras. Also, this property is directly related with the notion of zero product determined.

Definition 2.2. *We say that a Banach algebra $\mathfrak{A}$ has the property $\mathbb{B}$ if every bounded bilinear operator $\phi : \mathfrak{A}\times\mathfrak{A} \to \mathbb{C}$ in which preserves zero products is balanced, i.e,*

$$\phi(x_1x_2, x_3) = \phi(x_1, x_2x_3) \qquad (x_1, x_2, x_3 \in \mathfrak{A}).$$

It is easy to verify that all zero product determined Banach algebras have the property $\mathbb{B}$. However, there are many classes of Banach algebras with property $\mathbb{B}$ which are not zero product determined. Also, it is proved in [1, Lemma 2.3] that every Banach algbera with a left bounded approximate identity which has the property $\mathbb{B}$ is zero product determined. It is worthwhile to mention that all group algebras and C^*-algebras have the property $\mathbb{B}$, see [4, Theorem 5.19, Theorem 5.21].

In the sequel, we prove that this notion inherits from $\mathfrak{X}$ to $\mathfrak{A}$.

Theorem 2.3. *Assume that $\mathfrak{X}$ is an abstract Segal algebra with respect to $\mathfrak{A}$ such that $\mathfrak{X}$ has the property $\mathbb{B}$. Then $\mathfrak{A}$ has also the property $\mathbb{B}$.*

Proof. Assume that $\phi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{C}$ is a bounded bilinear operator satisfying $\phi(x_1, x_2) = 0$ whenever $x_1, x_2 \in \mathfrak{A}$ are such that $x_1x_2 = 0$. Now, we consider the operator $\psi : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathbb{C}$ by

$$\psi(y_1, y_2) = \phi(y_1, y_2), \qquad (y_1, y_2 \in \mathfrak{X}).$$

By the property (ii) of Definition 2.1, for each $y_1, y_2 \in \mathfrak{X}$ we have

$$\begin{aligned}|\psi(y_1, y_2)| &= |\phi(y_1, y_2)| \leq \|\phi\| \|y_1\|_{\mathfrak{A}} \|y_2\|_{\mathfrak{A}} \\ &\leq M_1^2 \|\phi\| \|y_1\|_{\mathfrak{X}} \|y_2\|_{\mathfrak{X}},\end{aligned}$$

and so the map ψ is continuous. Moreover, it is easy to verify that the bilinear operator ψ preserves zero products and so is balanced. For all $x, y, z \in \mathfrak{A}$ we note that the sequences $(x_n)_{n\in\mathbb{N}}, \{(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}\}$ can be chosen such that

$$x_n \overset{\|\cdot\|_{\mathfrak{A}}}{\longrightarrow} x, \quad y_n \overset{\|\cdot\|_{\mathfrak{A}}}{\longrightarrow} y, \quad z_n \overset{\|\cdot\|_{\mathfrak{A}}}{\longrightarrow} z.$$

Therefore, we have

$$\begin{aligned}\phi(zx, y) &= \lim_n \phi(z_nx_n, y_n) = \lim_n \psi(z_nx_n, y_n) \\ &= \lim_n \psi(z_n, x_ny_n) = \lim_n \phi(z_n, x_nz_n) \\ &= \phi(z, xy).\end{aligned}$$

Hence, the Banach algebra $\mathfrak{A}$ also has the property $\mathbb{B}$, as required. □

Theorem 2.4. *Suppose that $\mathfrak{X}$ is an abstract Segal algebra with respect to $\mathfrak{A}$. If $\mathfrak{X}$ has the property $\mathbb{B}$ and $\mathfrak{A}$ has a bounded left approximate identity, then $\mathfrak{A}$ is zero product determained.*

Proof. It obviously follows from Theorem 2.3 and [4, theorem 5.5]. □

Corollary 2.5. *Assume that $\mathfrak{X}$ is an abstract Segal algebra with respect to $\mathfrak{A}$ such that $\mathfrak{A}$ has a bounded left approximate identity. If the $\|\cdot\|_{\mathfrak{X}}$-closed linear span generates by $\mathfrak{X}^2$ has the porperty $\mathbb{B}$, then $\mathfrak{A}$ is zero product determined.*

Proof. Since $\mathfrak{A}$ has a bounded left approximate identity, we deduce that $\mathfrak{A} = \{x_1x_2 : x_1, x_2 \in \mathfrak{A}\}$. Now, It follows from [14, Lemma 3.1] that the $\|\cdot\|_{\mathfrak{X}}$-closed linear span generates by $\mathfrak{X}^2$ is also an abstract Segal algebra with respect to $\mathfrak{A}$. Hence by applying Theorem 2.4, we conclude that $\mathfrak{A}$ is a zero product determined Banach algebra. □

Definition 2.6. *An abstract Segal algebra $\mathfrak{X}$ is called essential with respect to $\mathfrak{A}$ if it is symmetric and*

$$\overline{\mathfrak{A}\mathfrak{X}}^{\|\cdot\|_{\mathfrak{X}}} = \overline{\mathfrak{X}\mathfrak{A}}^{\|\cdot\|_{\mathfrak{X}}} = \mathfrak{X}.$$

Theorem 2.7. *Suppose that $\mathfrak{X}$ is an essential symmetric abstract Segal algebra with respect to $\mathfrak{A}$. If $\mathfrak{A}$ has the property $\mathbb{B}$, then the Banach algebra $\mathfrak{X}$ is so.*

Proof. Choose a bounded bilinear operator $\phi : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathbb{C}$ in which $\phi(y_1, y_2) = 0$ whenever $y_1, y_2 \in \mathfrak{X}$ are such that $y_1y_2 = 0$. First, we consider the two arbitrary elements $y_0, y_0' \in B$. Then define the bilinear operator $\psi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{C}$ as follows:

$$\psi(x_1, x_2) = \phi(y_0x_1, x_2y_0') \qquad (x_1, x_2 \in \mathfrak{A}).$$

We note that for all $x_1, x_2 \in \mathfrak{A}$ we have

$$\begin{aligned}|\psi(x_1, x_2)| &= |\phi(y_0x_1, x_2y_0')| \\ &\leq \|\phi\| \|y_0x_1\|_{\mathfrak{X}} \|x_2y_0'\|_{\mathfrak{X}} \\ &\leq M_2^2 \|\phi\| \|y_0\|_{\mathfrak{X}} \|x_1\|_{\mathfrak{A}} \|x_2\|_{\mathfrak{A}} \|y_0'\|_{\mathfrak{X}},\end{aligned}$$

and so the ψ is continuous. Furthermore , for all $x_1, x_2 \in \mathfrak{A}$ such that $x_1x_2 = 0$ we have $y_0x_1x_2y_0' = 0$ and so

$$\psi(x_1, x_2) = \phi(y_0x_1, x_2y_0') = 0.$$

This follows that the map ψ preserves zero products. On the other hand, since $\mathfrak{A}$ has the property $\mathbb{B}$, we conclude that ψ is balanced. Thus,

$$\begin{aligned}\phi(y_0x_1x_2, x_3y_0') &= \psi(x_1x_2, x_3)\\ &= \psi(x_1, x_2x_3)\\ &= \phi(y_0x_1, x_2x_3y_0'), \qquad (x_1, x_2, x_3 \in \mathfrak{A}).\end{aligned}$$

Now, since ϕ is a continuous bilinear operator and $\mathfrak{X}$ is essential, it follows that

$$\phi(za, w) = \phi(z, aw) \qquad (z, w \in \mathfrak{X}, a \in \mathfrak{A}).$$

Hence, the map ϕ is balanced. This shows that the Banach algebra $\mathfrak{X}$ has also the property $\mathbb{B}$. □

In the consequence, we obtain some conditions on a locally compact group $\mathfrak{G}$ such that the Fourier algebra $A(\mathfrak{G})$ is zero product determined.

Assume that $f, g \in L^2(\mathfrak{G})$ and define

$$f^\flat(g) = f(g^{-1}), \quad \widehat{f}(g) = \overline{f(g^{-1})} \quad (g \in \mathfrak{G}).$$

Then it is easy to verify that $(f * \widehat{g})^\flat \in C_0(\mathfrak{G})$ and the Fourier algebras of $\mathfrak{G}$ is define as follows:

$$A(\mathfrak{G}) = \{(f * \widehat{g})^\flat : f, g \in L^2(\mathfrak{G})\} \subseteq C_0(\mathfrak{G}),$$

which is equipped with the pointwise product and the quotient norm induced from $L^2(\mathfrak{G})\widehat{\otimes}L^2(\mathfrak{G})$. Indeed, $A(\mathfrak{G})$ is a (commutative) Banach algebra. Moreover, using Leptin's theorem it is known that $A(\mathfrak{G})$ has a bounded approximate identity, whenever $\mathfrak{G}$ is amenable [13, Theorem 7.1.3]. For more details on the Fourier algebra $A(\mathfrak{G})$, we refer the reader to [6].

First, we have the following result.

Theorem 2.8. *Assume that $\mathfrak{G}$ is an amenable totally disconnected group. Then $A(\mathfrak{G})$ is zero product determined.*

Proof. Since $\mathfrak{G}$ is totally disconnected and the fact that the idempotents in $A(\mathfrak{G})$ are characteristic functions of open compact subsets in the coset ring of $\mathfrak{G}$, we conclude that $A(\mathfrak{G})$ generates by idempotents. This follows from [4, Theorem 5.14] that $A(\mathfrak{G})$ has the property $\mathbb{B}$. Moreover, since $\mathfrak{G}$ is a amenable so $A(\mathfrak{G})$ has a bounded approximate identity. Now, by [4, Proposition 5.5], we deduce that $A(\mathfrak{G})$ is zero product determined. □

Theorem 2.9. *If $\mathfrak{G}$ is almost abelian, then $A(\mathfrak{G})$ is zero product determined.*

Proof. Using [8, Theorem 2.3], it follows that the Fourier algebra $A(\mathfrak{G})$ is amenable and so has a bounded approximate identity. On the other hand, by [12, Example 5.1(ii)], we conclude that $A(\mathfrak{G})$ has the property $\mathbb{B}$. This completes the proof. □

Suppose that $\mathfrak{G}$ is a locally compact group and $1 \le p \le \infty$. We consider $SA^r(\mathfrak{G}) = A(\mathfrak{G}) \cap L^r(\mathfrak{G})$, equipped with the following norm

$$\|h\| = \|h\|_p + \|h\|_{A(\mathfrak{G})} \qquad (h \in SA^p(\mathfrak{G})).$$

It is easy to checked that $SA^p(\mathfrak{G})$ is an (symmetric) abstract Segal algebra with respect to $A(\mathfrak{G})$. In the sequel, using Theorem 2.10, we obtain another condition such that $A(\mathfrak{G})$ is zero product determined. In the case where $\mathfrak{G}$ is an amenable totally disconnected group, it follows from Theorem 2.8 and Theorem 2.7 that $SA^p(\mathfrak{G})$ has the property $\mathbb{B}$.

By this fact, the following result can be consider as a generalization of Theorem 2.8.

Theorem 2.10. *Assume that $\mathfrak{G}$ is amenable and $SA^p(\mathfrak{G})$ has the property $\mathbb{B}$. Then the Fourier algebra $A(\mathfrak{G})$ is zero product determined.*

Proof. Since $\mathfrak{G}$ is amenable, it follows that $A(\mathfrak{G})$ has a bounded approximate identity. Now using Theorem 2.4, we deduce that $A(\mathfrak{G})$ is zero product determined. □

Example 2.1. *Assume that $\mathfrak{G}$ is an amenable discrete group. Then for $p = 1$, we know that $SA^p(\mathfrak{G}) = \ell^1(\mathfrak{G})$ has the property $\mathbb{B}$. Now, it follows from Theorem 2.10 that $A(\mathfrak{G})$ is zero product determined. It is worthwhile to mention that this result can be obtained directly from Theorem 2.8.*

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