AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 7(1) (2026) 75-81
https://doi.org/10.22060/AJMC.2025.23749.1299

Original Article

Some trace functional inequalities for operator (p,h)-convex functions

Omid Pourbahri Rahpeyma[a], Mehdi Bakan[a], Abasalt Bodaghi*[b]

[a] *Department of Mathematics, Chalous Branch, Islamic Azad University, Chalous, Iran*
[b] *Department of Mathematics, West Tehran Branch, Islamic Azad University, Tehran, Iran*

ABSTRACT: In this paper, we present a theorem pertinent to singular value inequalities for positive and compact operators on a Hilbert space. Moreover, we obtain several trace inequalities for operator (p,h)-convex functions.

Review History:

Received:12 December 2024
Revised:14 May 2025
Accepted:15 May 2025
Available Online:01 January 2026

Keywords:

Operator (p,h)-convex function
Hermite-Hadamard inequality
Trace

1. Introduction

It is known that there are many important inequalities with miscellaneous applications in many areas of mathematics such as nonlinear analysis. One of them is the celebrated Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2},$$

where f is a convex function on $[a,b]$; for more information and details, we refer to [7], [9], [10] and [11]. In what follows, we state some historical notes, presented in the literature about some generalizations, modifications, refinements and improvements of the Hermite-Hadamard inequality used in this work.

Let $s \in (0,1)$. Recall from [5] that a real valued function f on an interval $I \subseteq [0,\infty)$ is said to be s-convex in the second sense if $f(rx+ty) \leq r^s f(x) + t^s f(y)$, for all $x,y \in I$ and $r,t \geq 0$ with $r+t=1$. In [5, Theorem 2.1], Dragomir and Fitzpatrick proved the following version of Hermite-Hadamard inequality for s-convex functions in the second sense: let $f : I \subseteq [0,\infty) \longrightarrow \mathbb{R}$ be a s-convex function, where $s \in (0,1]$ and $a,b \in I$ with $a<b$. If $f \in L^1(I)$, then

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}.$$

*Corresponding author.
E-mail addresses: omidpourbahri@gmail.com (O. P. Rahpeyma), m.bakan@gmail.com (M. Bakan), abasalt.bodaghi@gmail.com, abodaghi@iau.ir (A. Bodaghi)

Next, Zabandan et al. [14] presented a refinement of the Hermite-Hadamard inequality for s-convex functions in the case that $s \in [0, 1]$. In addition, they studied the Hermite-Hadamard inequality for the product of a r-convex function f and a s-convex function g.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex separable Hilbert space. We denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. In the following, we provide some definitions and notations.

• An operator $A \in \mathcal{B}(\mathcal{H})$ is *positive* if $\langle Ax, x\rangle \geq 0$ (denoted by $A \geq 0$) for all $x \in \mathcal{H}$. Moreover, a positive invertible operator A is naturally denoted by $A > 0$ and the set of all positive operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})^+$.

• For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ we write $B \geq A$ if $B - A \geq 0$.

• A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$ and Φ is said to be unital if $\Phi(I) = I$.

For a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$, it is known that there is a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on $Sp(A)$, the spectrum of A, and $C^*(A)$, the C^*-algebra generated by A and the identity operator $1_{\mathcal{B}(\mathcal{H})}$ on $\mathcal{H}$. This map is called the Gelfand map; see page 3 of [13]. Suppose that f is a continuous complex valued function on $Sp(A)$. We denote each element $\Phi(f)$ of $C^*(A)$ by $f(A)$ and call it the *continuous functional calculus* for a bounded self-adjoint operator A. Given A as a bounded self-adjoint operator, if f is a real-valued continuous function on $Sp(A)$ such that $f(t) \geq 0$ for all $t \in Sp(A)$, then $f(A) \geq 0$ and this means that $f(A)$ is a positive operator on $\mathcal{H}$. Furthermore, $f(A) \leq g(A)$ in the operator order in $\mathcal{B}(\mathcal{H})$ provided that both f and g are real-valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in sp(A)$.

An extension of the previously mentioned class of functions to operators was proposed by Ghazanfari [6] as follows: Given $s \in (0, 1]$. Let I be an interval in $[0, \infty)$. A continuous function $f : I \longrightarrow \mathbb{R}$ is called an *operator s-convex* on I provided that $f((1-\lambda)A + \lambda B) \leq (1-\lambda)^s f(A) + \lambda^s f(B)$ for all $\lambda \in [0, 1]$ and all A, B belong to $B(\mathcal{H})^+$ whose spectra are contained in I. Under these assumptions, he presented an inequality for operator s-convex functions as follows:

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(\lambda A + (1-\lambda)B)\, d\lambda \leq \frac{f(A)+f(B)}{s+1}.$$

Let $I, J \subseteq \mathbb{R}^+$, $(0, 1) \subseteq J$ and $h : J \longrightarrow \mathbb{R}$ be a nonnegative function that is not identically zero. Then

(1) (See [4].) A real-valued continuous function f on $K \subseteq [0, \infty)$ is called *operator convex* (*operator concave*) if

$$f(\lambda A + (1-\lambda)B) \leq (\geq) \lambda f(A) + (1-\lambda) f(B),$$

in the operator order in $\mathcal{B}(\mathcal{H})$, for all $\lambda \in [0, 1]$ and all bounded self-adjoint operators A and B in $\mathcal{B}(\mathcal{H})$ whose spectra are contained in K;

(2) (See [2] and [3].) We say that a continuous function $f : I \longrightarrow \mathbb{R}^+$ is an *operator h-convex function* (resp. *operator (p, h)-convex*) on I if

$$f(\lambda A + (1-\lambda)B) \leq h(\lambda) f(A) + h(1-\lambda) f(B)$$

$$(\text{resp. } f\big((\lambda A^p + (1-\lambda)B^p)^{\frac{1}{p}}\big) \leq h(\lambda) f(A) + h(1-\lambda) f(B)),$$

for all $A, B \in \mathcal{B}(\mathcal{H})^+$ whose spectra are contained in I, and for all $\lambda \in (0, 1)$.

The next inequalities, due to authors [2], provides a version of the Hermite-Hadamard inequalities for operator h-convex functions.

Let f be an operator h-convex function. Then

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\big(tA + (1-t)B\big)\, dt \leq \big(f(A) + f(B)\big) \int_0^1 h(t)\, dt,$$

for any self-adjoint operators A and B whose spectra lie in K.

Similarly, the following inequalities, establish the Hermite-Hadamard-type inequalities for operator (p, h)-convex functions [12].

Let $f : K \longrightarrow \mathbb{R}^+$ be a continuous operator (p, h)-convex function. Then

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\left(\frac{A^p + B^p}{2}\right)^{\frac{1}{p}}\right) &\leq \int_0^1 f\big((\lambda A^p + (1-\lambda)B^p)^{\frac{1}{p}}\big)\, d\lambda \\ &\leq \big(f(A) + f(B)\big) \int_0^1 h(\lambda)\, d\lambda, \end{aligned} \tag{1}$$

for any $A, B \in \mathcal{B}(\mathcal{H})^+$ with spectra in K and $h : J \longrightarrow \mathbb{R}^+$ be an continuous non-zero function.

An interval $K \subseteq \mathbb{R}^+$ is called a *p-convex set* if

$$sp\big((\lambda A^p + (1-\lambda)B^p)^{\frac{1}{p}}\big) \subseteq K,$$

for all positive operators $A, B \in \mathcal{B}(\mathcal{H})^+$ with spectra contained in K, for all $\lambda \in (0,1)$ and $p > 0$.

In this paper, we present some inequalities related to positive operators in $K(\mathcal{H})$, the C^*-algebra of compact operators on $\mathcal{H}$. As a consequence, we will show that if f is an operator (p,h)-convex function, then

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})} Tr\left(f\left(\left(\frac{A^p+B^p}{2}\right)^{\frac{1}{p}}\right)\right) &\le \int_0^1 Tr\Big(f\big((tA^p+(1-t)B^p)^{\frac{1}{p}}\big)\Big)\,dt \\ &\le \Big(Tr\big(f(A)\big)+Tr\big(f(B)\big)\Big)\int_0^1 h(t)\,dt,\end{aligned}$$

for any self-adjoint operators A and B on $\mathcal{H}$. Furthermore, we establish the singular value inequalities for positive operators in $K(\mathcal{H})$. Lastly, we present several trace inequalities for positive operators on $\mathcal{B}(\mathcal{H})$.

2. Main Results

We begin by restating some definitions, notations, commonly used terminologies and conventions from operator theory, as presented in the literature.

Let $\mathcal{H}$ be a Hilbert space and let $K(\mathcal{H})$ be the two sided ideal of compact operator in $\mathcal{B}(\mathcal{H})$. For any $A \in \mathcal{B}(\mathcal{H})$, the operator norm is defined by $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. Given $A, B \in \mathcal{B}(\mathcal{H})$, the direct sum $A \oplus B$ denotes the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ defined on $\mathcal{H} \oplus \mathcal{H}$; see [15] for more details. It is clear that $\|A \oplus B\| = \max\{\|A\|, \|B\|\}$. It is know that the operator A^*A is always positive and it has a unique positive square root, denote via $|A|$. The eigenvalues of $|A|$ counted with multiplicities are called the *singular values* of A. We will always enumerate these in descent order, and denoted through $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A)$. Note that $\|A\| = s_1(A) = \|A^*A\|^{\frac{1}{2}}$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. We say that $A \in \mathcal{B}(\mathcal{H})$ is *trace class* if

$$\|A\|_1 = \sum_{i\in I}\langle |A|e_i, e_i\rangle < \infty.$$

It is important to note that the definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote the set of all trace class operators in $\mathcal{B}(\mathcal{H})$ by $\mathcal{B}_1(\mathcal{H})$. In addition, the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ is defied via

$$Tr(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle,$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$. Note that $Tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(\mathcal{H})$ with $\|Tr\| = 1$. A useful result from [8] asserts that if $A, B \in K(\mathcal{H})$, then $s_j\big(\frac{A+B}{2}\big) \le s_j(A \oplus B)$ for all $j \in \mathbb{N}$.

Lemma 2.1. [[1], P. 75] *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that A is a compact operator. Then*

$$s_j(AB) \le \|B\| s_j(A),$$

for all $j \in \mathbb{N}$.

From now on, we set

$$\mathcal{M}(\mathcal{H}) := \{(A,B) \in \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+ | AB + BA \ge 0\}.$$

In the upcoming result, we state the singular value inequalities for positive operators in $K(\mathcal{H})$.

Theorem 2.2. *Let A, B be positive operators in $K(\mathcal{H})$ such that $AB + BA \ge 0$ and $\frac{1}{2} \le s \le p$. If $\mathcal{T}$ is an arbitrary operator in $\mathcal{B}(\mathcal{H})$, then*

$$\frac{1}{2}s_j\big((A+B)^{\frac{1}{2}}\mathcal{T}\big)^{\frac{2s}{p}} \le s_j\Big(\int_0^1 \big(\mathcal{T}^*(\lambda A + (1-\lambda)B)\mathcal{T}\big)^{\frac{s}{p}}\,d\lambda\Big) \le \frac{2p}{s+p}\|\mathcal{T}\|^{\frac{2s}{p}} s_j(A \oplus B)^{\frac{2s}{p}},$$

for all $\lambda \in (0,1)$ and all $j \in \mathbb{N}$.

Proof. Replacing (A,B) by $(A^{\frac{1}{p}},B^{\frac{1}{p}})$ in inequality (1), we obtain

$$\frac{1}{2h(\frac{1}{2})}f\left(\left(\frac{A+B}{2}\right)^{\frac{1}{p}}\right)\le\int_0^1 f\big((\lambda A+(1-\lambda)B)^{\frac{1}{p}}\big)\,d\lambda\le\big(f(A^{\frac{1}{p}})+f(B^{\frac{1}{p}})\big)\int_0^1 h(\lambda)\,d\lambda.$$

Switching A,B with $\Phi(A),\Phi(B)$, respectively, where Φ is positive unital operator in $\mathcal{B}(\mathcal{H})$, we obtain

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}f\left(\left(\frac{\Phi(A)+\Phi(B)}{2}\right)^{\frac{1}{p}}\right)&\le\int_0^1 f\big((\lambda\Phi(A)+(1-\lambda)\Phi(B))^{\frac{1}{p}}\big)\,d\lambda\\&\le\big(f(\Phi^{\frac{1}{p}}(A))+f(\Phi^{\frac{1}{p}}(B))\big)\int_0^1 h(\lambda)\,d\lambda.\end{aligned}\tag{2}$$

Put $f(x)=x^s$, $h(\lambda)=\lambda^{\frac{s}{p}}$ for $\frac{1}{2}\le s\le p$ and $\Phi(A)=\mathcal{T}^*A\mathcal{T}$ in (2). Then, we get

$$\begin{aligned}\frac{1}{2}(\mathcal{T}^*A\mathcal{T}+\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}&\le\int_0^1\big(\lambda\mathcal{T}^*A\mathcal{T}+(1-\lambda)\mathcal{T}^*B\mathcal{T}\big)^{\frac{s}{p}}\,d\lambda\\&\le\frac{p}{s+p}\big((\mathcal{T}^*A\mathcal{T})^{\frac{s}{p}}+(\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}\big).\end{aligned}\tag{3}$$

We compute the first part of inequality (3) as follows:

$$\begin{aligned}\frac{1}{2}(\mathcal{T}^*A\mathcal{T}+\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}&=\frac{1}{2}(\mathcal{T}^*(A+B)\mathcal{T})^{\frac{s}{p}}\\&=\frac{1}{2}\big(\mathcal{T}^*(A+B)^{\frac{1}{2}}(A+B)^{\frac{1}{2}}\mathcal{T}\big)^{\frac{s}{p}}\\&=\frac{1}{2}\big(\,|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|^2\,\big)^{\frac{s}{p}}=\frac{1}{2}\big(\,|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|\,\big)^{\frac{2s}{p}},\end{aligned}$$

and so

$$\frac{1}{2}(\mathcal{T}^*A\mathcal{T}+\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}=\frac{1}{2}(|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|\,)^{\frac{2s}{p}}.\tag{4}$$

For the third part of inequality (3), we find

$$\frac{p}{s+p}\big((\mathcal{T}^*A\mathcal{T})^{\frac{s}{p}}+(\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}\big)=\frac{p}{s+p}\Big(\,|\,A^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}+|\,B^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\,\Big).\tag{5}$$

It follows from (3), (5) and (4) that

$$\begin{aligned}\frac{1}{2}\big(\,|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|\,\big)^{\frac{2s}{p}}&\le\int_0^1\big(\lambda\mathcal{T}^*A\mathcal{T}+(1-\lambda)\mathcal{T}^*B\mathcal{T}\big)^{\frac{s}{p}}\,d\lambda\\&\le\frac{p}{s+p}\Big(\,|\,A^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}+|\,B^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\,\Big).\end{aligned}$$

Since for each j, s_j is a monotone operator, we arrive at

$$\begin{aligned}\frac{1}{2}s_j\Big(\big(\,|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|\,\big)^{\frac{2s}{p}}\Big)&\le s_j\Big(\int_0^1\big(\mathcal{T}^*(\lambda A+(1-\lambda)B)\mathcal{T}\big)^{\frac{s}{p}}\,d\lambda\Big)\\&\le s_j\Big(\frac{p}{s+p}\big(\,|\,A^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}+|\,B^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\,\big)\Big)\\&\le\frac{2p}{s+p}s_j\Big(\,|\,A^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\oplus|\,B^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\,\Big)\\&\le\frac{2p}{s+p}s_j\Big(\begin{pmatrix}A^{\frac{1}{2}}&0\\0&B^{\frac{1}{2}}\end{pmatrix}\oplus\begin{pmatrix}\mathcal{T}&0\\0&\mathcal{T}\end{pmatrix}\Big)^{\frac{2s}{p}}\\&\le\frac{2p}{s+p}\|\mathcal{T}\oplus\mathcal{T}\|^{\frac{2s}{p}}s_j\big(A^{\frac{1}{2}}\oplus B^{\frac{1}{2}}\big)^{\frac{2s}{p}}\\&=\frac{2p}{s+p}\|\mathcal{T}\|^{\frac{2s}{p}}s_j\big(A\oplus B\big)^{\frac{2s}{p}}.\end{aligned}$$

Therefore, the proof is now complete. □

In the next corollary, we include [2, Example 3.2] as a direct consequence of Theorem 2.2.

Corollary 2.3. *Let $\mathcal{T}$ be an operator in $\mathcal{B}(\mathcal{H})$. Then, for each pair of positive compact operators A and B in $M(\mathcal{H})$, we have*

$$\frac{1}{2}s_j\big((A+B)^{\frac{1}{2}}\mathcal{T}\big)^{2s} \leq s_j\Big(\int_0^1 \big(\mathcal{T}^*(\lambda A+(1-\lambda)B)\mathcal{T}\big)^s\, d\lambda \leq \frac{2}{s+1}\|\mathcal{T}\|^{2s} s_j(A\oplus B)^s,$$

for all $j\in\mathbb{N}$, where $s\in[\frac{1}{2},1]$.

Proof. The desired result can be obtained by putting $p=1$ in Theorem 2.2. □

Proposition 2.4. *Let $f: K\to\mathbb{R}^+$ be an operator (p,h)-convex function and $h: J\to\mathbb{R}^+$ be a non-zero function. Then, $g(t)=Tr\big(f\big((tA^p+(1-t)B^p)^{\frac{1}{p}}\big)\big)$ is h-convex on $(0,1)$ for any self-adjoint operators A and B with spectra contained in K.*

Proof. Since the trace functional is convex and monotonic, for each $u,v\in(0,1)$ and $0<\alpha<1$, we obtain

$$\begin{aligned}
g(\alpha u+(1-\alpha)v) &= Tr\Big(f\big((\alpha u+(1-\alpha)v)A^p+(\alpha u+(1-\alpha)v)B^p\big)^{\frac{1}{p}}\Big)\\
&= Tr\Big(f\big(\alpha(uA^p+(1-u)B^p)+(1-\alpha)(vA^p+(1-v)B^p)\big)^{\frac{1}{p}}\Big)\\
&= Tr\Big(f\big(\alpha[((uA^p+(1-u)B^p)^{\frac{1}{p}})^p]+(1-\alpha)[((vA^p+(1-v)B^p)^{\frac{1}{p}})^p]\big)^{\frac{1}{p}}\Big)\\
&\leq Tr\Big(h(\alpha)f\big(uA^p+(1-u)B^p\big)^{\frac{1}{p}}+h(1-\alpha)f\big((vA^p+(1-v)B^p)^{\frac{1}{p}}\big)\Big)\\
&= h(\alpha)Tr\Big(f\big(uA^p+(1-u)B^p\big)^{\frac{1}{p}}\big)\Big)+h(1-\alpha)Tr\Big(f\big((vA^p+(1-v)B^p)^{\frac{1}{p}}\big)\Big)\\
&= h(\alpha)g(u)+h(1-\alpha)g(v).
\end{aligned}$$

Therefore, g is h-convex. □

We now state the second main result of this paper using the above proposition. In fact, the next theorem generalizes the trace Hermite-Hadamard inequality for operator (p,h)-convex functions.

Theorem 2.5. *Let $f: K\longrightarrow\mathbb{R}^+$ be an operator (p,h)-convex function and $h: J\longrightarrow\mathbb{R}^+$ be a non-zero function. Then, for each self-adjoint operators A and B with spectra contained in K, we have.*

$$\begin{aligned}
\frac{1}{2h(\frac{1}{2})}Tr\left(f\left(\left(\frac{A^p+B^p}{2}\right)^{\frac{1}{p}}\right)\right) &\leq \int_0^1 Tr\Big(f\big((tA^p+(1-t)B^p)^{\frac{1}{p}}\big)\Big)dt\\
&\leq \Big(Tr\big(f(A)\big)+Tr\big(f(B)\big)\Big)\int_0^1 h(t)dt, \qquad (6)
\end{aligned}$$

Proof. By Proposition 2.4, the function $g(t)=Tr\big(f(tA^p+(1-t)B^p)^{\frac{1}{p}}\big)$ is h-convex on $(0,1)$, and hence by inequality (1), we get

$$\frac{1}{2h(\frac{1}{2})}g\left(\frac{0+1}{2}\right)\leq\int_0^1 g(t)\,dt\leq\left(\frac{g(0)+g(1)}{2}\right)\int_0^1 h(t)\,dt.$$

This finishes the proof. □

The incoming corollaries are direct consequences of Theorem 2.5.

Corollary 2.6. *Let $p>0$ and $s\in[p,2p]$. Then*

$$2^{-\frac{s}{p}}Tr\big((A^p+B^p)^{\frac{s}{p}}\big)\leq\int_0^1 Tr\big((\lambda A^p+(1-\lambda)B^p)^{\frac{s}{p}}\big)\,d\lambda\leq\frac{1}{2}Tr(A^s+B^s),$$

for any self-adjoint operators $A,B\in\mathcal{B}_1(\mathcal{H})^+$ with spectra contained in K.

Proof. For $h(\lambda) = \lambda$ and $s \in [p, 2p]$, the function $f(t) = t^s$ is operator (p, h)-convex. By inequality (6) the result can be deduced. $\square$

Corollary 2.7. *For $p > 0$ and $0 < s \le p$, we have*

$$\frac{1}{2}Tr\big((A^p + B^p)^{\frac{s}{p}}\big) \le \int_0^1 Tr\big((\lambda A^p + (1-\lambda)B^p)^{\frac{s}{p}}\big)\, d\lambda \le \frac{p}{s+p} Tr(A^s + B^s), \tag{7}$$

for all positive trace class operators A and B in $M(\mathcal{H})$.

Proof. Set $h(\lambda) = \lambda^{\frac{s}{p}}$ with $0 < s \le p$. Then, the function $f(t) = t^s$ is operator (p, h)-convex on $M(\mathcal{H})$. Applying f in inequality (6), we find the result.

$\square$

Corollary 2.8. *Suppose that $0 < s \le p$ and Φ is a unital positive linear map. For each pair of positive trace class operators A and B in $M(\mathcal{H})$, we have*

$$\frac{1}{2}Tr\big(\Phi((A^p + B^p)^{\frac{s}{p}})\big) \le \frac{p}{s+p} Tr\big((\Phi(A))^s + (\Phi(B))^s\big).$$

Proof. Replacing $\Phi(A), \Phi(B)$ by A, B, respectively, in inequality (7), we get the result by Proposition 2.4. $\square$

Remark 2.9. *It should be note that if Φ is a unital positive trace preserving map, then by Corollary 2.8 we find*

$$\frac{1}{2}Tr\big((A^p + B^p)^{\frac{s}{p}}\big) \le \frac{p}{s+p} Tr\big((\Phi(A))^s + (\Phi(B))^s\big).$$

Some inequalities for positive operators A and B in $\mathcal{B}_1(\mathcal{H})^+$ are presented in the upcoming proposition.

Proposition 2.10. *Let f be an operator (p, h)-convex and $h : J \to \mathbb{R}^+$ be a non-zero function. Then, we have*

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\left(\frac{(Tr(A))^p + (Tr(B))^p}{2}\right)^{\frac{1}{p}}\right) &\le \int_0^1 f\big((\lambda (Tr(A))^p + (1-\lambda)(Tr(B))^p)^{\frac{1}{p}}\big)\, d\lambda \\ &\le \big(f(Tr(A)) + f(Tr(B))\big) \int_0^1 h(\lambda)\, d\lambda, \end{aligned} \tag{8}$$

for all positive operators A and B in $\mathcal{B}_1(\mathcal{H})^+$ with spectra contained in K.

Proof. Interchanging (A, B) by $(Tr(A), Tr(B))$ in inequality (1), we obtain (8). $\square$

Plugging [12, Example 1] and Proposition 2.10, we obtain a relation as follows. Since the proof is routine, we omit it.

Corollary 2.11. *Let $0 < s \le p$ and let $A, B \in \mathcal{B}_1(\mathcal{H})^+$ such that $AB + BA \ge 0$. Then*

$$\begin{aligned} \frac{1}{2}\left((Tr(A))^p + (Tr(B))^p\right)^{\frac{s}{p}} &\le \int_0^1 \big(\lambda (Tr(A))^p + (1-\lambda)(Tr(B))^p\big)^{\frac{s}{p}}\, d\lambda \\ &\le \big((Tr(A))^s + (Tr(B))^s\big)\Big(\frac{p}{s+p}\Big). \end{aligned}$$

Corollary 2.12. *Suppose that $0 < p$ and $s \in [p, 2p]$ and let $A, B \in \mathcal{B}_1(\mathcal{H})^+$. Then, we have*

$$\begin{aligned} \left(\frac{(Tr(A))^p + (Tr(B))^p}{2}\right)^{\frac{s}{p}} &\le \int_0^1 \big(\lambda (Tr(A))^p + (1-\lambda)(Tr(B))^p\big)^{\frac{s}{p}}\, d\lambda \\ &\le \frac{(Tr(A))^s + (Tr(B))^s}{2}. \end{aligned}$$

Proof. The result follows from [12, Proposition 5] and Proposition 2.10. $\square$

Acknowledgements. The authors sincerely thank the anonymous reviewers for their careful reading, constructive comments to improve the quality of the first draft of the paper substantially.

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Please cite this article using:

Omid Pourbahri Rahpeyma, Mehdi Bakan, Abasalt Bodaghi, Some trace functional inequalities for operator (p, h)-convex functions, AUT J. Math. Comput., 7(1) (2026) 75-81
https://doi.org/10.22060/AJMC.2025.23749.1299