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Original Article

Some trace functional inequalities for operator (p, h)-convex functions

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ABSTRACT: In this paper, we present a theorem pertinent to singular value inequalities for positive and compact operators on a Hilbert space. Moreover, we obtain several trace inequalities for operator (p, h)-convex functions.

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1. Introduction

It is known that there are many important inequalities with miscellaneous applications in many areas of mathematics such as nonlinear analysis. One of them is the celebrated Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2},$$

where f is a convex function on $[a, b]$; for more information and details, we refer to [7], [9], [10] and [11]. In what follows, we state some historical notes, presented in the literature about some generalizations, modifications, refinements and improvements of the Hermite-Hadamard inequality used in this work.

Let $s \in (0, 1)$. Recall from [5] that a real valued function f on an interval $I \subseteq [0, \infty)$ is said to be s-convex in the second sense if $f(rx + ty) \leq r^s f(x) + t^s f(y)$, for all $x, y \in I$ and $r, t \geq 0$ with $r + t = 1$. In [5, Theorem 2.1], Dragomir and Fitzpatrick proved the following version of Hermite-Hadamard inequality for s-convex functions in

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the second sense: let $f : I \subseteq [0,\infty) \longrightarrow \mathbb{R}$ be a s-convex function, where $s \in (0,1]$ and $a, b \in I$ with $a < b$. If $f \in L^1(I)$, then

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}.$$

Next, Zabandan et al. [14] presented a refinement of the Hermite-Hadamard inequality for s-convex functions in the case that $s \in [0,1]$. In addition, they studied the Hermite-Hadamard inequality for the product of a r-convex function f and a s-convex function g.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex separable Hilbert space. We denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. In the following, we provide some definitions and notations.

• An operator $A \in \mathcal{B}(\mathcal{H})$ is *positive* if $\langle Ax, x\rangle \geq 0$ (denoted by $A \geq 0$) for all $x \in \mathcal{H}$. Moreover, a positive invertible operator A is naturally denoted by $A > 0$ and the set of all positive operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})^+$.

• For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ we write $B \geq A$ if $B - A \geq 0$.

• A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$ and Φ is said to be unital if $\Phi(I) = I$.

For a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$, it is known that there is a $*$-isometric isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on $Sp(A)$, the spectrum of A, and $C^*(A)$, the C^*-algebra generated by A and the identity operator $1_{\mathcal{B}(\mathcal{H})}$ on $\mathcal{H}$. This map is called the Gelfand map; see page 3 of [13]. Suppose that f is a continuous complex-valued function on $Sp(A)$. We denote each element $\Phi(f)$ of $C^*(A)$ by $f(A)$ and call it the *continuous functional calculus* for a bounded self-adjoint operator A. Given A as a bounded self-adjoint operator, if f is a real-valued continuous function on $Sp(A)$ such that $f(t) \geq 0$ for all $t \in Sp(A)$, then $f(A) \geq 0$ and this means that $f(A)$ is a positive operator on $\mathcal{H}$. Furthermore, $f(A) \leq g(A)$ in the operator order in $\mathcal{B}(\mathcal{H})$ provided that both f and g are real-valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in sp(A)$.

An extension of the previously mentioned class of functions to operators was proposed by Ghazanfari [6] as follows: Let $s \in (0,1]$ and let I be an interval in $[0,\infty)$. A continuous function $f : I \longrightarrow \mathbb{R}$ is called *operator s-convex* on I provided that $f((1-\lambda)A + \lambda B) \leq (1-\lambda)^s f(A) + \lambda^s f(B)$ for all $\lambda \in [0,1]$ and all A, B belonging to $B(\mathcal{H})^+$ whose spectra are contained in I. Under these assumptions, he presented an inequality for operator s-convex functions as follows:

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(\lambda A + (1-\lambda)B)\,d\lambda \leq \frac{f(A)+f(B)}{s+1}.$$

Let $I, J \subseteq \mathbb{R}^+$, $(0,1) \subseteq J$ and $h : J \longrightarrow \mathbb{R}$ be a nonnegative function that is not identically zero. Then

(1) (See [4].) A real-valued continuous function f on $K \subseteq [0,\infty)$ is called *operator convex* (*operator concave*) if

$$f(\lambda A + (1-\lambda)B) \leq (\geq) \lambda f(A) + (1-\lambda) f(B),$$

in the operator order in $\mathcal{B}(\mathcal{H})$, for all $\lambda \in [0,1]$ and all bounded self-adjoint operators A and B in $\mathcal{B}(\mathcal{H})$ whose spectra are contained in K;

(2) (See [2] and [3].) We say that a continuous function $f : I \longrightarrow \mathbb{R}^+$ is an *operator h-convex function* (resp. *operator (p,h)-convex*) on I if

$$f(\lambda A + (1-\lambda)B) \leq h(\lambda) f(A) + h(1-\lambda) f(B)$$

$$(\text{resp. } f\big((\lambda A^p + (1-\lambda)B^p)^{\frac{1}{p}}\big) \leq h(\lambda) f(A) + h(1-\lambda) f(B)),$$

for all $A, B \in \mathcal{B}(\mathcal{H})^+$ whose spectra are contained in I, and for all $\lambda \in (0,1)$.

The next inequalities, due to authors [2], provides a version of the Hermite-Hadamard inequalities for operator h-convex functions.

Let f be an operator h-convex function. Then

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\big(tA + (1-t)B\big)\,dt \leq \big(f(A)+f(B)\big)\int_0^1 h(t)\,dt,$$

for any self-adjoint operators A and B whose spectra lie in K.

Similarly, the following inequalities, establish the Hermite-Hadamard-type inequalities for operator (p,h)-convex functions [12].

Let $f : K \longrightarrow \mathbb{R}^+$ be a continuous operator (p,h)-convex function. Then

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})} f\left(\left(\frac{A^p+B^p}{2}\right)^{\frac{1}{p}}\right) &\leq \int_0^1 f\big((\lambda A^p + (1-\lambda)B^p)^{\frac{1}{p}}\big)\,d\lambda \\ &\leq \big(f(A)+f(B)\big)\int_0^1 h(\lambda)\,d\lambda, \qquad (1)\end{aligned}$$

for any $A, B \in \mathcal{B}(\mathcal{H})^+$ with spectra in K and $h : J \longrightarrow \mathbb{R}^+$ be a continuous non-zero function.

An interval $K \subseteq \mathbb{R}^+$ is called a *p-convex set* if

$$Sp\big((\lambda A^p + (1-\lambda)B^p)^{\frac{1}{p}}\big) \subseteq K,$$

for all positive operators $A, B \in \mathcal{B}(\mathcal{H})^+$ with spectra contained in K, for all $\lambda \in (0,1)$ and $p > 0$.

In this paper, we present some inequalities related to positive operators in $K(\mathcal{H})$, the C^*-algebra of compact operators on $\mathcal{H}$. As a consequence, we will show that if f is an operator (p,h)-convex function, then

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})} Tr\left(f\left(\left(\frac{A^p+B^p}{2}\right)^{\frac{1}{p}}\right)\right) &\leq \int_0^1 Tr\Big(f\big((tA^p+(1-t)B^p)^{\frac{1}{p}}\big)\Big)\, dt \\ &\leq \Big(Tr\big(f(A)\big) + Tr\big(f(B)\big)\Big) \int_0^1 h(t)\, dt,\end{aligned}$$

for any self-adjoint operators A and B on $\mathcal{H}$. Furthermore, we establish the singular value inequalities for positive operators in $K(\mathcal{H})$. Lastly, we present several trace inequalities for positive operators on $\mathcal{B}(\mathcal{H})$.

2. Main Results

We begin by restating some definitions, notations, commonly used terminologies and conventions from operator theory, as presented in the literature.

Let $\mathcal{H}$ be a Hilbert space and let $K(\mathcal{H})$ be the two sided ideal of compact operators in $\mathcal{B}(\mathcal{H})$. For any $A \in \mathcal{B}(\mathcal{H})$, the operator norm is defined by $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. Given $A, B \in \mathcal{B}(\mathcal{H})$, the direct sum $A \oplus B$ denotes the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ defined on $\mathcal{H} \oplus \mathcal{H}$; see [15] for more details. It is clear that $\|A \oplus B\| = \max\{\|A\|, \|B\|\}$. It is known that the operator A^*A is always positive and it has a unique positive square root, denoted via $|A|$. The eigenvalues of $|A|$ counted with multiplicities are called the *singular values* of A. We will always enumerate these in descending order, and denoted through $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. Note that $\|A\| = s_1(A) = \|A^*A\|^{\frac{1}{2}}$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. We say that $A \in \mathcal{B}(\mathcal{H})$ is *trace class* if

$$\|A\|_1 = \sum_{i\in I} \langle |A|e_i, e_i\rangle < \infty.$$

It is important to note that the definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote the set of all trace class operators in $\mathcal{B}(\mathcal{H})$ by $\mathcal{B}_1(\mathcal{H})$. In addition, the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ is defined via

$$Tr(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle,$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$. Note that $Tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(\mathcal{H})$ with $\|Tr\| = 1$. A useful result from [8] asserts that if $A, B \in K(\mathcal{H})$, then $s_j\big(\frac{A+B}{2}\big) \leq s_j(A \oplus B)$ for all $j \in \mathbb{N}$.

Lemma 2.1. [[1], P. 75] *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that A is a compact operator. Then*

$$s_j(AB) \leq \|B\| s_j(A),$$

for all $j \in \mathbb{N}$.

From now on, we set

$$\mathcal{M}(\mathcal{H}) := \{(A,B) \in \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+ | AB + BA \geq 0\}.$$

In the upcoming result, we state the singular value inequalities for positive operators in $K(\mathcal{H})$.

Theorem 2.2. *Let A, B be positive operators in $K(\mathcal{H})$ such that $AB + BA \geq 0$ and $\frac{1}{2} \leq s \leq p$. If $\mathcal{T}$ is an arbitrary operator in $\mathcal{B}(\mathcal{H})$, then*

$$\frac{1}{2} s_j\big((A+B)^{\frac{1}{2}}\mathcal{T}\big)^{\frac{2s}{p}} \leq s_j\Big(\int_0^1 \big(\mathcal{T}^*(\lambda A + (1-\lambda)B)\mathcal{T}\big)^{\frac{s}{p}}\, d\lambda\Big) \leq \frac{2p}{s+p} \|\mathcal{T}\|^{\frac{2s}{p}} s_j(A \oplus B)^{\frac{2s}{p}},$$

for all $\lambda \in (0,1)$ and all $j \in \mathbb{N}$.

Proof. Replacing (A,B) by $(A^{\frac{1}{p}},B^{\frac{1}{p}})$ in inequality (1), we obtain

$$\frac{1}{2h(\frac{1}{2})}f\left(\left(\frac{A+B}{2}\right)^{\frac{1}{p}}\right)\le\int_0^1 f\big((\lambda A+(1-\lambda)B)^{\frac{1}{p}}\big)\,d\lambda\le\big(f(A^{\frac{1}{p}})+f(B^{\frac{1}{p}})\big)\int_0^1 h(\lambda)\,d\lambda.$$

Switching A,B with $\Phi(A),\Phi(B)$, respectively, where Φ is a positive unital operator in $\mathcal{B}(\mathcal{H})$, we obtain

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}f\left(\left(\frac{\Phi(A)+\Phi(B)}{2}\right)^{\frac{1}{p}}\right)&\le\int_0^1 f\big((\lambda\Phi(A)+(1-\lambda)\Phi(B))^{\frac{1}{p}}\big)\,d\lambda\\&\le\big(f(\Phi^{\frac{1}{p}}(A))+f(\Phi^{\frac{1}{p}}(B))\big)\int_0^1 h(\lambda)\,d\lambda.\end{aligned}\tag{2}$$

Put $f(x)=x^s$, $h(\lambda)=\lambda^{\frac{s}{p}}$ for $\frac{1}{2}\le s\le p$ and $\Phi(A)=\mathcal{T}^*A\mathcal{T}$ in (2). Then, we get

$$\begin{aligned}\frac{1}{2}(\mathcal{T}^*A\mathcal{T}+\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}&\le\int_0^1\big(\lambda\mathcal{T}^*A\mathcal{T}+(1-\lambda)\mathcal{T}^*B\mathcal{T}\big)^{\frac{s}{p}}\,d\lambda\\&\le\frac{p}{s+p}\big((\mathcal{T}^*A\mathcal{T})^{\frac{s}{p}}+(\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}\big).\end{aligned}\tag{3}$$

We compute the first part of inequality (3) as follows:

$$\begin{aligned}\frac{1}{2}(\mathcal{T}^*A\mathcal{T}+\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}&=\frac{1}{2}(\mathcal{T}^*(A+B)\mathcal{T})^{\frac{s}{p}}\\&=\frac{1}{2}\big(\mathcal{T}^*(A+B)^{\frac{1}{2}}(A+B)^{\frac{1}{2}}\mathcal{T}\big)^{\frac{s}{p}}\\&=\frac{1}{2}\big(\,|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|^2\,\big)^{\frac{s}{p}}=\frac{1}{2}\big(\,|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|\,\big)^{\frac{2s}{p}},\end{aligned}$$

and so

$$\frac{1}{2}(\mathcal{T}^*A\mathcal{T}+\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}=\frac{1}{2}(|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|\,)^{\frac{2s}{p}}.\tag{4}$$

For the third part of inequality (3), we find

$$\frac{p}{s+p}\big((\mathcal{T}^*A\mathcal{T})^{\frac{s}{p}}+(\mathcal{T}^*B\mathcal{T})^{\frac{s}{p}}\big)=\frac{p}{s+p}\Big(\,|\,A^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}+|\,B^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\,\Big).\tag{5}$$

It follows from (3), (5) and (4) that

$$\begin{aligned}\frac{1}{2}\big(\,|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|\,\big)^{\frac{2s}{p}}&\le\int_0^1\big(\lambda\mathcal{T}^*A\mathcal{T}+(1-\lambda)\mathcal{T}^*B\mathcal{T}\big)^{\frac{s}{p}}\,d\lambda\\&\le\frac{p}{s+p}\Big(\,|\,A^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}+|\,B^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\,\Big).\end{aligned}$$

Since for each j, s_j is a monotone operator, we arrive at

$$\begin{aligned}\frac{1}{2}s_j\Big(\big(\,|\,(A+B)^{\frac{1}{2}}\mathcal{T}\,|\,\big)^{\frac{2s}{p}}\Big)&\le s_j\Big(\int_0^1\big(\mathcal{T}^*(\lambda A+(1-\lambda)B)\mathcal{T}\big)^{\frac{s}{p}}\,d\lambda\Big)\\&\le s_j\Big(\frac{p}{s+p}\big(\,|\,A^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}+|\,B^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\,\big)\Big)\\&\le\frac{2p}{s+p}s_j\Big(\,|\,A^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\oplus|\,B^{\frac{1}{2}}\mathcal{T}\,|^{\frac{2s}{p}}\,\Big)\\&\le\frac{2p}{s+p}s_j\Big(\begin{pmatrix}A^{\frac{1}{2}}&0\\0&B^{\frac{1}{2}}\end{pmatrix}\oplus\begin{pmatrix}\mathcal{T}&0\\0&\mathcal{T}\end{pmatrix}\Big)^{\frac{2s}{p}}\\&\le\frac{2p}{s+p}\|\mathcal{T}\oplus\mathcal{T}\|^{\frac{2s}{p}}s_j\big(A^{\frac{1}{2}}\oplus B^{\frac{1}{2}}\big)^{\frac{2s}{p}}\\&=\frac{2p}{s+p}\|\mathcal{T}\|^{\frac{2s}{p}}s_j\big(A\oplus B\big)^{\frac{2s}{p}}.\end{aligned}$$

Therefore, the proof is now complete. □

In the next corollary, we include [2, Example 3.2] as a direct consequence of Theorem 2.2.

Corollary 2.3. *Let $\mathcal{T}$ be an operator in $\mathcal{B}(\mathcal{H})$. Then, for each pair of positive compact operators A and B in $M(\mathcal{H})$, we have*

$$\begin{aligned}\frac{1}{2}s_j\big((A+B)^{\frac{1}{2}}\mathcal{T}\big)^{2s} &\leq s_j\Big(\int_0^1 \big(\mathcal{T}^*(\lambda A+(1-\lambda)B)\mathcal{T}\big)^s\,d\lambda\\ &\leq \frac{2}{s+1}\|\mathcal{T}\|^{2s}s_j(A\oplus B)^s,\end{aligned}$$

for all $j\in\mathbb{N}$, where $s\in[\frac{1}{2},1]$.

Proof. The desired result can be obtained by putting $p=1$ in Theorem 2.2. □

Proposition 2.4. *Let $f:K\to\mathbb{R}^+$ be an operator (p,h)-convex function and $h:J\to\mathbb{R}^+$ be a non-zero function. Then, $g(t)=Tr\big(f\big((tA^p+(1-t)B^p)^{\frac{1}{p}}\big)\big)$ is h-convex on $(0,1)$ for any self-adjoint operators A and B with spectra contained in K.*

Proof. Since the trace functional is convex and monotonic, for each $u,v\in(0,1)$ and $0<\alpha<1$, we obtain

$$\begin{aligned}g(\alpha u+(1-\alpha)v) &= Tr\Big(f\big((\alpha u+(1-\alpha)v)A^p+(\alpha u+(1-\alpha)v)B^p\big)^{\frac{1}{p}}\Big)\\ &= Tr\Big(f\big(\alpha(uA^p+(1-u)B^p)+(1-\alpha)(vA^p+(1-v)B^p)\big)^{\frac{1}{p}}\Big)\\ &= Tr\Big(f\big(\alpha[((uA^p+(1-u)B^p)^{\frac{1}{p}})^p]+(1-\alpha)[((vA^p+(1-v)B^p)^{\frac{1}{p}})^p]\big)^{\frac{1}{p}}\Big)\\ &\leq Tr\Big(h(\alpha)f\big(uA^p+(1-u)B^p\big)^{\frac{1}{p}}\big)+h(1-\alpha)f\big((vA^p+(1-v)B^p)^{\frac{1}{p}}\big)\Big)\\ &= h(\alpha)Tr\Big(f\big(uA^p+(1-u)B^p\big)^{\frac{1}{p}}\big)\Big)+h(1-\alpha)Tr\Big(f\big((vA^p+(1-v)B^p)^{\frac{1}{p}}\big)\Big)\\ &= h(\alpha)g(u)+h(1-\alpha)g(v).\end{aligned}$$

Therefore, g is h-convex. □

We now state the second main result of this paper using the above proposition. In fact, the next theorem generalizes the trace Hermite-Hadamard inequality for operator (p,h)-convex functions.

Theorem 2.5. *Let $f:K\longrightarrow\mathbb{R}^+$ be an operator (p,h)-convex function and $h:J\longrightarrow\mathbb{R}^+$ be a non-zero function. Then, for each self-adjoint operators A and B with spectra contained in K, we have.*

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}Tr\left(f\left(\left(\frac{A^p+B^p}{2}\right)^{\frac{1}{p}}\right)\right) &\leq \int_0^1 Tr\Big(f\big((tA^p+(1-t)B^p)^{\frac{1}{p}}\big)\Big)dt\\ &\leq \Big(Tr\big(f(A)\big)+Tr\big(f(B)\big)\Big)\int_0^1 h(t)dt. \qquad (6)\end{aligned}$$

Proof. By Proposition 2.4, the function $g(t)=Tr\big(f(tA^p+(1-t)B^p)^{\frac{1}{p}}\big)$ is h-convex on $(0,1)$, and hence by inequality (1), we get

$$\frac{1}{2h(\frac{1}{2})}g\left(\frac{0+1}{2}\right)\leq\int_0^1 g(t)\,dt\leq\left(\frac{g(0)+g(1)}{2}\right)\int_0^1 h(t)\,dt.$$

This finishes the proof. □

The incoming corollaries are direct consequences of Theorem 2.5.

Corollary 2.6. *Let $p>0$ and $s\in[p,2p]$. Then*

$$2^{-\frac{s}{p}}Tr\big((A^p+B^p)^{\frac{s}{p}}\big)\leq\int_0^1 Tr\big((\lambda A^p+(1-\lambda)B^p)^{\frac{s}{p}}\big)\,d\lambda\leq\frac{1}{2}Tr(A^s+B^s),$$

for any self-adjoint operators $A,B\in\mathcal{B}_1(\mathcal{H})^+$ with spectra contained in K.

Proof. For $h(\lambda) = \lambda$ and $s \in [p, 2p]$, the function $f(t) = t^s$ is operator (p, h)-convex. By inequality (6) the result can be deduced. □

Corollary 2.7. *For $p > 0$ and $0 < s \leq p$, we have*

$$\frac{1}{2}Tr\big((A^p + B^p)^{\frac{s}{p}}\big) \leq \int_0^1 Tr\big((\lambda A^p + (1-\lambda)B^p)^{\frac{s}{p}}\big)\, d\lambda \leq \frac{p}{s+p}Tr(A^s + B^s), \tag{7}$$

for all positive trace class operators A and B in $M(\mathcal{H})$.

Proof. Set $h(\lambda) = \lambda^{\frac{s}{p}}$ with $0 < s \leq p$. Then, the function $f(t) = t^s$ is operator (p, h)-convex on $M(\mathcal{H})$. Applying f in inequality (6), we find the result.

□

Corollary 2.8. *Suppose that $0 < s \leq p$ and Φ is a unital positive linear map. For each pair of positive trace class operators A and B in $M(\mathcal{H})$, we have*

$$\frac{1}{2}Tr\big(\Phi((A^p + B^p)^{\frac{s}{p}})\big) \leq \frac{p}{s+p}Tr\big((\Phi(A))^s + (\Phi(B))^s\big).$$

Proof. Replacing $\Phi(A), \Phi(B)$ by A, B, respectively, in inequality (7), we get the result by Proposition 2.4. □

Remark 2.9. *It should be note that if Φ is a unital positive trace preserving map, then by Corollary 2.8 we find*

$$\frac{1}{2}Tr\big((A^p + B^p)^{\frac{s}{p}}\big) \leq \frac{p}{s+p}Tr\big((\Phi(A))^s + (\Phi(B))^s\big).$$

Some inequalities for positive operators A and B in $\mathcal{B}_1(\mathcal{H})^+$ are presented in the upcoming proposition.

Proposition 2.10. *Let f be an operator (p, h)-convex and $h : J \to \mathbb{R}^+$ be a non-zero function. Then, we have*

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}f\left(\left(\frac{(Tr(A))^p + (Tr(B))^p}{2}\right)^{\frac{1}{p}}\right) &\leq \int_0^1 f\big((\lambda(Tr(A))^p + (1-\lambda)(Tr(B))^p)^{\frac{1}{p}}\big)\, d\lambda \\ &\leq \big(f(Tr(A)) + f(Tr(B))\big)\int_0^1 h(\lambda)\, d\lambda,\end{aligned} \tag{8}$$

for all positive operators A and B in $\mathcal{B}_1(\mathcal{H})^+$ with spectra contained in K.

Proof. Interchanging (A, B) by $(Tr(A), Tr(B))$ in inequality (1), we obtain (8). □

Plugging [12, Example 1] and Proposition 2.10, we obtain a relation as follows. Since the proof is routine, we omit it.

Corollary 2.11. *Let $0 < s \leq p$ and let $A, B \in \mathcal{B}_1(\mathcal{H})^+$ such that $AB + BA \geq 0$. Then*

$$\begin{aligned}\frac{1}{2}\left((Tr(A))^p + (Tr(B))^p\right)^{\frac{s}{p}} &\leq \int_0^1 \big(\lambda(Tr(A))^p + (1-\lambda)(Tr(B))^p\big)^{\frac{s}{p}}\, d\lambda \\ &\leq \big((Tr(A))^s + (Tr(B))^s\big)\left(\frac{p}{s+p}\right).\end{aligned}$$

Corollary 2.12. *Suppose that $0 < p$ and $s \in [p, 2p]$ and let $A, B \in \mathcal{B}_1(\mathcal{H})^+$. Then, we have*

$$\begin{aligned}\left(\frac{(Tr(A))^p + (Tr(B))^p}{2}\right)^{\frac{s}{p}} &\leq \int_0^1 \big(\lambda(Tr(A))^p + (1-\lambda)(Tr(B))^p\big)^{\frac{s}{p}}\, d\lambda \\ &\leq \frac{(Tr(A))^s + (Tr(B))^s}{2}.\end{aligned}$$

Proof. The result follows from [12, Proposition 5] and Proposition 2.10. □

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