An improved flux wave-HLLE approach for the solution of traffic flow models based on transition velocities

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Abstract

In this study, a new numerical method is presented to solve the nonlinear Partial Differential Equations of second-order one-dimensional non-homogeneous traffic flow models based on transition velocities. The proposed Improved Flux Wave-HLLE (IFW-HLLE) method utilizes a particular type of approximate Riemann speed, that is, a unique combination of characteristic speeds and the Roe speed, to reach a solution with positive velocity and density. This method provides an equilibrium between the source terms and flux variations for steady-state conditions when solving the Riemann problem. The spatial variations in traffic density were also based on the transition velocities. For evaluating its performance, the proposed numerical solution is also compared with the results of the Original Roe Method (ORM) for solving widely-used Payne–Whitham (PW), Zhang, and Khan–Gulliver models. Moreover, both straight and circular paths with periodic boundary conditions were modelled to analyse and investigate the traffic flow of a bottleneck. Results demonstrate that the IFW-HLLE method captures more realistic traffic behaviours compared to ORM. Notably, negative and unrealistic velocity values observed in ORM—for the PW and Zhang models (ranging from -120 to 400 m/s and -600 to 1200 m/s)—were effectively corrected with the proposed method (ranges from 16 to 25 m/s and 8 to 16 m/s). Euclidean error norms calculated for 2D velocity profiles showed maximum errors of 2.6976×10^{-2} and 4.0835×10^{-3} for straight and circular paths, respectively, confirming the improved accuracy.

Keywords Flux wave formula, Transition velocities, non-homogeneous traffic flow model, Riemann problem, Wave propagation algorithm.

1 Introduction

Due to ever-increasing population growth, the simulation of natural traffic flow on the road is crucial for reducing traffic problems such as congestion. The first study of the macroscopic traffic flow model was conducted by Lighthill and Whitham [1] and Richards [2], and is known as the first-order Lighthill-Whitham-Richards (LWR) model. In contrast to this study, subsequent research demonstrated that speed is not always in equilibrium with density, and ignoring vehicle acceleration can lead to unrealistic results [3]. Thus, Payne [4] introduced a second-order model considering an independent dynamic equation for vehicle velocity. Whitham [5] independently introduced the Payne-Whitham (PW) model. This model is one of the most prevalent second-order traffic models because of its capability to simulate traffic flow using a minimum number of variables accurately. The PW model does not conserve the non-isotropic nature of traffic flow under certain conditions, such as driving in reverse [6], which causes unrealistic behaviour (often oscillatory) at traffic discontinuities. Moreover, the PW model assumes that the regulation of traffic flow occurs at a constant speed, which can result in velocities greater than the maximum or less than zero, both of which are impossible [7]. Various studies have struggled to address the mentioned shortcomings of the PW model. For example, Catillo et al [8] investigated the impact of the drivers' reaction time on the stability of traffic flow. Daganzo [9], Aw, and Rascle [10] proposed other second-order traffic flow models, and Berg et a [11] utilised the continuum approach. In the other study, Zhang [12] replaced the constant velocity with the derivative of the equilibrium speed distribution. Khan et al. [7] introduced a new model by presenting a traffic constant as a function of the driver's physiological-psychological behaviour. They also incorporated variables, such as driver response, maximum speed, and distance between vehicles, into the traffic pressure function component of the motion equation [13]. They later considered the average traffic velocity in transition, maximum speed, and transition distance to enhance their model [14].

PW-type traffic flow models are classified as nonlinear systems of hyperbolic Partial Differential Equations (PDEs), categorized as non-homogeneous in the presence of a source term. The continuity and momentum equations in these models are not analytically solvable because of their inherent PDE system, which necessitates numerical methods [15]. One of the most commonly used approaches is the Finite Volume Method (FVM). FVM is based on the integral form of physical laws, which makes it capable of addressing issues arising from discontinuous problems. Explicit FVM, such as the wave propagation algorithm [16], used for shock capturing, belongs to the class of Godunov-type methods, providing non-oscillatory results in discontinuous problems. In general, two different methods are used to calculate the wave propagation speed in the Riemann problem. The exact Riemann solver calculates the wave speed using analytical solutions [17]. This method is accurate and solves nonlinear equations simultaneously in each time step, which is time-consuming and costly. The approximate Riemann solution calculates the speed of shock, rarefaction, and contact discontinuity waves [18,19]. It is recommended to use approximate Riemann solvers, such as Roe, Harten-Lax-van Leer (HLL), Harten-Lax-van Leer-Einfeldt (HLLE), Harten-Lax-van Leer Contact (HLLC), and the hybrid method. The differences between these methods can be summarized in their discretization schemes, considering factors such as the direction or propagation speed [16,20]. Mohammadian et al. [21] investigated the performance of the HLL, HLLC, and Rusanov Riemann solvers. They evaluated them for the homogeneous Aw-Rascle-Zhang model with a continuous solution area from free-flow to congestion zones. Araghi et al. [22] developed a new type of wave propagation algorithm for solving three non-homogeneous PW-type traffic flow models based on driver physiological response, reaction velocity [23], and driver physiological-psychological behaviour [24]. They used only the characteristic wave speed to calculate the right- and left-going fluctuations. Their proposed algorithm provided a stable and realistic response without numerical diffusion. However, it showed an assertive fluctuating behaviour on a circular road for all models.

In this study, the flux wave formula for the wave propagation algorithm was improved using the HLLE approximate Riemann solver by comparing the characteristic velocities and Roe speed. The proposed approach is named IFW-HLLE. This improvement aims to stabilize the numerical solution of the macroscopic traffic flow models based on transition velocities [12,14] and reduce the fluctuating behaviour near discontinuities. Furthermore, the Van Leer limiter has been utilized, which produces more suitable results for the proposed approach [10]. To our knowledge, IFW-HLLE has not been applied to non-homogeneous PW-type traffic flow models. Moreover, the applicability of the proposed method against the Original Roe Method (ORM) for solving the PW, Zhang, and Khan-Gulliver (KG) models is evaluated in an inactive bottleneck on straight and circular roads with different constraints and conditions imposed on the traffic flow status.

2 Methods

The non-homogeneous second-order PW model [4, 5, 25], conceptually describes the behaviour of a compressible gas [4]. The PW model assumes that drivers react similarly to different conditions and that only minor changes occur in speed and density. This is an insufficient description of driver behaviour, which can lead to unrealistic and oscillatory traffic behaviour over short distances [13]. Another standard heterogeneous traffic flow model is Zhang's model. This model was formulated for a single type of vehicle through a homogenous path without lateral access. Zhang [12] aimed to improve the PW model by deriving an acceleration relationship based on a microscopic model, which led to a better representation of the non-isotropic properties of traffic flow. Khan and Gulliver [14] incorporated transitional speeds into the formulation of the PW model, representing traffic flow as a function of current speed deviation from the equilibrium speed. Table 1 presents all the three traffic models and their eigenvalues and eigenvectors.

Table 1 Characteristics of non-homogeneous traffic flow models of Payne-Witham type

MODEL	**PW**	**ZHANG**	**KG**
Mathematical form	$\begin{cases}\rho_t+(\rho u)_x=0\\ u_t+uu_x=\frac{C_0^2}{-\rho}\rho_x+\left(\frac{V_e(\rho)-u}{\tau}\right)\end{cases}$	$\begin{cases}\rho_t+(\rho u)_x=0\\ u_t+(u+\rho\acute{V}_e(\rho))u_x=0\end{cases}$	$\begin{cases}\rho_t+(\rho u)_x=0\\ u_t+uu_x=\frac{\upsilon_m^2-u^2}{2\rho d_{tr}}\rho_x+\left(\frac{V_e(\rho)-u}{\tau}\right)\end{cases}$
Eigenvalues	$\lambda_{1,2}=u\pm C_0$	$\lambda_1=u$ $\lambda_2=u+\rho u'(\rho)$	$\lambda_{1,2}=u\pm\sqrt{\frac{\upsilon_m^2-u^2}{2d_{tr}}}$
Eigenvectors	$r_1=\begin{pmatrix}1\\ u-C_0\end{pmatrix}$ $r_2=\begin{pmatrix}1\\ u+C_0\end{pmatrix}$	$r_1=\begin{pmatrix}1\\ u-u(\rho\ )-\rho u'(\rho)\end{pmatrix}$ $r_2=\begin{pmatrix}1\\ u-u(\rho)\end{pmatrix}$	$r_1=\begin{pmatrix}1\\ u-\sqrt{\frac{\upsilon_m^2-u^2}{2d_{tr}}}\end{pmatrix}$ $r_2=\begin{pmatrix}1\\ u+\sqrt{\frac{\upsilon_m^2-u^2}{2d_{tr}}}\end{pmatrix}$

where ρ is traffic flow density (veh/m). u and υ_m represent the traffic flow velocity (m/s) and maximum flow velocity (m/s), respectively. In addition, $V_e(\rho)$ is the equilibrium velocity of vehicles. The velocity constant, C_0, represents the drivers' response to traffic flow density. Moreover, λ and r are eighenvalue and eighenvector, respectively. The transfer distance, d_{tr} is defined as:

$$d_{tr}=\tau\upsilon_m+l_s \tag{1}$$

Where l_s is the distance between vehicles at rest.

3 The proposed numerical method: IFW-HLLE

In response to hyperbolic systems such as the three models presented in Table 1, a simple jump discontinuity may propagate along the characteristic line [26]:

$$U(x,0)=\begin{cases}U_l & x<0\\ U_r & x>0\end{cases} \tag{2}$$

The initial value of Eq. (2) is known as the Riemann problem. The wave propagation algorithm [17] is an approach for re-averaging the Riemann problem in the neighbouring cell for the FVM. The second-order Godunov-type wave propagation algorithm is as follows [20].

$$U_i^{n+1}=U_i^{n}-\frac{\Delta t}{\Delta x}\left(\sum_{k:s_1<0}\xi_{k,i-1/2}+\sum_{k:s_2>0}\xi_{k,i-1/2}\right)-\frac{\Delta t}{2\Delta x}\sum_{k=1}^{m}\left(I-\frac{\Delta t}{\Delta x}|\lambda_k|\right)|\lambda_k|\varphi(\theta)(W_{k,i-1/2}-W_{k,i-1/2}) \quad (3)$$

In which $U=[\rho‚\rho u]^T$ is the vector of unknowns, $W_{k,i-1/2}$ is the k-th wave, λ_k represents the eigenvalue, and $\xi_{k,i-1/2}$ is the k-th created wave from the i-1/2 cell interface, calculable by multiplying the eigenvector $r_{i-1/2}$ by eigenvalue $\beta_{i-1/2}$. The third term on the right-hand side of Eq. (3) is necessary to achieve a higher-order solution accuracy. The first-order Godunov method is obtained if this term is set to zero. An appropriate high-order limiter function $\varphi(\theta)$ should be chosen to calculate this term. In this study, the Van Leer limiter [27] was utilized because of its performance near discontinuities [2]:

$$\varphi(\theta)=\frac{\theta+|\theta|}{1+|\theta|} \quad (4)$$

In which "θ" is a scalar coefficient. Generally, finding an exact solution for nonhomogeneous Riemann problems is difficult owing to the effect of the source terms on the characteristic speed. A well-balanced method balances the source terms and flux differencing components. The well-balanced wave flux method [28] was first proposed to solve gas dynamics problems. This method has been modified for one- and two-dimensional shallow water equations and for the Euler equation [18]. The summation of the relevant fluxes must be equal to the flux difference between two adjacent cells and the source term, as follows:

$$\begin{bmatrix}\rho_i\tilde{u}_i\\ \frac{(\rho_i\tilde{u}_i)^2}{\rho_i}+(\frac{v_m^2-\tilde{u}_i^2}{2d_{tr}})\rho_i\end{bmatrix}-\begin{bmatrix}\rho_{i-1}\tilde{u}_{i-1}\\ \frac{(\rho_{i-1}\tilde{u}_{i-1})^2}{\rho_{i-1}}+\left(\frac{v_m^2-\tilde{u}_{i-1}^2}{2d_{tr}}\right)\rho_{i-1}\end{bmatrix}-\Delta x\begin{bmatrix}0\\ \tilde{\rho}_{i-\frac{1}{2}}\left(\frac{V_e\left(\tilde{\rho}_{i-\frac{1}{2}}\right)-\tilde{u}_{i-\frac{1}{2}}}{\tau}\right)\end{bmatrix}=$$

$$\beta_1\begin{bmatrix}1\\ \tilde{u}_i-\sqrt{\frac{v_m^2-u^2}{2d_{tr}}}\end{bmatrix}+\beta_2\begin{bmatrix}1\\ \tilde{u}_i+\sqrt{\frac{v_m^2-u^2}{2d_{tr}}}\end{bmatrix} \quad (5)$$

In which $\tilde{u}_{i-1/2}$ is the velocity at the i-1/2 cell interface, which can be calculated from the combination of exact and approximate Riemann wave speeds. The above equation is based on characteristic speeds. Characteristic speeds in various fields refer to a speed that's significant for understanding a system's behaviour. In vehicle dynamics, it defines the speed at which an understeer vehicle's control sensitivity is reduced. The HLLE method calculates the wave speeds by comparing the characteristics and Roe speeds [20]. Roe speeds are the eigenvalues of the Roe-averaged flux Jacobian matrix, important for wave propagation in numerical solutions. They enhance accuracy and stability in compressible fluid dynamics, especially for shock waves. It can be extended to nonhomogeneous models based on the transfer velocities, as follows:

$$s_{1,i-\frac{1}{2}} = \min\left(u_{i-1} - \sqrt{\frac{v_{\mathrm{m}}^2 - u_{i-1}^2}{2\mathrm{d}_{\mathrm{tr}}}}, s_{Roe1}\right) \quad and \quad s_{2,i-\frac{1}{2}} = \max(u_i, s_{Roe2}) \tag{6}$$

Roe speeds $s_{Roe1,i-1/2}$ and $s_{Roe2,i-1/2}$ were calculated based on the following relations:

$$s_{Roe1,i-1/2} = \tilde{u}_{i-\frac{1}{2}} - \sqrt{\frac{v_{\mathrm{m}}^2 - u_{i-\frac{1}{2}}^2}{2\mathrm{d}_{\mathrm{tr}}}} \quad and \quad s_{Roe2,i-\frac{1}{2}} = \tilde{u}_{i-\frac{1}{2}} \tag{7}$$

where the approximate density and velocity at cell interface i-1/2 are obtained as follows:

$$\tilde{\rho}_{i-\frac{1}{2}} = \sqrt{\rho_{i-1}\rho_i} \;,\; \tilde{u}_{i-1/2} = \frac{\sqrt{\rho_{i-1}}\tilde{u}_{i-1} + \sqrt{\rho_i}\tilde{u}_i}{\sqrt{\rho_{i-1}} + \sqrt{\rho_i}} \tag{8}$$

Eq. (5) can be rewritten as follows:

$$\begin{bmatrix} 1 & 1 \\ \tilde{u}_i - \sqrt{\frac{v_{\mathrm{m}}^2 - u^2}{2\mathrm{d}_{\mathrm{tr}}}} & \tilde{u}_i + \sqrt{\frac{v_{\mathrm{m}}^2 - u^2}{2\mathrm{d}_{\mathrm{tr}}}} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \rho_i\tilde{u}_i - \rho_{i-1}\tilde{u}_{i-1} \\ \frac{(\rho_i\tilde{u}_i)^2}{\rho_i} + \left(\frac{v_{\mathrm{m}}^2 - \tilde{u}_i^2}{2\mathrm{d}_{\mathrm{tr}}}\right)\rho_i - \frac{(\rho_{i-1}\tilde{u}_{i-1})^2}{\rho_{i-1}} - \left(\frac{v_{\mathrm{m}}^2 - \tilde{u}_{i-1}^2}{2\mathrm{d}_{\mathrm{tr}}}\right)\rho_{i-1} - \Delta x.\tilde{\rho}_{i-\frac{1}{2}}\left(\frac{V_e\left(\tilde{\rho}_{i-\frac{1}{2}}\right) - \tilde{u}_{i-\frac{1}{2}}}{\tau}\right) \end{bmatrix} \tag{9}$$

The coefficients β_1 and β_2 are calculated by solving the linear system of Eq. (9). The calculation process of β_1 and β_2 in Zhang and PW models is similar to the KG model.

4 Results and discussion

The focus of this part is on evaluating the performance of the IFW-HLLE algorithm in solving the second-order macroscopic traffic flow models for one-dimensional PW, Zhang, and KG models. The grid dimensions and variables were defined like that used by Khan and Gulliver [14]. The initial distribution of the density at t=0s is presented in Eq. (10):

$$\rho_0 = \begin{cases} 0.01 & x \leq 30 \\ 0.3 & 30 < x < 60 \\ 0.1 & x > 60 \end{cases} \tag{10}$$

Free-flow conditions dominate the first 30 m and the last 40 m of the route, whereas congestion exists between 30m and 60m. Two discontinuities are observed in the initial density profile, resulting in two simultaneous Riemann problems in this example. The first situation leads to the upward propagation of shock waves, and the second discontinuity results in expansion waves moving both upward and downwards over time. The Greenshields equilibrium equation is used for the velocity-density equilibrium relationship:

$$V_e(\rho)=v_m(1-\frac{\rho}{\rho_m}) \quad (11)$$

where ρ_m and ρ are the maximum and average traffic density, respectively. The maximum velocity on the road is presented by v_m. The CFL number was defined as follows to examine the stability of this method [29]:

$$CFL=\frac{\max\left(\mathbf{s}_{k,i-\frac{1}{2}}\right).dt}{dx} \quad (12)$$

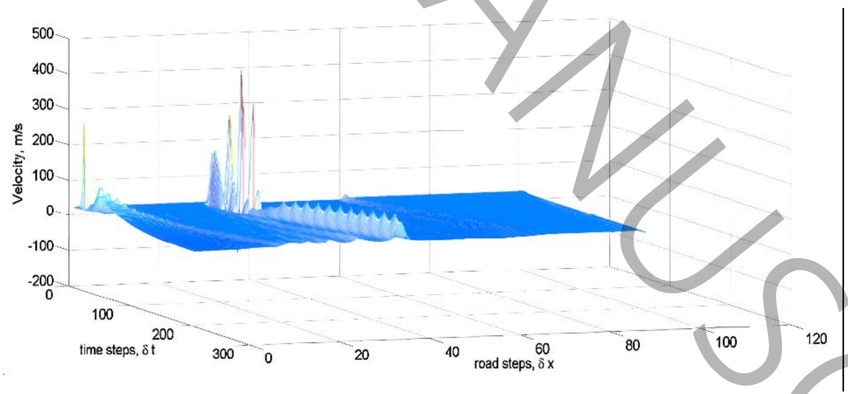
The Euclidean error norm is defined as follows to evaluate the accuracy of the results [30]:

$$\|e\| = \frac{\sqrt{\sum_{i=1}^{n}(\Delta y_i)^2}}{n} \quad (13)$$

where Δy_i represents the difference in the vertical parameter (e.g., velocity) at a specific horizontal position, and n is the total number of comparison points.

4.1 A straight path

A straight road with a length of 100 m was considered to study the performance of the proposed approach. The initial velocity is C_0=25 m/s. The velocity profiles of the PW and KG models obtained using the original Roe method (ORM) and the proposed IFW-HLLE approach are presented in Fig. 1. Fig. 2 compares 2D velocity profiles of mentioned models using ORM and IFW-HLLE methods at times 0.006, 0. 6, and 1.2 seconds.



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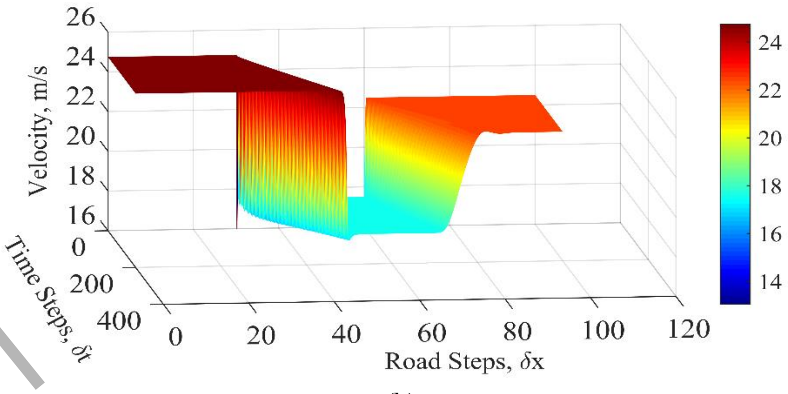

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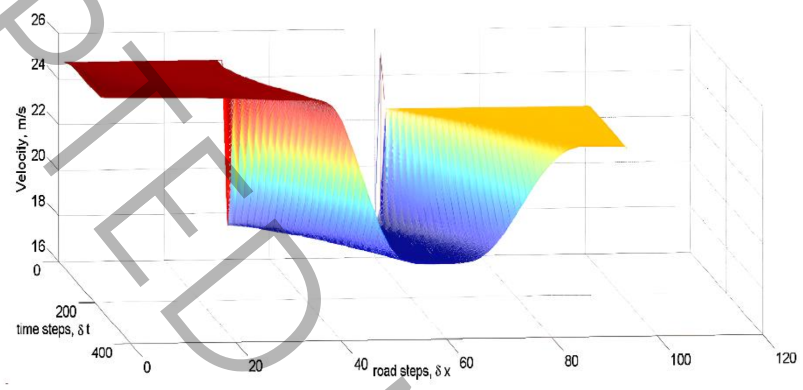

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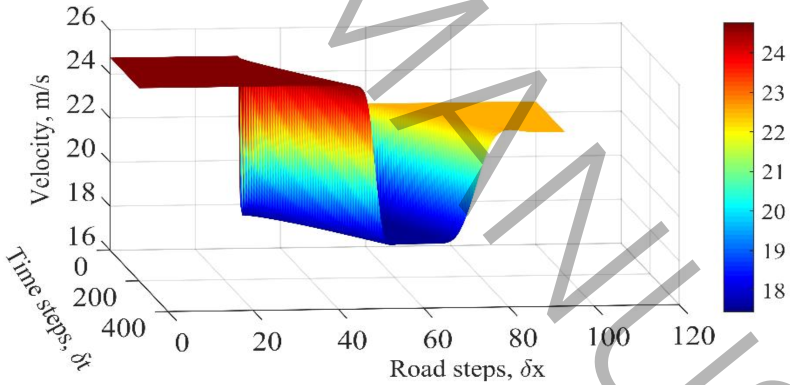


(d)

Fig. 1 The velocity profiles of PW and KG traffic flow models on a 100-meter length straight road with C_0=25 m/s using the ORM method (a, c) [14] and the proposed IFW-HLLE approach (b, d)

As shown in Fig 1(a), the response of the PW model using the ORM method is not realistic for sudden traffic density changes. Specifically, oscillatory behaviour is observed with velocities above 400 m/s and below -120 m/s, which violates maintaining positive speed values. By contrast, realistic velocities ranging from 0 to 25 m/s were achieved in the KG model using the same method. However, some oscillations were observed during the initial seconds of movement (Figs. 1(c) and 2). Moreover, it is evident in Figs 1(b), 1(d), and 2 that the proposed method significantly outperforms the ORM in controlling numerical diffusion errors and exhibits a more realistic velocity behaviour of the vehicle in traffic flow (within a range of 16 m/s to 25 m/s). In addition, the oscillations in the KG model using the ORM method decreased noticeably with the proposed model.

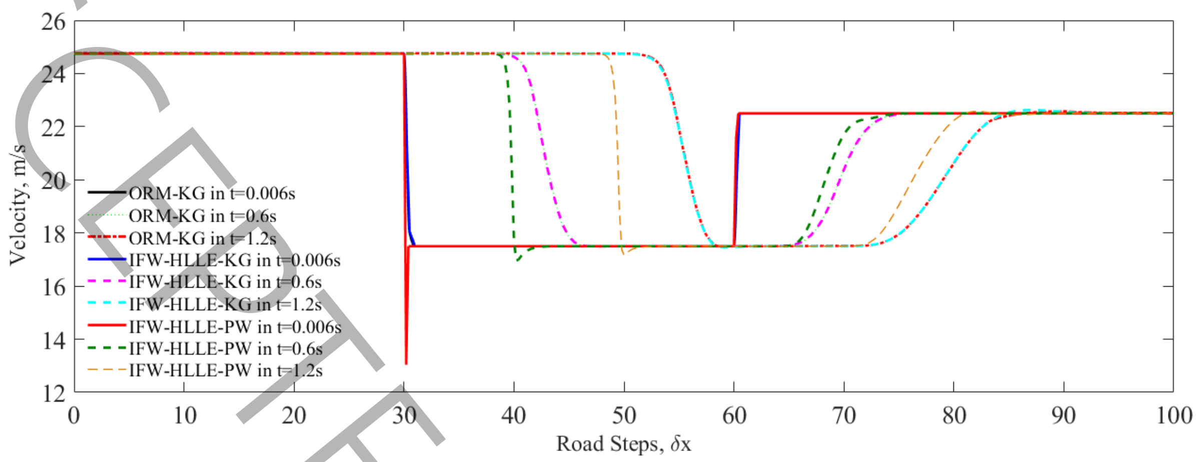


Fig. 2 Comparison between velocity profiles of PW and KG traffic flow models on a straight road with C_0=25 m/s using the ORM method [14] and the proposed IFW-HLLE approach

As shown in Fig. 2, the velocity profile variations were faster during the initial seconds of the numerical simulation. As time passed, these variations gradually became smoother. The Euclidean error norms for the KG model, calculated between the ORM and IFW-HLLE approaches at 0.006, 0.6, and 1.2 seconds, are presented in Table 2-a.

4.2 A circular path

Fig. 3 depicts the results of applying the ORM and IFW-HLLE methods to the Zhang model. The velocity profile in Fig. 3(a) shows that the ORM violates the maintenance of positive speed values. The velocity varies from -600 m/s to 400 m/s. On the other hand, the velocity profile of the proposed IFW-HLLE approach (Fig. 3(b)) demonstrates a consistent maintenance of positive values. The velocity increases from 10 m/s to 15.10 m/s within the first 21.5 meters length of the road. Subsequently, a decreasing trend starts and reaches 10 m/s at x=30 m. At this point, the velocity progressively increases to 13.72 m/s as it moves towards the end of the road. It is clear from Fig 3(b) that a discontinuity exists between x=25m and x=33m, where the velocity decreases.

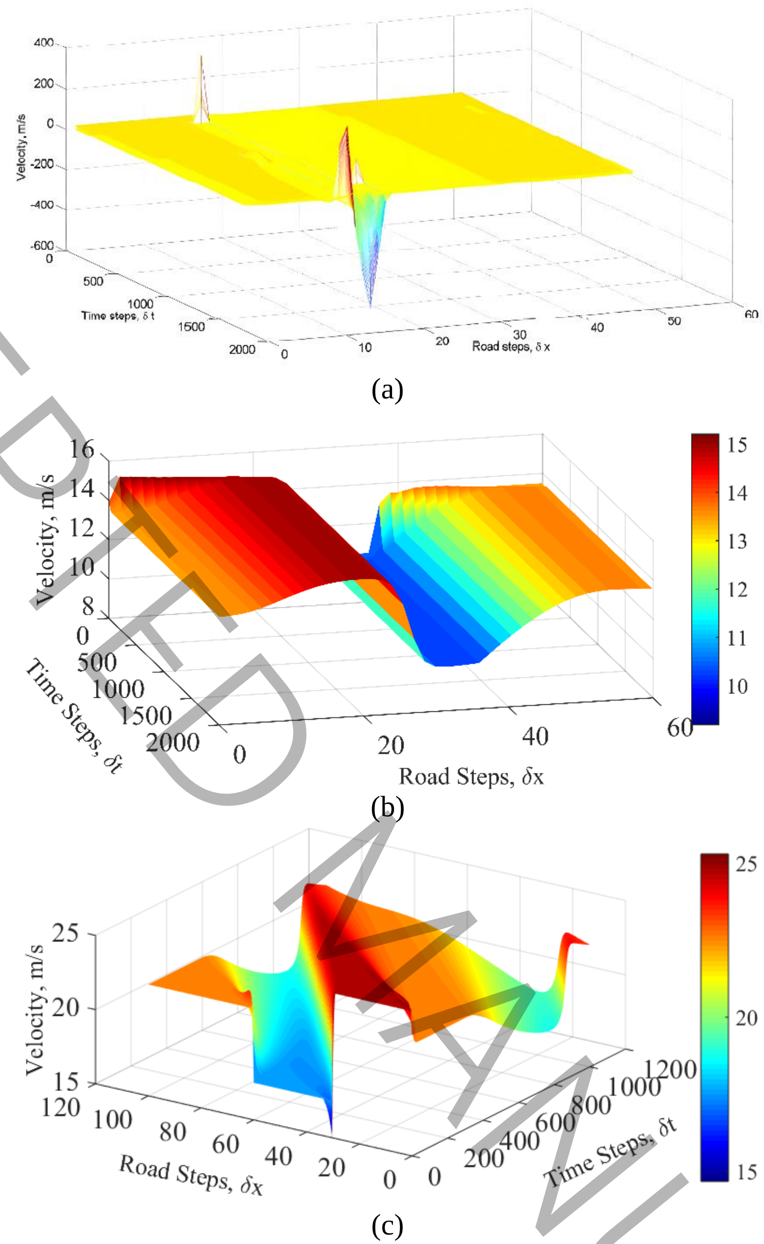
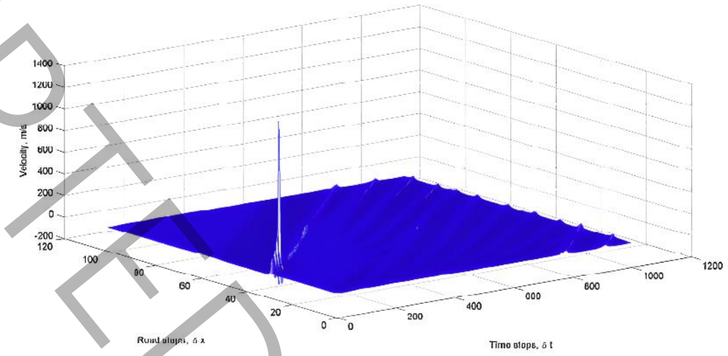


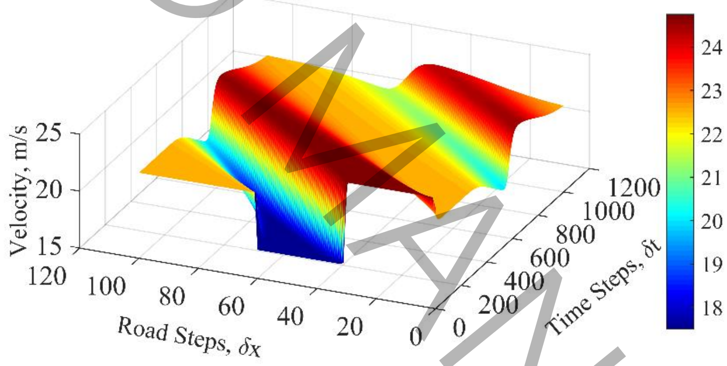
Fig. 3 The velocity profile of the Zhang model on a 100-meter length circular road with v_m=15 m/s (a-b) and v_m=25 m/s (c) using the ORM method (a) [14] and the proposed IFW-HLLE approach (b-c)

Fig. 3(c) illustrates the velocity profile of the Zhang model using the proposed method with maximum v_m=25 m/s and the same boundary conditions as the KG and PW models. It can be observed that the Zhang model produces more realistic behaviour compared to a model with v_m=15 m/s. Specifically, the traffic velocity varied within an acceptable range: 17.5 m/s to 24.77 m/s. Moreover, the negative and unrealistic velocity values observed in the ORM were rectified, and the velocity irregularities in the initial seconds of movement and discontinuities were entirely resolved using the proposed method.

The velocity profiles of the PW, KG, and Zhang models using the ORM and IFW–HLLE methods are depicted in Fig. 4. As can be seen in Fig. 4(a), the velocity varies from 1400 m/s to -120 m/s after 0.4 s, which is unrealistic. On average, the velocity variation was 30 m/s within the clusters through a distance of 5 m, which is unrealistic. The worst scenario occurred near x=20 m, where the velocity changed drastically from 23 m/s to 64 m/s and then decreased to 14 m/s near x=25 m. In contrast, the velocities obtained using the IFW-HLLE method were within the minimum and maximum limits.



(a)



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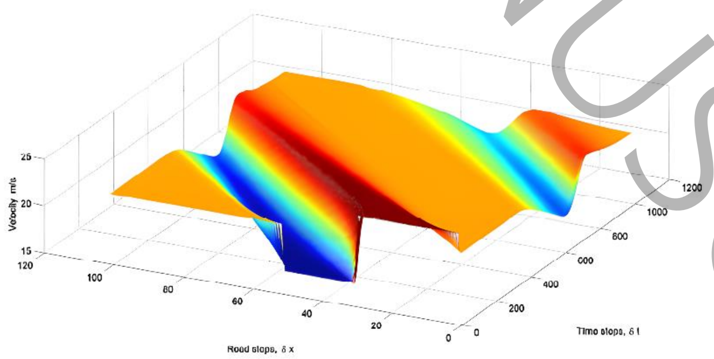

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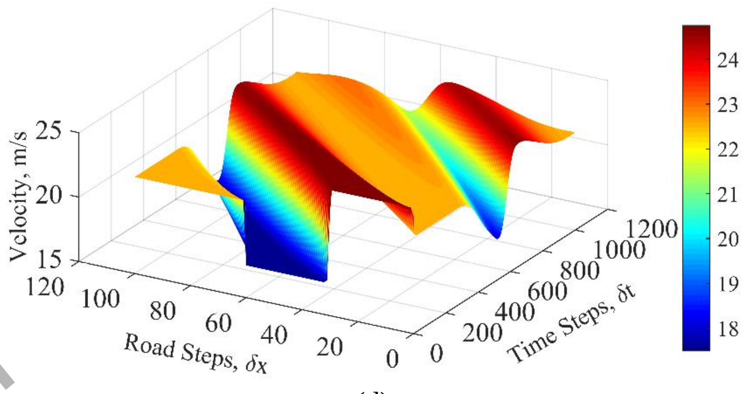

(d)

Fig. 4 The velocity profiles of PW and KG traffic flow models on a 100-meter length circular road with C_0=25 m/s using the ORM method (a, c) [14] and the proposed IFW-HLLE method (b, d)

The velocity profile of the KG model obtained using the ORM method is shown in Fig. 4(c). From x=0 m to x=10 m and from x=75 m to x=100 m, the velocity was almost constant at approximately 22.5 m/s, whereas it varied from 20 m/s to 23.3 m/s within the cluster. This traffic behaviour is realistic and falls within the maximum and minimum values. This pattern is also evident in Fig. 4(d). As presented in Table 2-b, the Euclidean error norms for the KG model on the circular path are calculated equal to 7.6269×10^{-4} and 4.0835×10^{-3} at t=3 s and t=6 s, respectively.

Table 2 The Euclidean error norms between the ORM and IFW-HLLE methods for the KG model in the case of a) a straight path, and b) a circular path at different times

(a)

Time (s)	Error Norm (m)
0.006	1.0885×10^{-2}
0.6	1.0772×10^{-2}
1.2	2.6976×10^{-2}

(b)

Time (s)	Error Norm (m)
3	7.6269×10^{-4}
6	4.0835×10^{-3}

Based on Fig. 5, the velocity variation became smoother and gradual from maximum to minimum and vice versa in all models. In addition, the KG model showed good agreement with both methods. The behaviour of the PW model using the proposed method was similar to that of KG, except at the initial 15 m and final five meters of the path. Delayed behaviour was also observed in the Zhang model at discontinuities compared to the PW and KG models.

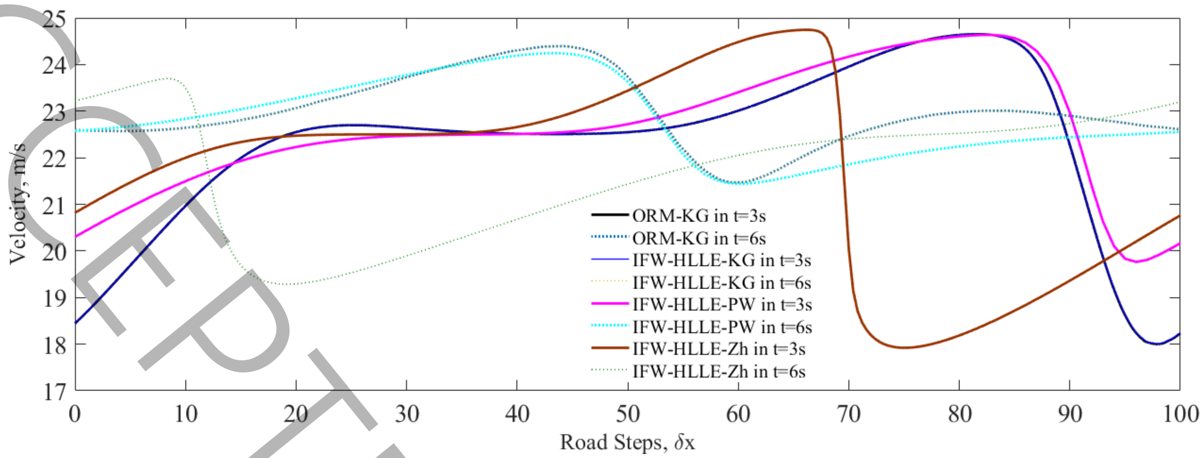

Fig. 5 Comparison between the velocity profiles of PW, KG, and Zhang traffic flow models on a circular road with C_0=25 m/s using the ORM method [14] and the proposed IFW-HLLE approach.

5 Conclusions

This paper presented the IFW-HLLE numerical method, based on a combination of characteristic velocities and the Roe velocity, to solve three well-known non-homogeneous second-order macroscopic traffic flow models: PW, KG, and Zhang. The approach consisted of the embedding of relaxation source terms into flux differences in a well-balanced finite-volume method. The results demonstrated that the application of the original Roe method (ORM) to the PW and Zhang models has the tendency to produce unrealistic velocity profiles with significant non-physical oscillations, where the velocities exceed plausible ranges (e.g., values greater than 400 m/s or negative velocities) particularly near discontinuities. The oscillations violate the physical bounds and yield sudden, discrete velocity jumps contrary to the nature of traffic flow. In contrast, the KG model when solved with ORM exhibited relatively smoother and physically more acceptable velocity variations but still had initial oscillatory artifacts.

On the other hand, the IFW-HLLE method steadily yielded stable and sensible velocity profiles for the three models under study. Numerical diffusion and oscillation damping were very effective, particularly in the vicinity of discontinuities, and velocity values were maintained within physically acceptable limits (e.g., between approximately 16 m/s to 25 m/s on straight roads and 10 m/s to 25 m/s on curved trajectories). Quantitatively, the Euclidean error norms of the KG model between ORM and IFW-HLLE reached a minimum of $7.63\times10{-}47.63\times10^{-4}$ at t=3 s, thereby validating the enhanced numerical accuracy and reliability of the proposed methodology. The IFW-HLLE method also exhibited improved control of wave propagation and relaxation effects, and smother velocity transitions that more accurately capture actual traffic flows. Extending these encouraging results, avenues for future research are:

- Implementing the IFW-HLLE approach on multilane highways, urban intersections, and heterogeneous traffic conditions characterized by a combination of vehicle types, in order to test its robustness and scalability towards more realistic environments.
- Implementation of this methodology in traffic simulators and control systems for predictive traffic management, focusing on real-time application and adaptive traffic control.
- Incorporating stochastic elements and driver behavior models to capture traffic variability and improve model precision.
- Optimizing the algorithm for high-performance computing to facilitate large-scale network simulations with better spatial and temporal resolution.

Declarations

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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