AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 7(1) (2026) 69-74
https://doi.org/10.22060/AJMC.2025.23648.1278

Original Article

Finiteness of fundamental groups in extended complete Einstein-type manifolds

Ayoub Mohamadalipour[a], Behroz Bidabad[*a], Mohamad Yar Ahmadi[b]

[a]*Department of Mathematics and Computer Sciences, Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran*
[b]*Department of Mathematics, Faculty of the Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran*

ABSTRACT: Here, an extension of the complete non-compact Einstein-type manifolds is studied and shows that it has a finite fundamental group. This paper generalizes Wylie's results to the setting of extended Einstein-type manifolds for certain parameters. Some direct corollaries of this result are also pointed out. For instance, the sphere bundle SM has a finite fundamental group. These findings not only generalize previous results but also offer new insights into the applications of Einstein-type manifolds across mathematics and physics, particularly in the structure of associated bundles and the behavior of geometric flows.

Review History:

Received:02 November 2024
Revised:26 June 2025
Accepted:21 August 2025
Available Online:01 January 2026

Keywords:

Fundamental group
Einstein-type manifold
Quasi-Einstein
De Rham cohomology

MSC (2020):

53C21; 53C25; 53C60

1. Introduction

1.1. General Background and Applications

The study of Einstein manifolds, which are Riemannian manifolds characterized by the property that their Ricci tensor is proportional to the metric, has significantly influenced the fields of differential geometry and theoretical physics. These manifolds are of particular interest because they generalize the notion of spaces with constant curvature and appear naturally as solutions in General Relativity. Broadly speaking, Einstein manifolds form the backbone of many advances in both mathematics and physics, playing a central role in the study of geometric structures, topology, and even quantum field theories.

The exploration of fundamental groups of Einstein manifolds, which reflect the connectivity and global geometry of these spaces, reveals deeper topological features that influence geometric behavior. Previous work by McKean and others, particularly in the context of negatively curved and non-compact spaces, has shown that certain conditions on the curvature of a manifold imply finiteness of the fundamental group. More recently, Wylie's generalizations

*Corresponding author.
E-mail addresses: a.mohamadalipour@aut.ac.ir (A. Mohammadalipour), bidabad@aut.ac.ir (B. Bidabad), m.yarahmadi@scu.ac.ir (M. Yar Ahmadi)

expanded the scope to a broader class of Einstein-type manifolds, which involve adjustments to the Ricci curvature condition using additional parameterizations and potential functions. However, the extent to which these properties apply in more generalized settings, such as complete non-compact Einstein-type manifolds with certain parameter constraints, has remained an open question.

This paper extends Wylie's results to a wider class of Einstein-type manifolds by establishing that complete non-compact Einstein-type manifolds exhibit finite fundamental groups under specific parameter choices. This generalization offers not only a new understanding of the topological structure of these manifolds but also reveals implications for related geometric structures. In particular, one immediate corollary is that the sphere bundle SM, associated with such manifolds, has a finite fundamental group. Sphere bundles over manifolds are significant in both mathematics and physics; they serve as critical structures in fields ranging from fiber bundle theory to geometric quantization in quantum mechanics.

Applications of the results presented here span several domains. In theoretical physics, Einstein-type manifolds are foundational to various generalizations of spacetime in relativity, where understanding the topology and global properties of the manifold sheds light on the nature of singularities and possible topologies of the universe. Additionally, finite fundamental group results contribute to stability analyses in geometric flow theory, as manifolds with such properties often exhibit controlled behavior under Ricci and other geometric flows. This work also contributes to the broader framework of geometric group theory, linking manifold geometry with algebraic properties of fundamental groups, which has implications in the study of discrete subgroups of Lie groups and their representations.

1.2. Historical Background

In 1968, J. Milnor [14] conjectured that a complete, non-compact Riemannian manifold with non-negative Ricci curvature has a finitely generated fundamental group, proven in dimension 3 and below. But counterexamples have been found in dimensions 6 and 7. This conjecture has drawn significant attention, yielding notable results for Einstein manifolds and their generalizations in Riemannian geometry. While this list is not exhaustive, we highlight several impacts findings, particularly concerning complete Riemannian manifolds with bounded Ricci curvature.

The manifolds bounded below Ricci curvature are studied by J. Cheeger, and T. Colding, especially those relevant to the collapsing of Riemannian manifolds are discussed by K. Fukaya and M. Gromov, see for instance [6, 7, 8], [11] and [12].

To analyze diffusion processes on Riemannian manifolds, D. Bakry and M. Emery [3] introduced a generalized Ricci tensor, defined by

$$\tilde{Ric} = Ric - Hess(\ln\phi), \tag{1}$$

where ϕ is a smooth positive function on M. The Bakry-Emery tensor (1) has since been linked to logarithmic Sobolev inequalities, isoperimetric inequalities, and heat semigroups (see [2] for details on these connections). J. Lott [13] showed that a closed Riemannian manifold with a gradient-shrinking Ricci soliton has a finite fundamental group. Similarly, A. Derdzinski [9] proved that any compact shrinking Ricci soliton has only finite conjugacy classes, and F. Lopez and G. Rio [10] demonstrated that a compact shrinking Ricci soliton also has a finite fundamental group. Extending these results, W. Wylie [15] proved the same for "complete" shrinking Ricci solitons.

Bidabad and Yar Ahmadi [4] further established that a complete, non-compact Riemannian shrinking Yamabe soliton has a finite fundamental group, provided the scalar curvature is strictly bounded above. Setting $\phi = e^{-\frac{f}{m}}$ for a smooth function f on M, equation (1) reformulates as

$$Ric_f^m = Ric + Hess(f) - \frac{1}{m}df \otimes df,$$

known as the m-Bakry-Emery Ricci tensor for $0 < m < \infty$. A triple (M, g, f), where f is a real function on M, is called an *m-quasi-Einstein* manifold if it satisfies

$$Ric_f^m = \lambda g,$$

for some $\lambda \in \mathbb{R}$. This equation becomes particularly significant when $m = \infty$, yielding the gradient Ricci soliton equation. When m is a positive integer, it corresponds to the warped product Einstein metrics.

A generalization of this equation, known as an *Einstein-type manifold*, is given in [5] as:

$$\alpha R_{ij} + \beta \mathcal{L}_X g_{ij} + \gamma X_i X_j = (\rho S + \mu(x))g_{ij},$$

for real constants $\alpha, \beta, \gamma, \rho$ and $\mu \in C^\infty(M)$, where $\mathcal{L}_X$ denotes the Lie derivative along a vector field X, and S is the scalar curvature. For convenience, we assume $\alpha \neq 0$ and rewrite this equation as:

$$R_{ij} + \beta \mathcal{L}_X g_{ij} + \gamma X_i X_j = \lambda(x) g_{ij}, \tag{2}$$

where $\lambda(x) = \rho S + \mu(x)$. For certain extensions of this work see also [1, 16, 17]. This work extends the results of Wylie for Einstein-type manifolds [15].

1.3. New Results

In this paper, we consider a vast extension of the above equation to the following inequality:

$$R_{ij} + \beta \mathcal{L}_X g_{ij} + \gamma X_i X_j \geq \lambda(x) g_{ij}, \tag{3}$$

and call it a wide *extended Einstein-type* manifold. First, we present an estimation of the distance function on an extension of the non-compact complete Einstein-type manifolds, as follows.

Theorem 1.1. *Let (M, g) be a complete Riemannian manifold satisfying the inequality (3), then for any $p, q \in M$ we have the following inequalities;*

1. *If $\gamma \leq 0$ and $\lambda(x) \geq \Lambda_1$, for some real positive number Λ_1,*

$$d(p,q) \leq \max\{1, \frac{1}{\Lambda_1}(2(n-1) + H_p + H_q + 2|\beta|(|X|_p + |X|_q))\}.$$

2. *If $\gamma > 0$ and $\lambda(x) - \gamma|x|^2 \geq \Lambda_2$, for some real positive number Λ_2,*

$$d(p,q) \leq \max\{1, \frac{1}{\Lambda_2}(2(n-1) + H_p + H_q + 2|\beta|(|X|_p + |X|_q))\}.$$

Next, we show that the following version of Milnor's conjecture is true for a large extension of Einstein-type Riemannian manifolds.

Theorem 1.2. *Let (M, g) be a complete extended Einstein-type manifold and cases (1) and (2) of Theorem 1.1 hold. Then the fundamental group (M, g) is finite.*

2. An Estimation for Distance Function

Let (M, g) be a Riemannian manifold and X a vector field on M. We consider the usual norm $|X|_x = \sqrt{g_{ij}(x) X^i X^j}$, at every point $x \in M$ and set

$$H_x = \max\{0, \sup\{Ric_y(V,V) : y \in B_x(1), g(V,V) = 1\}\}.$$

Using which we have the following lemma.

Lemma 2.1 ([15]). *Let (M, g) be a complete Riemannian manifold and $p, q \in M$ such that $d(p,q) > 1$. If $\theta(t)$ is the minimal geodesic from p to q parameterized by the arc-length t, then*

$$\int_0^T Ric(\theta', \theta') dt \leq 2(n-1) + H_p + H_q.$$

Recall that the Cauchy-Schwartz inequality, on a Riemannian manifold (M, g), is given by

$$|g(X,Y)|^2 \leq g(X,X) g(Y,Y), \qquad \forall X, Y \in \mathcal{X}(M). \tag{4}$$

Now we are in a position to prove the theorem 1.1.

Proof of Theorem 1.1. If the distance $d(p,q)$ is bounded below by 1, that is, $d(p,q) \leqslant 1$, then we have the assertion. Assume that $T := d(p,q) > 1$. Let $\theta : [0, +\infty) \longrightarrow M$ be a minimal geodesic parameterized by the arc length t emanating from a point p and passing through q. Along the geodesic θ, we have

$$Ric(\theta', \theta') \geq \lambda(x) - \beta \mathcal{L}_X g(\theta', \theta') - \gamma g(\theta', X)^2. \tag{5}$$

On the other hand, using lie derivative definition and metric compatibility, we have

$$\mathcal{L}_X g(\theta', \theta') = 2g(\nabla_{\theta'} X, \theta') = 2\frac{d}{dt} g(X, \theta').$$

Replacing the above equation in (5), we obtain by integrating

$$\int_0^T Ric(\theta', \theta') dt \geq \int_0^T (\lambda(x) - \gamma g(\theta', X)^2) dt - 2\beta g(X, \theta')\Big|_0^T. \tag{6}$$

Let $\theta(0) = p$ and $\theta(T) = q$. Using the Cauchy-Schwartz inequality (4), we have

$$\left|g\Big(\theta'(t), X(\theta'(t)\Big)\right|^2 \leq g(\theta'(t), \theta'(t))g(X(\theta'(t)), X(\theta'(t)).$$

Since θ is parameterized by the arc length t, we have $g(\theta'(t), \theta'(t)) = 1$ and therefore

$$-|X|_p \leq g(\theta'(0), X(\theta(0))) \leq |X|_p,$$

$$-|X|_q \leq g(\theta'(T), X(\theta(T))) \leq |X|_q.$$

Subtracting these inequalities yields

$$-|X|_q - |X|_p \leq g(\theta'(T), X(\theta(T))) - g(\theta'(0), X(\theta(0))) \leq |X|_p + |X|_q.$$

If $\beta \geqslant 0$, by multiplying -2β to the above inequality we have

$$-2\beta(|X|_q + |X|_p) \leq -2\beta g(\theta', X)\Big|_0^T \leq 2\beta(|X|_p + |X|_q).$$

If $\beta < 0$, by multiplying -2β to the above inequality gives

$$2\beta(|X|_q + |X|_p) \leq -2\beta g(\theta', X)\Big|_0^T \leq -2\beta(|X|_p + |X|_q),$$

from which (6) becomes

$$\int_0^T Ric(\theta', \theta')dt \geq \int_0^T (\lambda(x) - \gamma g(\theta', X)^2)dt - 2|\beta|(|X|_p + |X|_q). \tag{7}$$

Therefore by Lemma (2.1), the inequality (7) turns to be

$$\int_0^T (\lambda(x) - \gamma g(\theta', X)^2)dt \leq 2(n-1) + H_p + H_q + 2|\beta|(|X|_p + |X|_q). \tag{8}$$

If $\gamma \leq 0$, using the assumption $\lambda(x) \geq \Lambda_1 > 0$ we have

$$\Lambda_1 T \leqslant \int_0^T \lambda(x)dt \leq \int_0^T (\lambda(x) - \gamma g(\theta', X)^2)dt,$$

and, (8) becomes

$$\Lambda_1 T \leq 2(n-1) + H_p + H_q + 2|\beta|(|X|_p + |X|_q).$$

If $\gamma \geq 0$, the equation (4) leads to $g(\theta', X)^2 \leq |X|^2$, hence we have $-\gamma g(\theta', X)^2 \geq -\gamma|X|^2$. Therefore

$$\int_0^T (\lambda(x) - \gamma|X|^2)dt \leq \int_0^T (\lambda(x) - \gamma g(\theta', X)^2)dt.$$

According to the assumption (2) we have $\lambda(x) - \gamma|x|^2 \geq \Lambda_2 > 0$, and (8) becomes

$$\Lambda_2 T \leq 2(n-1) + H_p + H_q + 2|\beta|(|X|_p + |X|_q),$$

as we have claimed. This completes the proof of Theorem 1.1. □

3. The Fundamental Group of Einstein-type manifolds

Following the Theorem 1.1, we give a proof for the main result of this paper.

Proof of Theorem 1.2. Let $\tilde{M}$ be the universal covering of (M, g), with the smooth covering map $p : \tilde{M} \longrightarrow M$. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold, where $\tilde{g} := p^*g$ is the Riemannian pullback metric of g. Assume that $\tilde{\theta}(t)$ is a geodesic starting from the point $\tilde{x} = \tilde{\theta}(0)$ on $\tilde{M}$, where $\theta(t) = p(\tilde{\theta}(t))$. Recall that $(\tilde{M}, \tilde{g})$ and (M, g) are locally isometric, therefore $\theta(t)$ is also a geodesic on M. Since M is a complete manifold, the parameter t in the geodesic $\theta(t)$ can be extended to $t \in [-\infty, \infty]$. Being locally isometric, make $\tilde{\theta}(t)$ extendible, as well. Hence $(\tilde{M}, \tilde{g})$ is a geodesically complete manifold. Letting $\tilde{X} := p^*X$, since the projection $p : (\tilde{M}, \tilde{g}) \longrightarrow (M, g)$ is locally isometric, (3) implies

$$p^*(R_{ij} + \beta\mathcal{L}_X g_{ij} + \gamma X_i X_j) \geq p^*(\lambda(x) g_{ij}).$$

Linearity of p^* leads to

$$p^*R_{ij} + \beta p^*\mathcal{L}_X g_{ij} + \gamma p^*X_i p^*X_j \geq p^*\lambda(x) p^*g_{ij}.$$

Denoting $\tilde{R}_{ij} = p^*R_{ij}$ and using the Lie derivative property $p^*\mathcal{L}_X g_{ij} = \mathcal{L}_{\tilde{X}}\tilde{g}_{ij}$, we have

$$\tilde{R}_{ij} + \beta\mathcal{L}_{\tilde{X}}\tilde{g}_{ij} + \gamma\tilde{X}_i\tilde{X}_j \geq \lambda(p(\tilde{x}))\tilde{g}_{ij}. \tag{9}$$

Recall that a deck transformation $h : \tilde{M} \longrightarrow \tilde{M}$ is an isometry such that, $p \circ h = p$. Hence the group of all deck transformations on $\tilde{M}$ is isometric to the fundamental group of M. Let $h \in \pi_1(\tilde{M})$ be a deck transformation on $\tilde{M}$. Note that $B(\tilde{p}, 1)$ and $B(h(\tilde{p}), 1)$ are isometric, and thus $H_{\tilde{p}} = H_{h(\tilde{p})}$. Also $|\tilde{X}|_{\tilde{p}} = |\tilde{X}|_{h(\tilde{p})}$. Hence Theorem 1.1 implies that

$$d(\tilde{p}, h(\tilde{p})) \leq \max\{1, \frac{2}{\Lambda}[(n-1) + H_{\tilde{p}} + 2\beta|\tilde{X}|_{\tilde{p}}]\},$$

where Λ can be replaced by Λ_1 or Λ_2 for both cases mentioned in Theorem 1.1. Therefore, the set $p^{-1}(x)$, where $x = p(\tilde{x})$, is bounded. The geodesically completeness and the Hopf-Rinow's theorem yields that the closed and bounded subset $p^{-1}(x)$ of $\tilde{M}$ is compact. On the other hand, since the set $p^{-1}(x)$ is discrete, so it is finite. By connectedness assumption of M the fundamental group $\pi_1(M, x)$ are isomorphic. Since $\tilde{M}$ is a universal covering and $\pi_1(M, x)$ is in bijective relation with $p^{-1}(x)$, $\pi_1(M)$ is finite and consequently the first de Rham cohomology group vanishes $H^1_{dR}(M) = 0$. This completes the proof of Theorem 1.2. □

Corollary 3.1. *Any complete non-compact Einstein-type manifold, satisfying the conditions of Theorem 1.1, is of finite fundamental group and the de Rham cohomology group vanishes.*

Let us denote the sphere bundle by $SM := \bigcup_{x \in M} S_xM$, where $S_xM := \{v \in T_xM | \ g(v, v) = 1\}$. We have The following corollary.

Corollary 3.2. *Let (M, g) be an n-dimensional $(n > 2)$ Riemannian manifold satisfying the assumptions of Theorem 1.2. The sphere bundle SM has finite fundamental group and the de Rham cohomology group $H^1_{dR}(SM)$ vanishes.*

Proof. By a method similar to the proof of Theorem 1.1 and recalling the exactness of the following homotopic sequence of the fibre bundles $(S\tilde{M}, \tilde{\pi}, \tilde{M}, S^{n-1})$:

$$\cdots \longrightarrow \pi_1(S^{n-1}) \longrightarrow \pi_1(S\tilde{M}) \longrightarrow \pi_1(\tilde{M}) \longrightarrow \cdots$$

since $\pi_1(\tilde{M}) = \pi_1(S^{n-1}) = 0$, we have $\pi_1(S\tilde{M}) = 0$. Let $p_* : S\tilde{M} \longrightarrow SM$ be a covering map which implies that $S\tilde{M}$ is the universal covering manifold of SM. Since $p^{-1}(x)$ is a finite set for all $x \in M$, $p_*^{-1}(y) = \bigcup_{\tilde{x} \in p^{-1}(x)} S_{\tilde{x}}\tilde{M}$ is compact and discrete. Thus $p_*^{-1}(y)$ is a finite set, for all $y \in S_xM$. Hence the proof is complete. □

In summary, we have investigated the fundamental groups of a broad class of Einstein-type manifolds, extending classical results concerning Milnor's conjecture to new geometric structures. Einstein-type manifolds, as generalizations of Einstein metrics, Ricci solitons, almost Ricci solitons, and (m-)quasi Einstein manifolds, are of fundamental interest in both geometry and physics due to their varied applications and connections to important topics like diffusion processes, isoperimetric inequalities, and geometric flows.

The primary result, Theorem 1.2, establishes that Milnor's conjecture holds for a wide range of complete non-compact Einstein-type manifolds, demonstrating that these spaces have a finitely generated fundamental group under certain parameter conditions. This work builds upon and extends the results of Wylie [15] by proving similar topological properties for a broader class of Einstein-type manifolds.

These findings underscore the strength of Einstein-type metrics in exploring fundamental geometric and topological properties, especially within the framework of finite fundamental groups. Moreover, the insights gained here open avenues for further research on the role of Einstein-type structures in the stability of geometric flows, the topology of associated bundles, and their applications in theoretical physics and differential geometry.

As it's mentioned earlier, Einstein-type manifolds are a generalization of Einstein metrics, Ricci solitons, almost Ricci solitons and (m-)quasi Einstein manifolds. Theorem 1.2 states that Milnor's conjecture is true for a wide extension of Einstein-type manifolds. This work extends the results of Wylie for Einstein-type manifolds, [15].

References

[1] M. Ahmad Mirshafeazadeh and B. Bidabad, *On rigidity and scalar curvature of Einstein-type manifolds*, Int. J. Geom. Methods Mod. Phys., 15 (2018), pp. 1850073, 16.

[2] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer, *Sur les inégalités de Sobolev logarithmiques*, vol. 10 of Panoramas et Synthèses [Panoramas and Syntheses], Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.

[3] D. Bakry and M. Émery, *Diffusions hypercontractives*, in Séminaire de probabilités, XIX, 1983/84, vol. 1123 of Lecture Notes in Math., Springer, Berlin, 1985, pp. 177–206.

[4] B. Bidabad and M. Yar Ahmadi, *On complete Yamabe solitons*, Adv. Geom., 18 (2018), pp. 101–104.

[5] G. Catino, P. Mastrolia, D. D. Monticelli, and M. Rigoli, *On the geometry of gradient Einstein-type manifolds*, Pacific J. Math., 286 (2017), pp. 39–67.

[6] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom., 46 (1997), pp. 406–480.

[7] ——, *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom., 54 (2000), pp. 13–35.

[8] ——, *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom., 54 (2000), pp. 37–74.

[9] A. Derdzinski, *A Myers-type theorem and compact Ricci solitons*, Proc. Amer. Math. Soc., 134 (2006), pp. 3645–3648.

[10] M. Fernández-López and E. García-Río, *A remark on compact Ricci solitons*, Math. Ann., 340 (2008), pp. 893–896.

[11] K. Fukaya and K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math., 87 (1987), pp. 517–547.

[12] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, vol. 152 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[13] J. Lott, *Some geometric properties of the Bakry-Émery-Ricci tensor*, Comment. Math. Helv., 78 (2003), pp. 865–883.

[14] J. Milnor, *A note on curvature and fundamental group*, J. Differential Geometry, 2 (1968), pp. 1–7.

[15] W. Wylie, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc., 136 (2008), pp. 1803–1806.

[16] M. Yar Ahmadi, *On the gradient Finsler Yamabe solitons*, AUT J. Math. Comput., 1 (2020), pp. 229–233.

[17] M. Yar Ahmadi and S. Hedayatian, *Finite topological type of complete Finsler gradient shrinking Ricci solitons*, Turkish J. Math., 45 (2021), pp. 2419–2426.

Please cite this article using:

Ayoub Mohamadalipour, Behroz Bidabad, Mohamad Yar Ahmadi, Finiteness of fundamental groups in extended complete Einstein-type manifolds, AUT J. Math. Comput., 7(1) (2026) 69-74
https://doi.org/10.22060/AJMC.2023.22115.1134