Analytical Modeling of Nonlinear Oscillators Using the Energy Balance Method: Applications to Relativistic and Duffing Systems

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Abstract

Accurate determination of the frequency–amplitude relationship in strongly nonlinear oscillators remains a critical challenge in engineering dynamics, particularly when classical linearization and traditional perturbation methods fail due to the absence of small parameters. This limitation complicates the analysis and design of systems where nonlinear effects dominate, such as mechanical vibrations, structural dynamics, and nonlinear control applications. This study investigates the applicability and effectiveness of the Energy Balance Method (EBM) for analytical modeling of relativistic and Duffing-type oscillators. The method provides reliable and computationally efficient expressions for system frequencies across a wide range of oscillation amplitudes without relying on restrictive perturbation assumptions. The formulation assumes conservative systems undergoing periodic motion, limiting its application to undamped oscillators; within these constraints, EBM offers a simple and robust solution strategy. The approach constructs an energy balance over a single oscillation period using a single-term trial function, requiring only one iteration to obtain the frequency–amplitude relationship. The main novelty lies in demonstrating that such a minimal framework delivers highly accurate results even in strongly nonlinear regimes. Quantitative comparisons with exact solutions and established analytical methods—including the Harmonic Balance Method, Variational Iteration Method, Homotopy Perturbation Method, and the method of multiple scales—show excellent agreement, with negligible relative errors over a broad amplitude range. These results confirm that EBM achieves comparable or superior accuracy with significantly reduced computational effort. Future work may extend the method to damped, forced, and multi-degree-of-freedom systems, further broadening its applicability to complex engineering problems.

Keywords: Energy Balance Method; Relativistic oscillator; Approximate frequency; Nonlinear oscillations.

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1. Introduction

Nonlinear oscillations are widely encountered in engineering and physical systems and play a fundamental role in the analysis and design of structural, mechanical, and electromechanical components. In many practical applications, such as large-amplitude structural vibrations, nonlinear mechanical devices, and high-energy dynamic systems, the assumption of linear behavior is no longer valid. Nonlinear oscillators exhibit complex phenomena including amplitude-dependent frequencies, internal resonances, bifurcations, and chaotic responses, making accurate prediction of their dynamic behavior essential for ensuring safety, reliability, and optimal performance. Consequently, the development of efficient and accurate analytical tools for modeling nonlinear oscillatory systems remains a critical issue in civil and mechanical engineering.

To address nonlinear governing equations, a variety of analytical and numerical solution methodologies [1, 2] have been proposed in the literature. Classical perturbation-based techniques [18], such as the method of multiple scales and averaging methods, have been extensively used due to their mathematical simplicity; however, their applicability is restricted to weakly nonlinear systems because they rely on the presence of a small parameter. In strongly nonlinear problems, where such parameters do not exist, these traditional approaches often fail or yield inaccurate results. To overcome these limitations, several alternative analytical methods have been developed, including the Variational Iteration Method (VIM) [3, 4], Homotopy Perturbation Method (HPM) [5-7], Exp-function method [8-11], Green's function method [12], and the Harmonic Balance Method [13]. These techniques have been successfully applied to a wide range of nonlinear problems, including Duffing-type [14] and relativistic oscillators [15, 16]. In particular, harmonic balance–based approaches have been used to derive periodic solutions and to study the

fundamental characteristics of nonlinear oscillatory systems. Comprehensive reviews of these methods have highlighted their effectiveness as well as their inherent assumptions and limitations when applied to strongly nonlinear dynamics.

Among the available analytical techniques, the Energy Balance Method (EBM) [14, 17] has attracted considerable attention as a simple yet powerful alternative for solving nonlinear oscillation problems without requiring small perturbation parameters. The theoretical foundation of EBM is based on the principle of energy conservation in conservative systems and employs a variational framework combined with a collocation strategy to balance the residual energy over a portion of the oscillation period. By introducing an appropriate trial function, the nonlinear differential equation is reduced to an algebraic form, allowing direct determination of the frequency–amplitude relationship. Previous studies have demonstrated that EBM is applicable to both weakly and strongly nonlinear systems and that even its lowest-order approximations can yield highly accurate results over a wide range of oscillation amplitudes. These characteristics make EBM particularly attractive for engineering applications involving strong nonlinearities, motivating its use in the present study.

2. Basic idea of the Energy balance method

The analysis begins with the Duffing equation [14], which serves as a fundamental model for characterizing the non-linear dynamic response of structures subjected to extreme loading conditions, such as severe earthquakes and high winds. Unlike linear models that assume constant stiffness, the Duffing equation incorporates geometric and material non-linearities—including the hardening or softening of structural elements—that arise when buildings, bridges, and tall towers undergo large deformations. This capability facilitates the accurate prediction of critical

phenomena, such as resonance, frequency shifts, and internal energy dissipation, which are essential for assessing structural integrity and preventing collapse during seismic events. The fundamental principles of the Variational Iteration Method (VIM) are demonstrated using the following differential equation [14]:

$$u'' + \omega_0^2 u + \varepsilon f(u) = 0 \tag{1}$$

With initial conditions:

$$u(0) = A, \qquad u'(0) = 0 \tag{2}$$

Where f is a nonlinear function of u'', u', and u. In the present preliminary analysis, the simplest case is considered, where f depends upon only the function of u. If f is taken as u^3, then Eqs. (1) and (2) take the following form:

$$u'' + u + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{3}$$

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 \right\} d\tau \quad , \tag{4}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 = \frac{1}{2}A^2 + \frac{1}{4}\varepsilon A^4 \tag{5}$$

Or:

$$\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 - \frac{1}{2}A^2 - \frac{1}{4}\varepsilon A^4 = 0 \tag{6}$$

In Eq. (5) and Eq. (6) the kinetic energy (E) and potential energy (T) can be respectively expressed as: $u'^2/2, u^2/2 + \varepsilon u^4/4$ throughout the oscillation, it holds $H = E + T$ constant. The following trial function is employed to determine the frequency ω.

$$u = A\cos(\omega t) \tag{7}$$

Substituting Eq. (7) into Eq. (6) yields the following residual equation:

$$R(t) = \omega^2 \sin^2 \omega t + \cos^2 \omega t + \frac{1}{2}\varepsilon A^2 \cos^4 \omega t - 1 - \frac{1}{2}\varepsilon A^2 \tag{8}$$

If the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (7) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocate at $\omega t = \pi/4$ to find the constant ω:

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2} \tag{9}$$

The cause for opting $\theta = \pi/4$ is: It is known that $\omega = \frac{d\theta}{dt} \rightarrow \omega \cong \frac{\theta}{t} \rightarrow \theta \cong \omega t$. In Fig. 1, the vertical axis represents the potential energy. When $\theta = \pi/2$, the total energy is entirely converted into potential energy, whereas when $\theta = 0$, the energy is purely kinetic. The governing equation is established at the point where the potential and kinetic energies are equal, resulting in a balanced energy condition. This point corresponds to the critical position in $\theta = \pi/4$ indicated in Fig. 1.

Various alternative techniques, such as the Least Squares Method and the Galerkin Method, can be applied to determine the constant ω.

The corresponding period can then be expressed as:

$$T = \frac{2\pi}{\sqrt{1 + \frac{3}{4}\varepsilon A^2}} \tag{10}$$

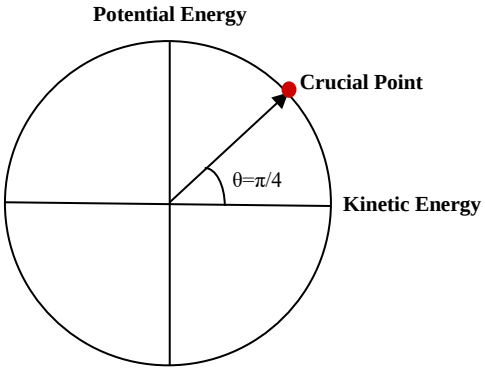

Fig. 1. The critical point at $\omega t = \pi/4$ corresponds to equal potential and kinetic energies, representing a balanced energy state in the nonlinear oscillator.

The approximate period obtained by the traditional Perturbation Method reads [18].

$$T_{pert} = 2\pi\left(1-\frac{3}{8}\varepsilon A^2\right) \tag{11}$$

Consequently, our theory, in case $\varepsilon << 1$, gives exactly the same result with those obtained by Perturbation Method. What is rather surprising about the remarkable range of validity of Eq. (10) is that the actual asymptotic period as $\varepsilon \to \infty$ is also of high accuracy.

$$\lim_{\varepsilon\to\infty}\frac{T_{ex}}{T} = \frac{2\sqrt{3/4}}{\pi}\int_0^{\pi/2}\frac{dx}{\sqrt{1-0.5\sin^2 x}} = 0.9294 \tag{12}$$

The lowest-order approximation provided by Eq. (10) remains within 7.6% of the exact frequency, independent of the amplitude εA^2.

In the absence of a small parameter in these equations, traditional perturbation methods cannot be applied directly [14].

3. Applications

To evaluate the accuracy and advantages of the Energy Balance Method, the following three examples are considered:

3.1. Example 1

The governing nondimensional nonlinear differential equation of motion for the relativistic oscillator is expressed as [15]:

$$\frac{d^2x}{dt^2}+\left[1-\left(\frac{dx}{dt}\right)^2\right]^{3/2}x=0 \tag{13}$$

Where x and t are dimensionless variables. While the relativistic oscillator equation originates in theoretical physics to describe systems with velocities approaching the speed of light, its mathematical structure offers significant utility in civil engineering, particularly within the realm of structural dynamics. In this context, the equation serves as a powerful analog for modeling the non-linear behavior of flexible structures—such as long-span bridges and tall buildings—subjected to extreme displacements or high-velocity impacts, where the restoring force exhibits saturation or stiffening rather than increasing linearly with deformation. Introducing the phase space variable (x, y), Eq. (13) can be written as follows:

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=-(1-y^2)^{3/2}x \tag{14}$$

The phase-space trajectories are determined from the solutions of the first-order ordinary differential equation:

$$\frac{dy}{dx}=\frac{-(1-y^2)^{3/2}x}{y}. \tag{15}$$

As noted by Mickens, because the physical solutions of both Eqs. (13) and (15) are real, the phase space exhibits a “strip” structure [16] that is,

$$-\infty<x<+\infty \quad and \quad -1<y<+1. \tag{16}$$

To apply the Energy Balance Method, a change of variable from y to u is introduced, such that $-\infty < u < +\infty,$, as follows:

$$y = \frac{u}{\sqrt{1+u^2}} \tag{17}$$

The corresponding second-order nonlinear differential equation for u is:

$$\frac{d^2u}{dt^2} + \frac{u}{\sqrt{1+u^2}} = 0. \tag{18}$$

The following initial conditions are applied to Eq. (19):

$$u(0) = A \quad and \quad u'(0) = 0. \tag{19}$$

Accordingly, its Hamiltonian can be expressed as:

$$H = \frac{1}{2}u'^2 + \sqrt{1+u^2} - \sqrt{1+A^2} = 0 \tag{20}$$

By selecting the trial function in Eq. (7), the following residual equation is obtained:

$$R(t) = \frac{1}{2}A^2\omega^2 \sin^2 \omega t + \sqrt{1+A^2\cos^2 \omega t} - \sqrt{1+A^2} = 0 \tag{21}$$

By collocating at $\omega t = \pi/4$, the following expression is obtained:

$$\omega = \sqrt{2}\left(\frac{2\sqrt{1+A^2} - \sqrt{4+2A^2}}{A^2}\right)^{1/2} \tag{22}$$

The corresponding period can be expressed as:

$$T = 2\sqrt{2}\pi\left(\frac{2\sqrt{1+A^2} - \sqrt{4+2A^2}}{A^2}\right)^{-1/2} \tag{23}$$

For comparison with the harmonic balance solution and the exact solution, the result reported by Beléndez and Pascual [19] is expressed as:

$$\omega_{hbm} = \left(1+\frac{3}{4}A^2\right)^{-1/4}, \quad T = \frac{2\pi}{\omega}. \tag{24}$$

$$\omega_{ex}=\frac{\pi}{2}\left(\int_0^A \frac{1+\frac{1}{2}\left(A^2-x^2\right)}{\sqrt{A^2-x^2+\frac{1}{4}\left(A^2-x^2\right)^2}}dx\right)^{-1},\quad T=\frac{2\pi}{\omega}. \tag{25}$$

It can be obtained the following approximate solution:

$$u=A\cos\left(\sqrt{2}\left(\frac{2\sqrt{1+A^2}-\sqrt{4+2A^2}}{A^2}\right)^{1/2}t\right) \tag{26}$$

Table 1 presents a comparison of the frequencies determined using the Energy Balance Method, the Harmonic Balance Method, and the exact solution. As the results indicate, the Energy Balance Method, despite its simplicity and rapid convergence, provides relatively accurate frequency estimates.

Table 1. Comparison of energy balance frequency with Harmonic Balance frequency and exact solution.

A	Energy balance frequency	Harmonic balance frequency [20]	Exact frequency
0.000	1.0000	1.0000	1.0000
0.100	0.9981	0.9981	0.9981
0.200	0.9926	0.9926	0.9926
0.300	0.9838	0.9838	0.9835
0.400	0.9721	0.9721	0.9712
0.500	0.9581	0.9579	0.9560
1.000	0.8706	0.8694	0.8513
5.000	0.4775	0.4744	0.3016
10.00	0.3411	0.3387	0.1555
50.00	0.1531	0.1519	0.0314
100.0	0.1082	0.1075	0.0157
500.0	0.0484	0.0481	0.0031
1000	0.0342	0.0340	0.0016

Figures 2 and 3 show the displacement time histories obtained from the Energy Balance and Harmonic Balance methods for amplitudes A=1 and A=100, respectively. The results demonstrate good agreement between the two methods across both amplitude ranges.

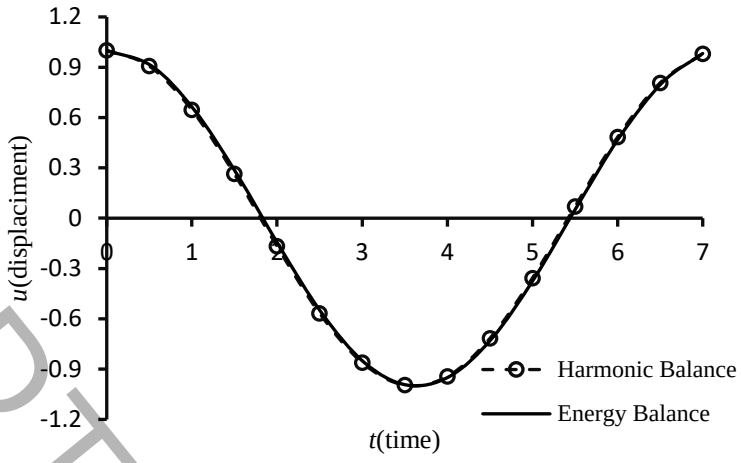

Fig. 2. Comparison of the Energy balance solution with the harmonic balance solution (A=1).

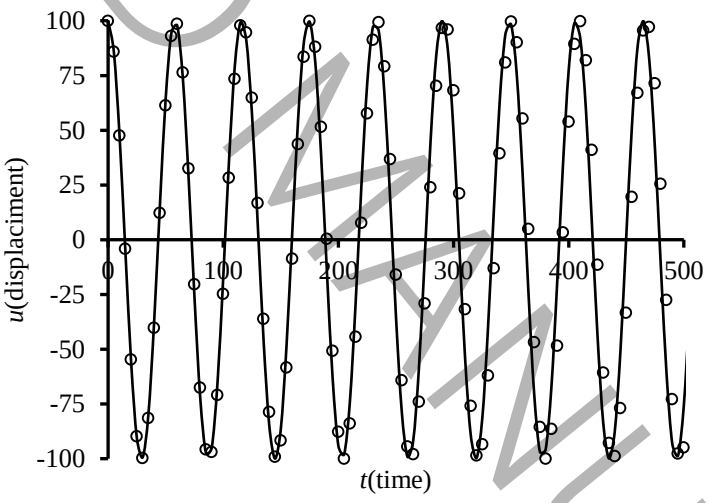

Fig. 3. Comparison of the Energy balance solution with the harmonic balance solution (A=100).

Figures 4 and 5 provide a comparison of the displacement variations over time for A=100 in two time intervals, $0 < t < 60$ s and $640 < t < 700$ s, using both the Energy Balance and Harmonic Balance methods. The results indicate that, as time progresses, the differences between the two methods become more pronounced at high amplitudes, although the predicted frequencies remain in close agreement.

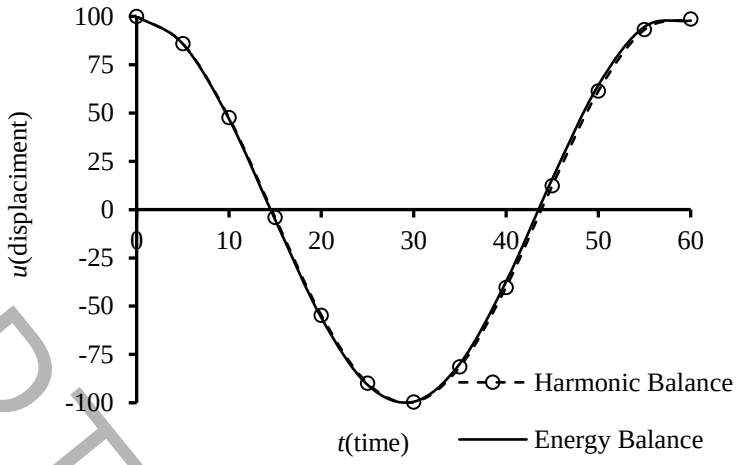

Fig. 4. Comparison of the Energy Balance solution with the Harmonic Balance solution (A=100, 0<*t*<60 s).

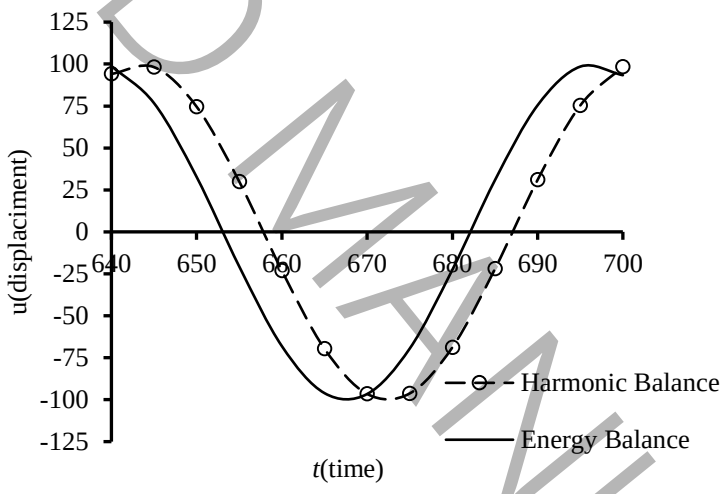

Fig. 5. Comparison of the Energy Balance solution with the Harmonic Balance solution (A=100, 640<*t*<700 s).

3.2. Example 2

The Duffing equation has been widely used to assess the effectiveness of perturbation techniques; herein, the Duffing equation with fifth-order nonlinearity is also employed to illustrate the general evaluation procedure of the proposed method [3].

$$u'' + u + \varepsilon u^5 = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{27}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{6}\varepsilon u^6 = \frac{1}{2}A^2 + \frac{1}{6}\varepsilon A^6 \tag{28}$$

Or:

$$\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{6}\varepsilon u^6 - \frac{1}{2}A^2 - \frac{1}{6}\varepsilon A^6 = 0 \tag{29}$$

By collocating at $\omega t = \pi/4$, the following expression is obtained:

$$R(t) = \frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}A^2 \cos^2 \omega t + \frac{1}{6}A^6 \cos^6 \omega t - \frac{1}{2}A^2 - \frac{1}{6}A^6 = 0 \tag{30}$$

By collocating at $\omega t = \pi/4$, the following expression is obtained:

$$\omega = \sqrt{1 + \frac{7}{12}\varepsilon A^4}, \quad T_{ebm} = 2\pi\left(1 + \frac{7}{12}\varepsilon A^4\right)^{-1/2} \tag{31}$$

The periods obtained using the Variational Iteration Method (VIM) [3] and the perturbation method [18] are given by:

$$T_{vim} = 2\pi\left(1 + \frac{5}{8}\varepsilon A^4\right)^{-1/2}, \qquad T_{pert} = 2\pi(1 - \frac{5}{16}\varepsilon A^4) \tag{32}$$

The exact solution can be obtained as follows:

$$T_{ex} = \frac{4}{\sqrt{1 + \frac{1}{3}\varepsilon A^4}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 + k\cos^2 x + k\cos^4 x}}, \qquad k = \frac{1}{3}\varepsilon A^4 \Big/ (1 + \frac{1}{3}\varepsilon A^4). \tag{33}$$

It should be noted that the perturbation formula T_{pert} in Eq. (32) is valid only for small values of the parameter ε, whereas T_{ebm} in Eq. (23) and T_{vim} remain valid for both small and large values of $\varepsilon A^4 \to \infty$, including cases of high amplitude. When comparing the periods obtained by different

methods, the T_{ebm} can be determined more easily in the Energy Balance Method due to the simplicity of the approach, compared to the other methods. The corresponding expressions are as follows:

$$\lim_{\varepsilon A^4 \to \infty} \frac{T_{ex}}{T} = \frac{2\sqrt{21/12}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1+\cos^2 x + \cos^4 x}} = \frac{2\sqrt{21/12}}{\pi} \times 1.14811 = 0.9669 \tag{34}$$

Therefore, for any value of ε, it can be easily proved that $0 \le \left|\left(T_{ex} - T_{ebm}\right)\right| / T_{ex} \le 0.033$ so the approximate solution obtained by the proposed method is uniformly valid for any value of ε.

Tables 2 and 3 present, respectively, a comparison of the energy balance frequency with the variational iteration method, the perturbation method, and the exact solution for small and large values of ε. As the numerical frequency values indicate, across both small and large amplitudes, the Energy Balance Method (EBM) exhibits smaller errors compared to the other two methods and provides results that are closer to the exact solution.

Table 2. Comparison of the energy balance frequency with the variational iteration method, perturbation method, and exact solution for small $\varepsilon = 0.01$.

A	Perturbation frequency	Variational iteration frequency	Energy balance frequency	Exact frequency
0	1	1	1	1
0.01	1	1	1	1
0.05	1.00000002	1.000000020	1.000000018	1.000000020
0.1	1.000000313	1.000000312	1.000000292	1.000000312
0.2	1.000005	1.00000500	1.000004667	1.000004999
0.3	1.000025313	1.000025312	1.000023625	1.000025312
0.4	1.000080006	1.000079997	1.000074664	1.000079996
0.5	1.000195351	1.000195293	1.000182275	1.000195285
1	1.003134796	1.003120132	1.002912426	1.003118037
5	-1.049180328	2.215005643	2.155419526	2.139018936
10	-	7.968688725	7.702813339	7.542516141

Figure 6 illustrates the variation of system frequency with respect to amplitude for a fixed value of ε in the Duffing equation, as determined by four different approaches: the perturbation method (PM), the variational iteration method (VIM), the energy balance method (EBM), and the exact solution. As observed, the error associated with the perturbation method increases with amplitude, whereas the EBM maintains a consistently low error across the entire amplitude range.

Table 3. Comparison of the energy balance frequency with the variational iteration method, perturbation method, and exact solution for large $\varepsilon = 10$.

A	Perturbation frequency	Variational iteration frequency	Energy balance frequency	Exact frequency
0	1	1	1	1
0.01	1.000000031	1.000000031	1.000000029	1.000000031
0.05	1.000019532	1.000019531	1.000018229	1.000019531
0.1	1.000312598	1.000312451	1.000291624	1.000312430
0.2	1.005025126	1.004987562	1.004655828	1.004982229
0.3	1.025969862	1.025	1.023352334	1.024871567
0.4	1.086956522	1.077032961	1.072069649	1.075933384
0.5	1.242718447	1.179247642	1.168153814	1.174282469
1	-	2.692582404	2.614064524	2.583642613
5	-	62.50799949	60.38901666	59.05187342
10	-	250.002	241.525016	236.1720725

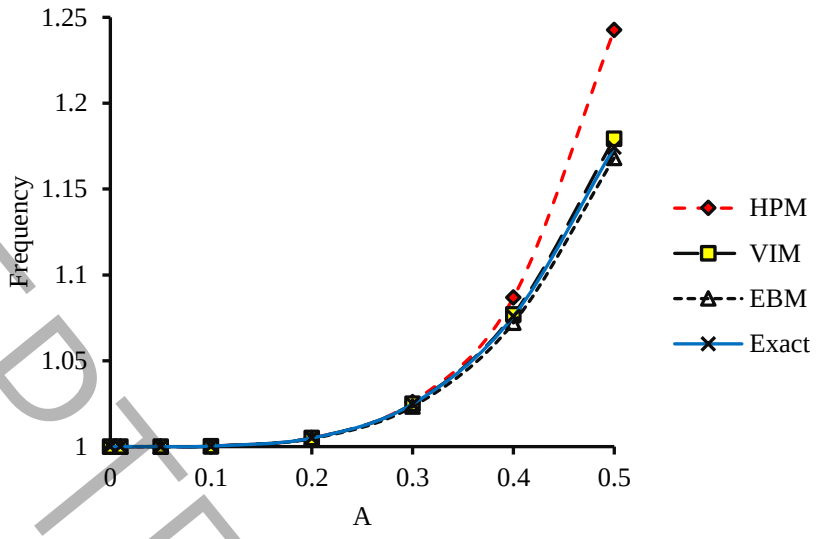

Fig. 6. Comparison of the energy balance frequency with other solution methods ($\varepsilon = 10$).

3.3. Example 3

Now consider another example [18]:

$$u'' + \frac{u}{1+\varepsilon u^2} = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{35}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{2\varepsilon}\ln(1+\varepsilon u^2) = \frac{1}{2\varepsilon}\ln(1+\varepsilon A^2) \tag{36}$$

By selecting the trial function in Eq. (7), the following residual equation is obtained:

$$R(t) = \frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2\varepsilon}\ln(1+\varepsilon A^2 \cos^2 \omega t) - \frac{1}{2\varepsilon}\ln(1+\varepsilon A^2) = 0 \tag{37}$$

By collocating at $\omega t = \pi/4$, the following expression is obtained:

$$\omega = \sqrt{\frac{2}{\varepsilon A^2}\ln\left(\frac{1+\varepsilon A^2}{1+\varepsilon A^2/2}\right)}, \quad T = 2\pi\left(\frac{2}{\varepsilon A^2}\ln\left(\frac{1+\varepsilon A^2}{1+\varepsilon A^2/2}\right)\right)^{-1/2} \tag{38}$$

The approximate period given by Eq. (38) is compared with the exact solution:

$$T_{ex} = 4\sqrt{\varepsilon}\int_0^A \frac{dx}{\sqrt{\ln[(1+\varepsilon A^2)/(1+\varepsilon x^2)]}}\,, \tag{39}$$

Whereas the period T in Eq. (38) is valid not only for small parameters but also for very large parameters, including the case of large $\varepsilon A^4 \to \infty$, the following expression is obtained:

$$\lim_{\varepsilon A^4 \to \infty} \frac{T_{ex}}{T} = \frac{2\sqrt{2\ln 2}}{\pi A}\int_0^A \frac{dx}{\sqrt{\ln[(1+\varepsilon A^2)/(1+\varepsilon x^2)]}} = 0.9394 \tag{40}$$

Therefore, for any values of ε, it can be easily proved that the maximal relative error is less than 6.06% on the whole solution domain ($0 < \varepsilon < \infty$).

4. Conclusions

This study presents a rigorous analytical investigation of nonlinear oscillators, including relativistic and Duffing-type systems, using He's Energy Balance Method (EBM). Accurately predicting the frequency–amplitude relationship of strongly nonlinear systems remains a critical challenge in engineering dynamics, particularly when conventional perturbation and linearization techniques are invalid due to the absence of small parameters. The results demonstrate that EBM provides a reliable and efficient analytical framework for addressing this challenge.

The analysis is conducted for conservative, undamped systems undergoing periodic motion, which defines the scope of the method. Within this framework, EBM avoids restrictive perturbation assumptions and requires only a single-term trial function and one iteration to obtain accurate frequency–amplitude relations, even in strongly nonlinear regimes. Comparisons with exact numerical solutions and established analytical methods confirm that the lowest-order EBM approximations achieve excellent accuracy over the entire amplitude range with significantly reduced computational effort.

The key contribution of this work is the demonstration that a minimal and computationally inexpensive formulation can yield highly accurate results for a wide class of nonlinear oscillators.

These findings establish the Energy Balance Method as a robust, simple, and effective analytical tool for nonlinear vibration analysis. Future work will focus on extending the method to damped, forced, and multi-degree-of-freedom systems to further enhance its applicability to practical engineering problems.

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