$$\hat{J}_c = -2f(x)(x+\rho_1 f(x)+\rho_1 y+\rho_2 x)+(\rho_1 f(x)+\rho_1 y+\rho_2 x)^2$$

$$J^{nc} = -2f(x)(x+\rho f(x)+\rho y)+(\rho f(x)+\rho y)^2$$

or

$$\hat{J}^c = -2f(x)(x+\rho_1 f(x)+\rho_1 y+\rho_2 x)+\rho_1^2 f^2(x)+\rho_1^2 y^2+\rho_2^2 x^2+2\rho_1^2 f(x)y+2\rho_1\rho_2 f(x)x+2\rho_1\rho_2 yx$$

$$= -2f(x)(x+\rho_1 f(x)+\rho_1 y+\rho_2 x)+\rho_1^2 f^2(x)+2\rho_1^2 f(x)y+2\rho_1\rho_2 f(x)x+(\rho_1^2 y^2+\rho_2^2 x^2+2\rho_1\rho_2 yx)$$

$$= 2(\rho_1\rho_2-1-\rho_2)f(x)x+2(\rho_1^2-\rho_1)f(x)y+2(\rho_1^2-\rho_1)f^2 6x)+(\rho_1^2 y^2+\rho_2^2 x^2+2\rho_1\rho_2 yx)$$

$$y_c = (\rho_1\rho_2-1-\rho_2)x f(x)+(\rho_1^2-\rho_1)y f(x)+(\rho_1^2-\rho_1)f^2(x)+(\rho_1^2 y^2+\rho_2^2 x^2+2\rho_1\rho_2 yx)$$

$$\hat{y}_{nc} = -2f(x)(x+pf(x)+\rho y)+\rho^2 f^2(x)+\rho^2 y^2+2p^2 f(x)y$$

$$= 2(\rho^2-\rho)f^2(x)+2(\rho^2-\rho)y f(x)-2xf(x)+\rho^2 y^2$$

$$\frac{1}{2}\hat{y}_c = -xf(x)+(\rho^2-\rho)yf(x)+(\rho^2-\rho)f^2(x)+\rho^2 y^2$$

As far as f is concerned the two functions J_c and J_{nc} have the form

$$\hat{j} = \alpha_1 xf(x)+\alpha_2 yf()+\alpha_3 f^2(x)$$

$$I(f,y) = \int\int(\alpha_1 xf(x)+\alpha_2 yf(x)+\alpha_3 f^2(x))\,p(x,y)dydx$$

$$= \int(\alpha_1 xf(x)+\alpha_3 f^2(x))\int P(x,y)\,dy\}\,dx+H$$

$$+\int \alpha_2 f(x)\{\int y\,P(x,y)\,dy\}\,dx$$

$$\int(\alpha_1 x f(x)\pi(x)+\alpha_3 f^2(x)\pi(x)\}\,dx+$$

$$\int \alpha_2 f(x)\,\omega(x)\,dx+H$$

Hence the Euter equ. becomes

$$2\alpha_3 f(x)\pi(x)+\alpha_1 x\pi(x)+\alpha_2\omega(x)=0$$

or

$$f(x) = -\frac{\alpha_1}{2\alpha_3}x-\frac{\alpha_2}{2\alpha_3}\frac{\omega(x)}{\pi(x)}$$

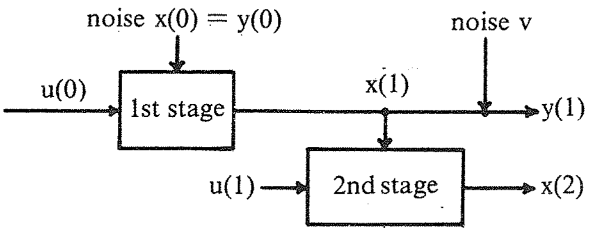
$$\omega(x)=\int y\,p(x,y)\,dy \qquad \pi(x)=\int p(x,y)\,dy$$

Now, since x and y are assumed independent, then $p(x,y)=P_1(x)\,P_2(y)$ and hence

$$\frac{\omega(x)}{\pi(x)} = \text{constant}$$

We thus arrive at the conclusion that if g is linear then an optimal linear solution for f exists for both the classical and the nonclassical cases.

Let us now aproach the problem from a different angle i.e. assume that f is linear. We shall see that in this case the function g_{nc} for the non-classical case is not necessarily linear. The process is shown schematically in the diagram.



Schematic Diagram of The Process

First stage

As we have assumed that f is linear then at this stage for both the classical and the non-classical cases we have

$$u(0) = \alpha\,y(0)$$

Second stage

Let us first consider the decision process in the classical case.

$$x(2) = x(1) - u(1)$$
$$= x(0) + u(0) - u(1)$$
$$= y(0) + \alpha\,y(0) - u(1)$$

Hence, by substituting for $\beta = 1+\alpha$, the input u(1) at this stage is the minimzer of the function

$$\text{Min}\ U(1)\ \{\alpha^2 y^2(0) + \{\beta\,y(0) - u(1)\}^2$$

where the solution is $u(1) = \beta y(0)$ which is linear in y(0).

Now let us consider the non-classical case.

$$x(2) = x(1) - u(1)$$

However in the non- classical case x(1) is not known at this stage. We can only measure y(1), and therefore x(2) must be written as

$$x(2) = y(1) - v - u(1)$$

u(1) is chosen by the minimiser of the function

$$\text{Min}\ U(1)\ \{\alpha^2 y^2(0) + \{y(1) - v - u(1)\}^2$$

Solution is given by the minimiser of $\{y(1) - v - u(1)\}^2$ which is not necessarily linear. Hence, for the non-classical information structure, the optimal decision requence is not necessarily linear.

3. Conclusion

In the paper Witzenhausen's counter example has been studied. It has been shown, using a simpler proof, that non-linear solutions can arise if decision making units are provided with restricted amount of information.

References

1. Witzenhausen, H.S. «A counter example in stochastic optimum control», SIAM, J. Cont., 6, 1968.

2. Kwakernaak, H. ε Sivan, R. «Linear optimal control systems», Wiley-interscience, 1972.

derivation of the linear control law (3) a further fundamental assumption was made. The assumption was that the information pattern is classical, i.e. that the controller has access to all the information about the system. Specifically, it was assumed that the controller has access to all the past measurements of the output of the system. In the following it is shown, through a counter example introduced by Witzenhausen: (1969) that if the controller does not have access to all the past measurements of the system, the problem does not, in general, admit a linear solution.

2. The counter example

The counter example concerns a two-stage (N=2) single variable system (n=1 , m=1 , l=1) described by the equation

$$x(1) = x(0) + u(0)$$
$$x(2) = x(1) - u(1) \quad \ldots (4)$$

Where x(0) is a random variable. The system outputs are

$$y(0) = x(0)$$
$$\text{and} \quad y(1) = x(1) + v \quad \ldots (5)$$

where v is noise. The problem is to optimise the performance of the system with respect tot he performance measure.

$$J = k^2u^2(0) + x^2(2) \quad \ldots (6)$$

For the Classical case one has admissible control laws.

$$u(0) = \gamma_1(y(0)) \text{ 1st stage}$$
$$\text{and} \quad u^{nc}(1) = \gamma_2^{nc} \ (y(1) , y(0)) \text{ 2nd stage} \quad \ldots (7)$$

i.e. at the second stage the controller has access to the past measurement, namely, y(0). For the non-classical case one has the admissible control laws.

$$u(0) = \gamma_1 \ (y(0)) \text{ 1st stage}$$
$$u_{nc}(1) = \gamma_{2nc} \ (y(1)) \text{ 2nd stage} \quad \ldots (8)$$

Note that now the second stage the controller only has access to the current measurement of the output, i.e. y(1).

The stochastic optimal control prolbem can be stated as

$$\text{Min} \quad E(J)$$
$$\text{S.t.} \quad x(1) = x(0) + u(0)$$
$$x(2) = x(1) - u(1)$$
$$y(0) = x(0)$$
$$y(1) = x(1) + v$$

where E(J) denotes the expectation value of J. Now if the constraints are incorporated into the performance measure the above problem becomes

$$\text{Min} \quad E\{ k^2u^2(0) + (x(0) + u(0) - u(1))^2 \}$$

Let us call $J^c = k^2u^2(0) + (x(0) + u(0) - u^c(1))^2$

and $J^{nc} = k^2u^2(0) + (x(0) + u(0) - u^{nc}(1))^2$

where J^c and J^{nc} denote the performance measure in classical and non-classical cases, respectively. Thus for the classical case one has the admissible $u^c(1)$ given by equations (7) and for the non-classical one has the admissible $u^{nc}(1)$ given by equation (8). Using equations (7) and (8) we have.

$$J^c = k^2\gamma_1^2 (y(0)) + [x(0) + \gamma_1 (y(0)) - \gamma_2 \ (y(1), y(0)]^2$$

and

$$J^{nc} = k^2 \gamma_1^2 (y(0)) + [x(0) + \gamma_1 (y(0)) - \gamma_2^{nc} (y(1))]^2$$

Now if the transtormation

$$f = \gamma_1 \ (y(0) + y(0)$$
$$\text{and} \quad g^c = \gamma_2^c \ (y(1) , y(0))$$
$$g^{nc} = \gamma_2^{nc} \ (y(1))$$

is made then we have (note y(0) = x(0))

$$J^c = k^2 \ (f(x(0) - x(0))^2 + (f(x(0)) - g^c y(1), x(0))^2$$
$$J^{nc} = k^2 \ (f(x(0) - x(0))^2 + (f(x(0)) - g^{nc}(y(1))^2$$

And, finally, substituting for

$$Y(1) = x(1) + v$$
$$= x(0) + u(0) + v$$
$$x(0) + \gamma_1 (y(0)) + v$$
$$= f (y(0)) + v$$
$$= f(x(0)) + v$$

We have

$$J^c = k^2 (f(x(0)) - x(0))^2 + [f(x(0)) - g^c((f(x(0)) + v), x(0))]^2$$
$$J^{nc} = k^2 (f(x(0) - x(0))^2 + [f(x(0) - g^{nc}((f(x(0)) + v))]^2$$

The Stochastic optimal control problem is reduced to the minimization of the expectation value of J^c for the classical case and J^{nc} for the non-classical case, i.e. we have for the classical case

$$\text{Min} \int_{\infty}^{\infty} \int_{\infty}^{\infty} J^c (x(0) , v , f, g) P (x(0) , v) dx (0) dv$$

and for the non-classical case

$$\text{Min} \int_{\infty}^{\infty} \int_{\infty}^{\infty} J^{nc} (x(0) , v , f , g) P (x(0) , v) dx (0) dv$$

where P(x(0) , v) is the joint probability distribution for the random variables x(0) and v.

Let us put $x(0) = x$
$v = y$

then the two performance indices become

$$J^c = k^2 (f(x) - x)^2 + [f(x) - g^c ((f(x) + y) , x)]^2$$
$$J^{nc} = K^2 (f(x) - x)^2 + [f(x) - g^{nc} (f(x) + y]^2$$
$$J^c = k^2 (f^2(x) + x^2 - 2f(x)x) + f^2(x) + g^{2c} - 2f(x)g^{nc}$$
$$J^{nc} = k^2 (f^2(x) + x^2 - 2f(x)x) + f^2(x) + g^{2nc} - 2f(x) g^{nc}$$

which on rearranging gives.

$$J^c = (k^2 + 1) f^2(x) - 2f (x) (x + g^c) + g^{2c} + k^2x^2$$
$$J^{nc} = (k^2 + 1) f^2(x) - 2f (x) (x + g^{nc}) + g^{2nc} + k^2x^2$$

Now assume g linear

$$g^c = \rho_1 (f(x) + y) + \rho_2 x$$
$$g^{nc} = \rho (f(x) + y)$$

hence defining

$$\hat{J}^c = J^c - (k^2 + 1) f^2(x) - k^2 x^2$$
$$J^{nc} = J^{nc} - (k^2 + 1) f^2(x) - k^2x^2$$

we have

CLASSICAL VS. NON-CLASSICAL INFORMATION IN STOCHASTIC OPTIMAL CONTROL

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Abstract

The paper is concerned with a comparison of the optimal stochastic control problem (the Linear Quadratic case) for the cases where the information structure is centralised (the classical pattern) and where it is decentralised (the non-classical pattern). It is shown, using an example introduced by Witzenhausen, that unlike the decentralised case, one has non-linear, and still undetermined, soulutions. The main contribution of the report is to provide an alternative, and simpler, proof of the foregoing result.

1. Introduction

Consider the following stochastic process

$\underline{x}(k+1) = A(k)\,\underline{x}(k) + B(k)\,\underline{u}(k) + \underline{v}(k)$

$k = 0, 1, 2, \ldots$

$\underline{x}(0) = \underline{x}^0$

where $\underline{u} \varepsilon R^m$, $\underline{x} \varepsilon R^n$ and $\underline{v} \varepsilon R^n$. The equation describes a linear discretetime system corrupted by the noise vector $\underline{v}(k)$. The output of the system is also corrupted by the noise vector $\underline{w}(k) \varepsilon R^{\ell}$ and is given by

$\underline{y}(k) = C(k)\,\underline{x}(k) + W(k)$ ... (2)

Where $\underline{y}(k) \varepsilon R^{\ell}$ The problem of optimising the system described by (1) and (2) with respect to the expectation value of the performance measure

$$J = \sum_{K=0}^{N} \psi(\underline{Y}(k), \underline{U}(k))$$

is called the linear optimal stochastic control problem. If the performance function is quadratic, then the problem is called the Linear-Quadratic (L.Q.) problem.

Now it is well-known that the solution to this problem is given by a linear control law of the form

$\underline{U}(k) = K(k)\,\underline{X}(k)$ ... (3)

where K is an (mxn) matrix (see, for example. Kwakernaak & Sivan, (1972). However, the linearity of the solution is not simply brought about by the L.Q. form of the problem. A fact which has until recently not been emphasised is that in the