the well known result

$$\int_{-\infty}^{\infty} d^N y \exp(-\sum_{i,j} K_{ij} y_i y_j) = [\pi^N/\det(K)]^{1/2}$$

which holds for any positive definite matrix, leads to

$$\exp(-\sum_{ij} K_{ij} u_i u_j) = [\pi^N \det(K)]^{-1/2} \times$$

$$\int_{-\infty}^{\infty} d^N x \exp(-\sum_{ij} (K^{-1})_{ij} x_i x_j - 2i \sum_{j=1} x_j u_j)$$

Thus for $-1<S<0$ equation (10) reduces to

$$Z_N = \int_{-\infty}^{\infty} d^N u \left(\prod_{j=1}^{N} (2S+1)^{-1} \frac{-\sin 2\pi S}{\cosh 2\pi u_j - \cos 2\pi S} \right)$$

$$\times \exp(-\sum_{i,j} K_{ij} u_i u_j)$$

This is just equation (9) in zero field.

Finally we remark the limiting case of $S=-\frac{1}{2}$ for which the model takes a particularly simple from, i.e

$$Z_N = \left(\frac{\pi}{2}\right)^N \int_{-\infty}^{\infty} d^N u \left(\prod_{j=1}^{N} \operatorname{sech}^2 \pi u_j \right) \times$$

$$\exp(-\sum_{i,j}^{N} K_{ij} u_i u_j + iL \sum_{j=1}^{N} u_j)$$

In view of the spin independence of critical behaviour (whose domain is now extended to $S>-1$ [1]), the choice $S=-\frac{1}{2}$ may present an alternative to the usual $S=\frac{1}{2}$, since they posses identical critical behaviour. It will be of significance to see if the $S=-\frac{1}{2}$ Ising model can be solved exactly, particularly in two dimensions with a non-zero field; an important task which has not been achieved with the $S=\frac{1}{2}$ Ising model. Apart from these, it has the advantage of being a continuous spin model involving the matrix K and not its inverse (as in (10)). This may be favorable for certain considerations.

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For $-1<S<0$ we have $[S]=-1$ by definition. The set $\mathcal{S}$ is the empty set for this particular rage of spin and we are left with

$$(2S+1)\ \mathrm{tr}_i F(S_i^z) =$$

$$\int_{-\infty}^{\infty} -\frac{\sin 2\pi(S+1)}{\cosh 2\pi u - \cos 2\pi(S+1)} F(iu)\, du \qquad (7)$$

In particular we have [2] for the moments $\mathrm{tr}_i(S_i^z)^r$; $r=0,1,2,\ldots$;

$$(2S+1)\ \mathrm{tr}_i(S_i^z)^r =$$

$$\int_{-\infty}^{\infty} -\frac{\sin 2\pi(S+1)}{\cosh 2\pi u - \cos 2\pi(S+1)} i^r u^r\, du =$$

$$\begin{cases} 0, & r=2n+1 \\ \dfrac{2}{2n+1} B_{2n+1}(S+1), & r=2n \end{cases} \qquad (8)$$

which holds for $-1<S<0$. Here $B_r(x)$ denotes the Bernoulli polynomial of degree r. The above equation, of course, holds for the standard Ising model [1]. Following the line of arguments presented in reference [1], the fact that equation (8) is formally valid also for $-1<S<0$ is sufficient to demonstrate the validity of our model for the range of spin under consideration. Hence equations (4) and (5)actually define a generalization of the Ising model for any $S>-1$. Now (7)reads

$$(2S+1)\ \mathrm{tr}_i F(S_i^z) =$$

$$\int_{-\infty}^{\infty} -\frac{\sin 2\pi S}{\cosh 2\pi u - \cos 2\pi S} F(iu)\, du$$

The partition function for the generalized Ising model thus becomes

$$Z_N = \int_{-\infty}^{\infty} d^N u \left(\prod_{i=1}^{N} (2S+1)^{-1} \frac{-\sin 2\pi S}{\cosh 2\pi u_i - \cos 2\pi S} \right) \times$$

$$\exp\left(-\sum_{i,j} K_{ij} u_i u_j + iL \sum_{j=1}^{N} u_j\right) \qquad (9)$$

for $-1<S<0$. Evidently. the interaction matrix K must be positive definite for the convergence of the multiple integral to be insured.

Equation (9) in zero field ($L=0$) is reminiscent of the generalized continuous spin model defined via

$$Z_N = [\pi^N \det(K)]^{-1/2} \int_{-\infty}^{\infty} d^N u \left(\prod_{i=1}^{N} (2S+1)^{-1} \frac{\sinh(2S+1)u_i}{\sinh u_i}\right)$$

$$\times \exp\left(-\sum_{i,j} (K^{-1})_{ij} u_i u_j\right) \qquad (10)$$

in which K is, as before, positive definite and $S>-1$. This is an alternative form for the partition function of the generalized zero-field Ising midel, obtained via the Kac-Hubbard-Stratonovich transformation [3,4,5] in the usual manner [1] (i.e as for the standard Ising model for which (10), of course holds). It is instructive to prove the equivalence of (9) in zero field and (10) for the spin range $-1<S<0$. This serves as a consistency check on the model as well. Substituting the Fourier transform formula

$$\int_{-\infty}^{\infty} \frac{-\sin 2\pi S}{\mathrm{coch}\, 2\pi u - \cos 2\pi S} e^{-2iux}\, du = \frac{\sinh (2S+1)x}{\sinh x}$$

which is only valid for $-1<S<0$, into (10) yields

$$Z_N = [\pi^N \det(K)]^{-1/2} \int_{-\infty}^{\infty} d^N u \prod_{j=1}^{N} (2S+1)^{-1}$$

$$\times \frac{-\sin 2\pi S}{\cosh 2\pi u_i - \cos 2\pi S}$$

$$\times \int_{-\infty}^{\infty} d^N x \exp\left(-\sum_{i,j} (K^{-1})_{ij} x_i x_j - 2i \sum_{j=1}^{N} x_j u_j\right)$$

In the second multiple integral, changing the variable of integration to $y_i = x_i + i\Sigma_j K_{ij} u_j$ and making use of

$S = \frac{-1}{2}$ ISING MODEL

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ABSTRACT

Following an article describing a novel generalization of the Ising model formally valid for all positive spin values, it is shown that this model also holds for $-1<S<0$. The model simplifies when $S= -\frac{1}{2}$. It is suggested, in view of the spin independence aspect of universality, that rather than the usual $S = \frac{1}{2}$ Ising model, the choice of $S= -\frac{1}{2}$. may be more favorable for certain considerations.

The standard spin S (where S takes one of the usual integral or odd - half integral values)Ising model has in general, in the usual notation, the partition function

$$Z_N = tr_1 \dots tr_N \exp \left(\sum_{i,j} K_{ij} S^z_i S^z_j + L \sum_{i=1}^{N} S^z_i \right) \tag{1}$$

where the partial trace tri is defined by

$$(2S+1)\, tr_i\, F(S_i^z) = \sum_{S_i^z \in \mathcal{S}} F(S_i^z) \tag{2}$$

in which F is any function of the spin components S_i^z and $\mathcal{S}$ is the set

$$\mathcal{S} = \{-S, -S+1, \dots , S-1, S\} \tag{3}$$

The normalising factor $(2S+1)$ is incorporated so that $tr_i 1 = 1$.

In an article [1] we presented a generalization of the Ising model which formally permits any arbitrary positive value for the spin S. This generalized model was defined for all $S \geq 0$ via replacing (2) and (3) respectively by

$$(2S+1)\, tr_i\, F(S_i^z) = \sum_{S_i^z \in \mathcal{S}} F(S_i^z) + \int_{-\infty}^{\infty} W_{S-[S]}(u)\, F(iu)\, du \tag{4}$$

$$\mathcal{S} = \{-S, -S+1, \dots , -S+[S], S-[S], \dots , S-1, S\} \tag{5}$$

where $[S]$ is the greatest integer $\leq S$, and the spin distribution function $W_{S-[S]}(u)$ is given in terms of the quantity $0 \leq S-[S] < 1$ by

$$W_{S-[S]}(u) = -\frac{\sin 2\pi\,(S-[S])}{\cosh 2\pi u - \cos 2\pi(S-[S])} \tag{6}$$

It was shown that this new model reduces to the standard Ising model when $S=\frac{1}{2}, 1, \frac{3}{2}, \dots$ and remains consistent for any other positive spin value. Everything derived from it is valid for any $S \geq 0$ [1]. It is the purpose of this paper to point out that the range of validity of this model also includes $-1<S<0$. Some consequences will also be discussed.