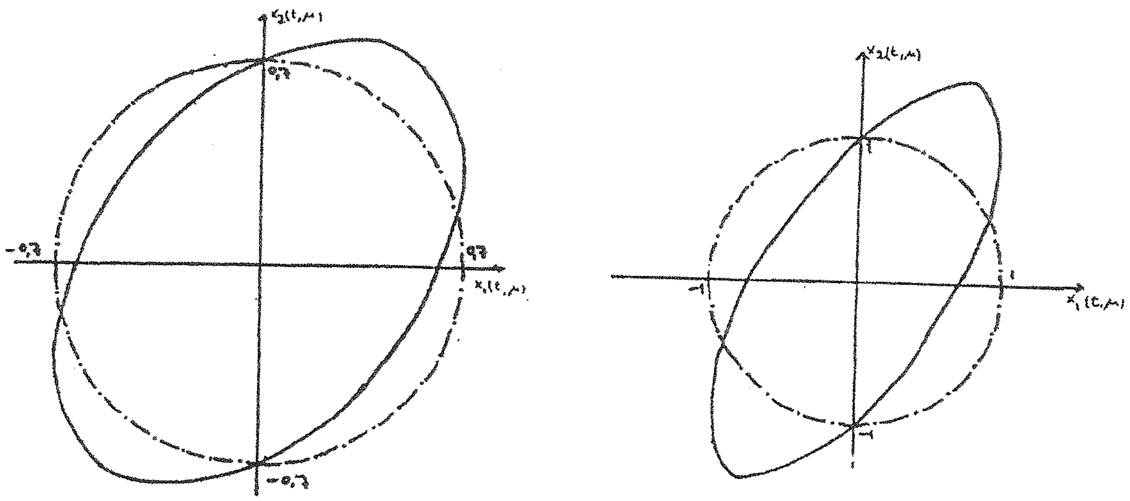

Fig. 2

4. Conclusion

We have shown that in the Hopf bifurcation theorem for the differential equation:

$x'(t, \mu) = f(x(t, \mu), \mu)$ at least one periodic solution and at least one limit cycle exists given the condition that $\sigma(0)<0$ and $\mu>0$ and μ is small enough.

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and that means for our transformed differential equation:

$$f(\xi,\mu) = (N(\mu))^{-1} \tilde{f}(N(\mu)\xi,\mu) \tag{3.19}$$

with $\alpha(\mu) = \mu/2$ we will have:

$$f(\xi,\mu)=(1/\omega(\mu))\begin{pmatrix} 0 & -\omega(\mu) \\ 1 & -\alpha(\mu) \end{pmatrix}\tilde{f}\left(\begin{pmatrix} -\alpha(\mu) & \omega(\mu) \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \mu\right)$$

$$= \begin{pmatrix} (\mu/2)\xi_1 + \omega(\mu)\xi_2 - [-(\mu/2)\xi_1 + \omega(\mu)\xi_2]^2 \xi_1 \\ -\omega(\mu)\xi_1 + (\mu/2)\xi_1 - (1/\omega(\mu))\{[-(\mu/2)\xi_1 + \omega(\mu)\xi_2]^3 + (\mu/2)[-(\mu/2)\xi_1 + \omega(\mu)\xi_2]^2 \xi_1\} \end{pmatrix} \tag{3.20}$$

now we can see that this differential equation has the same form as (1. 6) with $\alpha(\mu) = \mu / 2$ and $\omega(\mu) = (1 - (\mu / 2)^2)^{1/2}$, it means we can use the method described in section 2 or in problem 1 to find $r(t, \mu)$ and $T(\mu)$.

We will get:

$$r(t,\mu) = \mu^{1/2} + O(\mu^{3/2}) \tag{3.21}$$

$$T(\mu) = 2\pi + O(\mu^{3/2}) \tag{3.22}$$

For the solution $x(t, \mu)$ of the transformed differential equation (3.20) we get:

$$\begin{pmatrix} x_1(t,\mu) \\ x_2(t,\mu) \end{pmatrix} = r(t,\mu)\begin{pmatrix} \cos\varphi(t,\mu) \\ \sin\varphi(t,\mu) \end{pmatrix} = \mu^{1/2}\begin{pmatrix} \cos\varphi(t,\mu) \\ \sin\varphi(t,\mu) \end{pmatrix} + O(\mu^{3/2}) \tag{3.23}$$

and for our main differential equation (3.8) we get from (3.16):

$$\tilde{x}(t,\mu) = N(\mu)\,x(t,\mu) = \mu^{1/2}\begin{pmatrix} \sin\varphi(t,\mu) \\ -\cos\varphi(t,\mu) \end{pmatrix} + O(\mu^{3/2}) \tag{3.24}$$

Now I am going to draw two diagramms for two different values for μ (0. 50 and 1.0) shown in Fig. 2.

In each figure we can see two closed curves, one of them (drawn with -----) is determined by the formula from (3.21) and the other one (drawn with -----) is determined by programming using the Runge-kutta method. As we can see this curve changes when μ changes.

According to Runge-kutta method:

μ=0.50 $T(\mu)$ = 6.489

Angle	$x_1(t, \mu)$	$x_2(t, \mu)$
0.000	0.63246	-0.00000
30.000	0.70500	0.40703
60.000	0.44562	0.77183
90.000	-0.00092	0.70664
120.000	-0.28911	0.50075
150.000	-0.48105	0.27773
180.000	-0.63246	0.00000
210.000	-0.70500	-0.40703
240. 000	-0.44562	-0. 77183
270. 000	0. 00092	-0. 70664
300. 000	0.28911	-0.50075
330. 000	0. 48105	-0. 27773
360. 000	0. 63246	-0. 00000

Again:

μ=1 $T(\mu)$ = 7.255

Angle	$x_1(t, \mu)$	$x_2(t, \mu)$
0.000	0.70711	-0.00000
30.000	0.92110	0.53179
60.000	0.80690	1.39758
90.000	-0.0043	0.99957
120.000	-0.34372	0.59535
150.000	-0.53544	0.30914
180.000	-0.70710	0.00001
210.000	-0.92110	-0.53179
240.000	-0.80690	-1. 39758
270. 000	0. 00043	-0.99957
300. 000	0. 34372	-0. 59535
330. 000	0. 533544	-0. 30914
360. 000	0. 70711	-0. 00000

since $\alpha(\mu) = \mu$ and conversely $\alpha(\mu) = \alpha_1 \mu + O(\mu\, \mu^{1/2})$, we get $\alpha_1 = 1$

we know that $\omega(\mu) = 1$ and conversely $\omega(\mu)=\omega_0+\omega_1\mu+ O(\mu\, \mu^{1/2})$ and that means that $\omega_0 = 1$ and $\omega_1 = 0$.

We then find that:

$$P_0(-\alpha_1/\sigma(0))^{1/2} = (-1/-1)^{1/2} = 1 \qquad (3.4)$$

similarly we find for q_1 and s_1: $q_1=0$ and $s_1=0$

(3.5)

so we get for $r(t, \mu)$: $r(t,\mu) = \mu^{1/2} + O(\mu\, \mu^{1/2})$

(3.6)

and for $T(\mu)$ we get:

$$T(\mu) = 2\pi + O(\mu\, \mu^{1/2}) \quad \text{for } \mu > 0\ (2\pi - 6.283)$$

(3.7)

Problem 2:

We want to solve the following differential equation

$$x_1'(t) = x_2(t) - (x_1(t))^3$$

$$x_2'(t) = -x_1(t) + \mu x_2(t) - (x_1(t))^2 x_2(t)$$

$$\tilde{f}_1(\xi, \mu) = \xi_2 - \xi_1^3$$

$$\tilde{f}_2(\xi, \mu) = -\xi_1 + \mu \xi_2 - \xi_1^2 \xi_2 \qquad (3.8)$$

as we can see there is no similarity with (1.6) so we should make a transformation We therefore have to use the jacobian matrix

$$(\delta\tilde{f}/\delta\xi(\xi,\mu)) = \begin{pmatrix} -3\xi_1^2 & 1 \\ -1-2\xi_1\xi_2 & \mu-\xi_1^2 \end{pmatrix} \qquad (3.9)$$

$$A(\mu) = (\delta\tilde{f}/\delta\xi(\varnothing,\mu)) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \qquad (3.10)$$

we shall now determine the eigenvalue of the matrix $A(\mu)$:

$$|A(\mu) - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & \mu-\lambda \end{vmatrix} = \lambda^2 - \mu\lambda + 1\ (=0)$$

(3.11)

$$\lambda_1(\mu) = (\mu/2) + ((\mu/2)^2 - 1)^{1/2} \text{ and } \lambda_2(\mu) = (\mu/2) - ((\mu/2)^2 - 1)^{\frac{1}{2}}$$

for $|\mu| < 2$ i. e. $\mu \in]-2, +2[$ we have: $(\mu/2)^2 - 1 < 0]$

(3.12)

this means that $\lambda_1(\mu)$ and $\lambda_2(\mu)$ are for $\mu\in [-2+2]$ complex.

We consider that for j = 1,2: $\mathrm{Re}(\lambda_j(\mu)) = \mu/2 =: \alpha(\mu)$ and

$(\mathrm{Re}(\lambda_j(.)))'(0) = 1/2 > 0$

further we consider for j = 1, 2: $\mathrm{Im}(\lambda_j(\mu)) = (1 - (\mu/2))^{1/2} =: \omega(\mu)$.

With using a suitable regular martix $N(\mu)$ we can calculate the matrix $A(\mu)$ in real-numbers to get the following form:

$$(N(\mu))^{-1} A(\mu) N(\mu) = \begin{pmatrix} \alpha(\mu) & \omega(\mu) \\ -\omega(\mu) & \alpha(\mu) \end{pmatrix} \qquad (3.13)$$

we can define

$$N(\mu) := \begin{pmatrix} -\alpha(\mu) & \omega(\mu) \\ -1 & 0 \end{pmatrix} \qquad (3.14)$$

as we know from linear algebra that we can determine $(N(\mu))^{-1}$ as follows:

$$(N(\mu))^{-1} = (1/\det(N(\mu)))\,\mathrm{Adj}(N(\mu)) =$$

$$(1/\omega(\mu)) \begin{pmatrix} 0 & -\omega(\mu) \\ 1 & -\alpha(\mu) \end{pmatrix} \qquad (3.15)$$

with $N(\mu)$ from (3.14) we get:

$$\tilde{x}(t,\mu) := N(\mu)\, x(t,\mu) \qquad (3.16)$$

$$x(t,\mu) = (N(\mu))^{-1}\, \tilde{x}(t,\mu) \qquad (3.17)$$

it also means:

$$x'(t,\mu) = (N(\mu))^{-1}\, \tilde{x}'(t,\mu) = (N(\mu))^{-1}\, \tilde{f}(\tilde{x}(t,\mu),\mu) =$$

$$(N(\mu))^{-1}\, \tilde{f}(N(\mu)\, x(t,\mu),\mu) \qquad (3.18)$$

the following formula for the amplitude:

$$r(t,\mu)=p_0\mu^{1/2}+q_1\mu\cos\varphi(t,\mu)+s_1\mu\sin\varphi(t,\mu)+O(\mu\mu^{1/2}) \quad (2.7)$$

and for the period we will get the following formula:

$$T(\mu)=(2\pi/\omega_0)\{1+\mu\{-(\omega_1/\omega_0)-$$
$$[\alpha_1/(128\,\omega_0^3\sigma(0))]\{[2f^1_{12}(\varnothing,0)+f^2_{11}(\varnothing,0)+3f^2_{22}(\varnothing,0)]$$
$$[3f^2_{11}(\varnothing,\mu)+f^2_{22}(\varnothing,\mu)-2f^1_{12}(\varnothing,\mu)]$$
$$-[3f^1_{11}(\varnothing,0)+f^1_{22}(\varnothing,0)+2f^2_{12}(\varnothing,0)]$$
$$[2f^2_{12}(\varnothing,\mu)-f^1_{11}(\varnothing,\mu)-3f^1_{22}(\varnothing,\mu)]+$$
$$+10(f^2_{11}(\varnothing,\mu))^2+$$
$$+8\{[f^2_{12}(\varnothing,\mu)-(1/2)f^1_{11}(\varnothing,\mu)]^2+$$
$$+f^2_{11}(\varnothing,\mu)[(1/2)f^2_{22}(\varnothing,\mu)-f^1_{12}(\varnothing,\mu)]\}$$
$$+8\{[(1/2)f^2_{22}(\varnothing,\mu)-f^1_{12}(\varnothing,\mu)]^2\cdot$$
$$-f^1_{22}(\varnothing,\mu)[f^2_{12}(\varnothing,\mu)-(1/2)f^1_{11}(\varnothing,\mu)]\}$$
$$+10(f^1_{22}(\varnothing,\mu))^2\}\cdot$$
$$-[3\alpha_1/(48\,\omega_0\sigma(0))][f^2_{111}(\varnothing,\mu)+f^2_{122}(\varnothing,\mu)-f^1_{112}(\varnothing,\mu)-$$
$$-f^1_{222}(\varnothing,\mu))]\}\}+O(\mu^{3/2}) \quad (2.8)$$

in which ω_0, ω_1, ω_2, α_1, α_2, p_0, q_1 and s_1 are the coefficients hidden in A_2, A_3, B_2 and B_3. They can be obtained very easily.

3. Numeric Solution for Two Problems

Problem 1:

We want to solve a problem using the knowledge we have gained from the above.

We want to produce a limit cycle of the following differential equation:

$$x_1'(t)=\mu x_1(t)+x_2(t)-x_1(t)[x_1(t))^2+x_2(t))^2]$$
$$x_2'(t)=-x_1(t)+\mu x_2(t)-x_2(t)[(x_1(t))^2+(x_2(t))^2] \quad (3.1)$$

The function f has such a form:

$$f_1(\xi,\mu)=\mu\xi_1+\xi_2-\xi_1[\xi_1^2+\xi_2^2] \qquad (=\mu\xi_1+\xi_2-\xi_1 r^2)$$
$$f_2(\xi,\mu)=-\xi_1+\mu\xi_2-\xi_2[\xi_1^2+\xi_2^2] \qquad (=-\xi_1+\mu\xi_2-\xi_2 r^2) \quad (3.2)$$

if we compare (3.2) with (1.6), we will see that $\alpha(\mu)=\mu$ and $\omega(\mu)=1$.

Further, we can calculate the following derivation:

$$f^1_1(\xi_1,\xi_2,\mu)=\mu-r^2-2\xi_1^2 \Rightarrow f^1_1(0,0,\mu)=\mu$$
$$f^1_2(\xi_1,\xi_2,\mu)=1-2\xi_1\xi_2 \Rightarrow f^1_2(0,0,\mu)=1$$
$$f^2_1(\xi_1,\xi_2,\mu)=-1-2\xi_1\xi_2 \Rightarrow f^1_2(0,0,\mu)=-1$$
$$f^2_2(\xi_1,\xi_2,\mu)=\mu-r^2-2\xi_2^2 \Rightarrow f^2_2(0,0,\mu)=\mu$$
$$f^1_{11}(\xi_1,\xi_2,\mu)=-6\xi_1 \Rightarrow f^1_{11}(0,0,\mu)=0$$
$$f^1_{12}(\xi_1,\xi_2,\mu)=-2\xi_2 \Rightarrow f^1_{12}(0,0,\mu)=0$$
$$f^1_{21}(\xi_1,\xi_2,\mu)=-2\xi_2 \Rightarrow f^1_{21}(0,0,\mu)=0$$
$$f^1_{22}(\xi_1,\xi_2,\mu)=-2\xi_1 \Rightarrow f^1_{22}(0,0,\mu)=0$$
$$f^2_{11}(\xi_1,\xi_2,\mu)=-2\xi_2 \Rightarrow f^2_{11}(0,0,\mu)=0$$
$$f^2_{12}(\xi_1,\xi_2,\mu)=-2\xi_1 \Rightarrow f^2_{12}(0,0,\mu)=0$$
$$f^2_{21}(\xi_1,\xi_2,\mu)=-2\xi_1 \Rightarrow f^2_{21}(0,0,\mu)=0$$
$$f^2_{22}(\xi_1,\xi_2,\mu)=-6\xi_2 \Rightarrow f^2_{22}(0,0,\mu)=0$$
$$f^1_{111}(\xi_1,\xi_2,\mu)=-6 \Rightarrow f^1_{111}(0,0,\mu)=-6$$
$$f^1_{122}(\xi_1,\xi_2,\mu)=-2 \Rightarrow f^1_{122}(0,0,\mu)=-2$$
$$f^2_{112}(\xi_1,\xi_2,\mu)=-2 \Rightarrow f^2_{112}(0,0,\mu)=-2$$
$$f^2_{222}(\xi_1,\xi_2,\mu)=-6 \Rightarrow f^2_{222}(0,0,\mu)=-6$$
$$f^2_{111}(\xi_1,\xi_2,\mu)=0 \Rightarrow f^2_{111}(0,0,\mu)=0$$
$$f^2_{122}(\xi_1,\xi_2,\mu)=0 \Rightarrow f^2_{122}(0,0,\mu)=0$$
$$f^1_{112}(\xi_1,\xi_2,\mu)=0 \Rightarrow f^1_{112}(0,0,\mu)=0$$
$$f^1_{222}(\xi_1,\xi_2,\mu)=0 \Rightarrow f^1_{222}(0,0,\mu)=0$$

substituting the values determined above in (1.7) give us the following formula for $\sigma(0)$:

$$\sigma(0)=(1/16).0+(1/16)[-6-2-2-6]=-1<0 \quad (3.3)$$

and for $r(t,\mu)$ from (2.7) we have obtained:

$$r(t,\mu)=p_0\mu^{1/2}+q_1\mu\cos\varphi(t,\mu)+s_1\mu\sin\varphi(t,\mu)+O(\mu\mu^{1/2}) \quad (3.3a)$$

Hopf bifurcation:

1.2 Theorem of Hopf bifurcation

Under the conditions that:

f is a continuous function and f is 3 times continuous and differentiable in ξ,

$$f: D \times M \rightarrow \Re^2$$

$$(\xi, \mu) \rightarrow f(\xi, \mu) := \begin{pmatrix} f_1(\xi, \mu) \\ f_2(\xi, \mu) \end{pmatrix}$$

with $0 \in M$, $M \subseteq \Re$, M is open and $(0,0) = \varnothing \in D$, $D \subseteq \Re^2$, D is an open set and for every $\mu \in M$ we have $f(\varnothing, \mu) = \varnothing$, f has to have the following form:

$$f_1(\xi, \mu) = \alpha(\mu)\xi_1 + \omega(\mu)\xi_2 + O(|\xi|^2)$$

$$f_2(\xi, \mu) = -\omega(\mu)\xi_1 + \alpha(\mu)\xi_2 + O(|\xi|^2) \qquad (1.8)$$

in which the following functions are continuous and 3 times continuous and differentiable:

$\alpha: M \rightarrow \Re$, $\omega: M \rightarrow \Re$, $\alpha(0) = 0$, $\alpha'(0) > 0$, $\omega(0) \neq 0$

with $|\xi| := (\xi^2_1 + \xi^2_2)^{1/2}$ and

$$\sigma(0) = (1/(16\omega(0))\{f^1_{11}(\varnothing,0)[f^2_{11}(\varnothing,0) - f^1_{22}(\varnothing,0)] +$$
$$f^2_{22}(\varnothing,0)[f^2_{12}(\varnothing,0) - f^1_{22}(\varnothing,0)] +$$
$$[f^2_{12}(\varnothing,0)f^2_{11}(\varnothing,0) - f^1_{12}(\varnothing,0)\, f^1_{22}(\varnothing,0)]\} +$$
$$(1/(16)\,[f^1_{111}(\varnothing,0) + f^1_{122}(\varnothing,0) + f^2_{112}(\varnothing,0)] + f^2_{222}(\varnothing,0)] \qquad (1.9)$$

we will have:

(a) if $\sigma(0) < 0$ is then for $x'(t) = f(x(t), \mu)$, $\mu = 0$ becomes a supercritical Hopf-bifurcatio.
It means as long $\mu < 0$, $\varnothing$ is a sink and as soon as $\mu > 0$, $\varnothing$ is a source.
and there exists an orbital asymptic stable ω-limit cycle.

(b) if $\sigma(0) > 0$ is then for $x'(t) = f(x(t), \mu)$, $\mu = 0$ becomes a subcritical Hopf-bifurcation.
It means as long as $\mu > 0$, $\varnothing$ is a sink and as soon as $\mu < 0$, $\varnothing$ is a source.
and there exists an orbital asymptic unstable α-limit cycle.

We can show that the theorem of Hopf-bifurcation is not only valid for the special differential equation $x'(t) = f(x(t), \mu)$ but also for a general differential equation

$$\tilde{\tilde{x}}'(t, \tilde{\mu}) = \tilde{\tilde{f}}\,\tilde{\tilde{x}}(t, \tilde{\mu}), \tilde{\mu}).$$

2. Aproximation for Periodic Solutions

To get an approximation for periodic solutions we should concern ourselves with the differential equation $x'(t, \mu) = f(x(t, \mu), \mu)$ and when we use the polar coordinates

$$x_1(t, \mu) = r(t, \mu)\cos\varphi(t, \mu) \qquad (2.1)$$

$$x_2(t, \mu) = r(t, \mu)\sin\varphi(t, \mu) \qquad (2.2)$$

we will finally get the following formula:

$$r'(t, \mu) = \alpha(\mu)\, r(t, \mu) + (r(t, \mu))^2[A_2] + (r(t, \mu))^3[A_3] \qquad (2.3)$$

$$\varphi'(t, \mu) = -\omega(\mu) + r(t, \mu)[B_2] + (r(t, \mu)^2[B_3] \qquad (2.4)$$

A_2, A_3, B_2 and B_3 represent very long formula that I am not including here, as they contain some coefficients which have to be found.

If we try the usual way to get these coefficients, it will take long time and it will be very complicated, therefore we will use the method introduced by [2]

This method recommends for $\omega(\mu)$ and $\alpha(\mu)$:

$$\omega(\mu) := \omega_0 + \omega_1\mu + \omega_1\mu^2 + O(\mu^3) \qquad (2.5)$$

$$\alpha(\mu): \alpha_1\mu + \alpha_2\mu^2 + O(\mu^3) \qquad (2.6)$$

After a while of calculation we will get

$$f_2(\xi,\mu) = -\omega(\mu)\xi_1 + \alpha(\mu)\xi_2 + (1/2) f^2_{11}(\varnothing,\mu)\xi_1^2$$
$$+ f^2_{12}(\varnothing,\mu)\xi_1\xi + (1/2) f^2_{22}(\varnothing,\mu)\xi_2^2 + (1/6) f^2_{111}(\varnothing,\mu)\xi_1^3$$
$$+ (1/2) f^2_{112}(\varnothing,\mu)\xi_1^2\xi_2 + (1/2) f^2_{122}(\varnothing,\mu)\xi_1\xi_2^2$$
$$+ (1/6) f^2_{222}(\varnothing,\mu)\xi_2^3 + O(|\xi|^4) \quad (1.6)$$

We can see the function W has some coefficients such as a(μ), b(μ), c(μ), d(μ), e(μ), g(μ), h(μ), j(μ) and k(μ). They are to be determined useful. Therefore we should try to determine them so that the function W is (positive/negative) definite in a region about (0,0) = ∅ and at a distance far enough from ∅, for example along a closed curve C, with ∅ ∈ J (C), J(C) is inside of the closed curve C-W is (negative/positive) definite.

This task is not as fearsome as it seems, since it can be done in stages.

We shall first consider just the case where μ=0, recalling that we want sign definiteness even when α(μ) = 0, but that a(μ), b(μ), c(μ) and d (μ) are O (1), we see that we must equate the coefficients of ξ^3_1. ξ^3_2. ξ^2_1. ξ_2. and ξ_1 ξ^2_2 in the function W* (1.3) to zero.

This determines the coefficients a(μ) , b (μ), c(μ) and d(μ) uniquely. To make the quartic terms sign definite we shall certainly have to make the sum of the terms in ξ^4_1, ξ^2_1. ξ^2_2 and ξ^4_2 sign definite. Since a(μ), b(μ), c(μ) and d(μ) are now fixed all that matters is our choice of g(μ) and j(μ).

This means there is no loss of generality in choosing e(μ), h(μ), and k(μ) to make the terms in ξ^3_1 ξ_2 and ξ_1 ξ^3_2 vanish. The inequality constraints do not determine g(μ) and j(μ) completely, so we can try to derive a simpler yet equivalent constraint. Let us make the nonezero quartic terms in W* equal to a constant times a perfect square σ (μ) $(\xi^2_1 + \xi^2_2)^2$ where σ(μ) is a function of μ.

Substituting the values a(μ), b(μ), c(μ), d (μ), g(μ), and j(μ) determined above gives us the following formula for σ(μ) at criticality:

$$\sigma(0) = (1/16)\{a(0)[3 f^1_{11}(\varnothing,0) + f^1_{22}(\varnothing,0)] +$$
$$b(0)[3 f^2_{11}(\varnothing,0) + 4 f^1_{12}(\varnothing,0) + f^2_{22}(\varnothing,0)] +$$
$$c(0)[f^1_{11}(\varnothing,0) + 4 f^2_{12}(\varnothing,0) + 3 f^1_{22}(\varnothing,0)] +$$
$$d(0)[f^2_{11}(\varnothing,0) + 3 f^2_{22}(\varnothing,0)] +$$
$$f^1_{111}(\varnothing,0) + f^1_{122}(\varnothing,0) + f^2_{112}(\varnothing,0) + f^2_{222}(\varnothing,0)\}$$

For σ(0)=0, we can't say anything about the periodic solutions considering the character of W.

For this perpose we should improve the function W for order of four or higher and then again try to find out if W* is definite, but we will not do that in this work.

For μ≠0 we can determine the coefficients again in the same way as described above.

Since these coefficients are continuous we will again get the same formula for σ(0) if we set μ=0.

Lemma 1.1 [Asymptotic stablity of the solution 0 in x' (t) = f(x(t))]
f: D → $\Re^m$ is continuous and differentiable, D ⊆ $\Re^m$, D is an open set, m∈N, f(∅)=∅, {∅} ⊂ Ω ⊂ D, Ω is an open set and v: Ω → ℜ is continuous and differentialable in Ω, v is positive definite in Ω and v* is negative definite in Ω

⇒ ∅ is an asymptotic stable solution of x' (t)=f (x(t)).

Let us consider W* (ξ, μ) again and discuss the following points:

1) σ(0) < 0.
2) σ(0) > 0.

For σ(0)=0 we won't discuss because it is not the scope of this work.

About point 1): we learn from the lemma 1.1 that ∅ is an asymptotic stable solution for x'(t)=f(x(t), μ). It means for σ(0) < 0 we will get a super critical Hopf bifurcation when μ becomes 0(μ=0).

About point 2): "Changing the direction of time" we can show that for σ(0) >0 we will not get an ω- limit cycle but we might get an α-limit cycle. In this case if we change the direction of time, then we will get an ω-limit cycle but because of "change" it is actually an α-limit cycle of x' (t)=f(x(t), μ).

We have already shown the theorem of

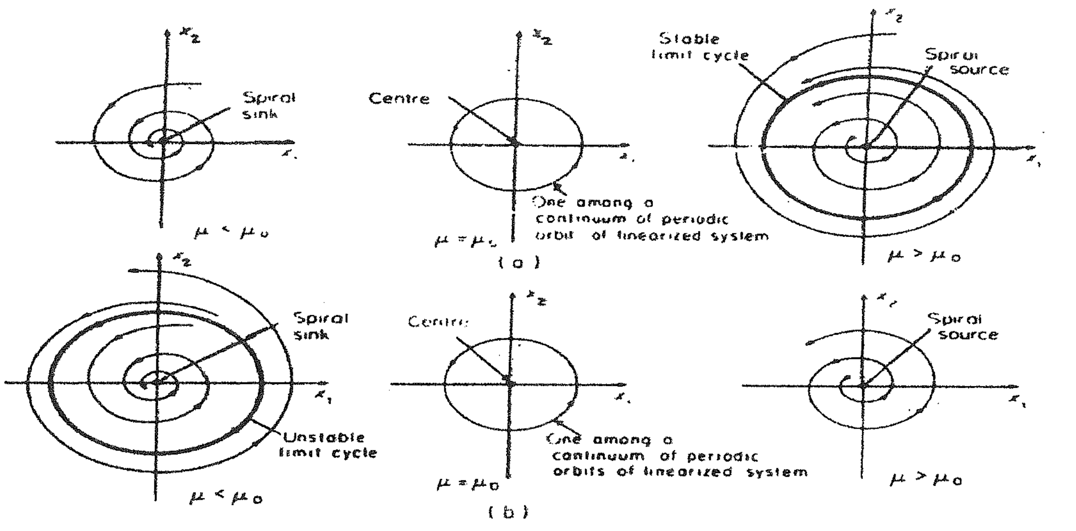

Fig. 1 as μ increases, a sink changes to a source, expelling or absorbing a limit cycle. (a) Type I (Supercritical) bifurcation (b) Type II (Subcritical) bifurcation.

1. Preliminaries

Let us to consider the following differntial equations:

$$x'(t)=f(x(t), \mu) \qquad (1.1)$$

in which f is a continuous function, and that f is 3 times continuous and differentiable in ξ.

$$f: D \times M \rightarrow \Re^2$$

$$(\xi,\mu) \rightarrow f(\xi,\mu) := \begin{pmatrix} f_1(\xi,\mu) \\ f_2(\xi,\mu) \end{pmatrix}$$

with $0 \in M$, $M \subseteq \Re$, M is open and $(0,0) = \varnothing \in D$, $D \subseteq \Re^2$, D is an open set and for every $\mu \in M$ we have $f(\varnothing, \mu) = \varnothing$.

Now we decide to use the Lyapunov method to prove the existence of periodic solutions. For this perpose, we choose for the Lyapunov function, the following function W and we wish:

1. Function W is positive definite,
2. The derivation function W* has some useful properties.

The question of which properties are useful will be answered later.

Because it is essential to take account of higher order derivatives than the first, a quadratic W will not do: to ensure sign definiteness of W* (at least when ξ is close enough to 0), we need quartic terms.

Thus we should choose:

$$W(\xi,\mu)=(1/2)(\xi_1^2+\xi_2^2)+(1/3)a(\mu)\xi_1^3+b(\mu)\xi_1^2\xi_2$$
$$+c(\mu)\xi_1\xi_2^2+(1/3)d(\mu)\xi_2^3+(1/4)e(\mu)\xi_1^4+g(\mu)\xi_1^3\xi_2$$
$$+(1/2)h(\mu)\xi_1^2\xi_2^2+j(\mu)\xi_1\xi_2^3+(1/4)k(\mu)\xi_2^4 \qquad (1.2)$$

and for derivation function W* we have the followig form:

$$W^*(\xi,\mu)=\partial W/\partial\xi_1(\xi,\mu).f_1(\xi,\mu)+\partial W/\partial\xi_2(\xi,\mu).f_2(\xi,\mu) \qquad (1.3)$$

In the following we use some abbreviations in the form:

$$f^i_{pq}(\varnothing,\mu)=(\partial^2 f_i/\partial\xi_p\partial\xi_q)(\varnothing,\mu) \qquad (1.4)$$

$$f^i_{pqr}(\varnothing,\mu)=(\partial^3 f_i/\partial\xi_p\partial\xi_q\partial\xi_r)(\varnothing,\mu) \qquad (1.5)$$

The differential equations can be more comprehensively written:

$$f_1(\xi,\mu)=\alpha(\mu)\xi_1+\omega(\mu)\xi_2+(1/2)f^1_{11}(\varnothing,\mu)\xi_1^2$$
$$+f^1_{12}(\varnothing,\mu)\xi_1\xi_2+(1/2)f^1_{22}(\varnothing,\mu)\xi_2^2$$
$$+(1/6)f^1_{111}(\varnothing,\mu)\xi_1^3+(1/2)f^1_{112}(\varnothing,\mu)\xi_1^2\xi_2$$
$$+(1/2)f^1_{122}(\varnothing,\mu)\xi_1\xi_2^2+(1/6)f^1_{222}(\varnothing,\mu)\xi_2^3+O(|\xi|^4)$$

The Hopf Bifurcation in Simple Situation

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Abstract

We can find a lot of conditions in nature and in engineering, which are called Hopf Bifurcation. We consider a special condition in which a system works well until a parameter changes. This system will lose its balance and we can get periodic solutions.

I am going to show how we can get such a condition in a simple situation and I will prove the theorem of HOPF BIFURCATION.

More importantly, I will demonstrate the method of getting asymptotic formula for form and period of limit cycle.

Introduction

The HOPF BIFURCATION THEOREM has become an important tool for understanding systems described by ordinary differential equations, because it is one of the few reliable methods of estabilishing the existence of limit cycles in high-dimentional systems. To use it effectively, one must be aware of both its advantages and disadvantages: for example, it is important to appreciate the local nature of the theorem, which only makes predictions for unspecified regions of parameter space and behaviour space. These predictions may be valid over regions which are very big or very small and the usual form of the theorem gives little help in determining their size.

Loosely, HOPF's theorem says that if an n-dimentional ordinary differential equation $x'(t)=f(x(t), \mu)$ depends on a real parameter μ and if on linearizing about an equilibrium point we find that pairs of complex conjugate eigenvalues of the linearized system cross the imaginary axis as μ varies through certain critical values, then for near - critical values of μ there exist limit cycles close to the equilibrium point. Just how near to-criticality μ has to be is not determined, and indeed unless a certain rather complicated expression (we shall call it the curvature coefficient) is nonzero, the usual statement of the theorem does not guarantee existence at all. The sign of the curvature coefficinet determines the stability of the limit cycle, and whether the limit cycle exists for subcritical ($\mu<\mu_0$) or supercritical $\mu>\mu_0$) parameter values. (We shall adpot the convention that near $\mu=\mu_0$ the real parts of the eigenvalues increase as μ increases.) Fig. 1 shows this argument.