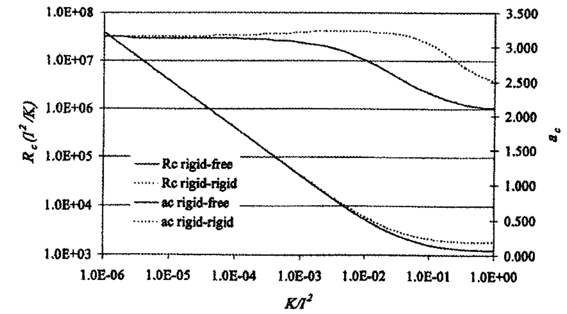

Figure (2) Comparison of variational results for rigid-rigid boundary conditions (M=N=1)

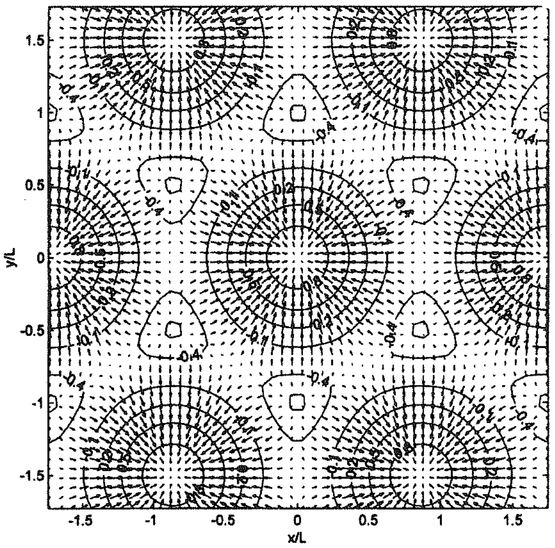

Figure (3) Hexagonal cell pattern

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as

$$w = \frac{1}{3} W(z) \left\{ 2 \cos \frac{2\pi}{L\sqrt{3}} x \cos \frac{2\pi}{3L} y + \cos \frac{4\pi}{3L} y \right\} \quad (A-1)$$

where L is length of the side of the hexagon and is related to the wave number k as $k = 4\pi/3L$. Hence, noting that $\alpha = kl$,

$$W = 0 \ @ \ z = \pm \frac{1}{2} \quad (A-2)$$

W (z) is given by equation (20) and can be tabulated when the variational equation (29) approaches a solution for a given l^2 / K.

It may be proved (see [15]) that horizontal velocity components are related to vertical velocity components as

$$u = \frac{1}{\alpha^2} \frac{\partial^2 w}{\partial x \partial z} \quad (A-3)$$

$$\upsilon = \frac{1}{\alpha^2} \frac{\partial^2 w}{\partial y \partial z} \quad (A-4)$$

Thus, one may plot the velocity components of a cell when R, α, and therefore W (z) are known. Fig. 3 shows a sample plot, where vertical velocity (w/W (z)) is plotted as contours and horizontal velocity (vectorial sum of $u \Big/ \frac{DW}{3\alpha^2} \frac{4\pi}{L\sqrt{3}}$ and $u \Big/ \frac{DW}{3\alpha^2} \frac{4\pi}{3\sqrt{3}L}$ is plotted as horizontal vectors. This theory has been verified experimentally by Arrayo et. al. [14].

Nomenclature

α	dimensionless wave number
C	concentration of diffusing fluid
D	d/dz operator
D_e	effective diffusivity in porous medium
E	porosity
g	gravity constant
K	permeability
l	fluid layer thickness
m	time constant
N and M	variational parameters
p	pressure
$R_{thermal}$	thermal Rayleigh number
u_j	velocity components of fluid
ω	z-component of the velocity
W(z)	dimensionless z functionality of w
x_j	position in fluid layer
X_i	external force in i direction
z	vertical coordinate

Greeks

α	coefficient of expansion
β	inverse concentration gradient
θ	perturbation in concentration
(z)	dimensionless z functionalitiy of θ
κ	thermal diffusivity
V	kinematic viscosity
μ	fluid viscosity
ρ	fluid density

superscripts

prime	perturbed

subscripts

c	critical property
0	property at reference concentration

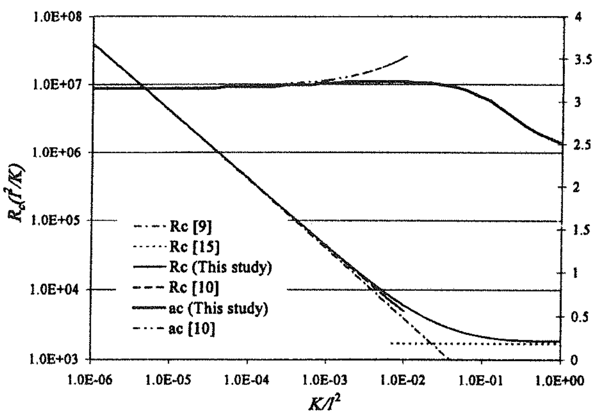

Figure (1) Comparison of variational results for rigid-rigid boundary conditions to Lapwood [9] and Katto and Masuoka [10]

Rayleigh number for rigid-free boundaries is 1/16th of odd solution to rigid-rigid one, and the corresponding wave number is one-half of that of odd solution. No such simple relations could be found between the solution to porous odd rigid-rigid and porous even rigid-free soultion. Therefore, an anlysis similar to the previous section was applied to rigid-free boundaries.

As mentioned earlier, the origin of coordinates system is placed at the lower surface. If the origin of coordinates system was assumed to be at the middle of the layer, from the boundary condition $W = 0 \; @ \; z = \pm \frac{1}{2}$ it would be concluded that W has an odd and an even part. However, boundary conditions (18) and (19) do not imply that the soultion is symmetric. Therefore, no simplification can be made to equation (26) and P_m, Q_m, P'_m, and Q'_m should be calculated in a way similar to rigid-rigid boundaries. It should be mentioned that since cos $[(2m + 1)\pi z]$ does not satisfy the boundary conditions (18) and (19), it is replaced everywhere by sin $[2m + 1) \pi z]$ in the analysis.

6 - Results and Discussions

Fig. 1 compares the results obtained in this study, by solving equation (29) for rigid-rigid boundaries when N and M were set to 1 with those proposed by Lapwood [9] and Katto and Masuoka [10]. The results for N = M = 1 are up to 6% different with respect to the results for N = M = 0, while only negligible modification to the calculated critical Rayleigh is obtained when N = M = 2 is used. It should be noted that the critical Rayleigh numbers obtained from the odd solution are higher than those of the even solution and therefore they are discarded.

The relative difference between critical Rayleigh numbers calculated in this study and those of Lapwood [19] , and Katto and Masuoka [10] for $K/l^2 = 0.01$ are 42% and 19%, respectively. The errors reduce as K/l^2 is reduced.

Fig. 2 compares the variational results for rigid-rigid boundaries to that of rigid-free boundaries, both obtained in this study. In a same manner to rigid-rigid boundaries, choosing M and N to be greater than 1 does not change the results noticeably in rigid-free boundaries. It may be observed that wave number a obtained for rigid-rigid boundaries are greater than those obtained for rigid-free boundaries. Therefore, it may be concluded that smaller cells are formed when both boundaries are rigid compared to when one of the boundaries is free (Appendix).

7 - Conclusion

The instability analysis described in this study tackles the problem in its general form without any simplifications. No improvements in results were obtained by increasing the values of variational parameters, i.e., M and N, above 1. Therefore, it may be concluded that the obtained results are more accurate than previously reported ones.

Appendix - Cell Pattern

As it may be deduced from symmetry considerations and it was first experimentally investigated by Benard [1] , the hexagonal cell pattern prevails in a layer of fluid when convection starts. The details of hexagonal pattern are given by Bisshopp [13]. The vertical component of velocity is given tiplying both sides of equation (21) by A_m and taking the summation over m we obtain

$$(D^2 - \alpha^2)(D^2 - \alpha^2 - l^2/K)\sum A_m W_m = \sum A_m \cos[(2m+1)\pi z] \quad (22)$$

which when compared to equation (14) gives

$$F = \sum A_m \cos[(2m+1)\pi z] \quad (23)$$

Therefore, equation (17) is also satisfied. Substitute W and F from equations (20) and (23) in equation (15) to get

$$F = \sum_m \frac{A_m}{\gamma_{2m+1}} \cos[(2m+1)\pi z] = R\alpha^2 \sum_m A_m W_m(z) \quad (24)$$

where

$$\gamma_{2m+1} = \frac{1}{(2m+1)^2\pi^2 + \alpha^2} \quad (25)$$

The solution to equation (21) is

$$W_m = P_m \cosh \alpha z + P'_m \sinh \alpha z + Q_m \cosh bz + Q'_m \sinh bz + \gamma_{2m+1}\gamma'_{2m+1} \cos[(2m+1)\pi z] \quad (26)$$

where

$$\gamma'_{2m+1} = \frac{1}{(2m+1)^2\pi^2 + b^2} \quad \text{and} \quad b = \sqrt{\alpha^2 + l^2/K}$$

It is obvious that W_m consist of an even part (summation of cosine terms) and an odd part (summation of sine terms). Application of boundary conditions (16) requires that either the even or the odd part to be zero. Consider the even part and therefore assume $P'_m = Q'_m = 0$. Applying boundary conditions (16) to equation (26) and solving the resulting system of equations yields P_m and Q_m.

To evaluate A_m, equation (24) is multiplied by cos [(2n + 1) πz] and integrated from

$z = -\frac{1}{2}$ to $z = \frac{1}{2}$ to give

$$\frac{1}{2}\frac{A_n}{\gamma_{2n+1}} = R\alpha^2 \sum (n|m) A_m \qquad (n = 0, 1, \ldots, N,\ m = 0, 1, \ldots, M) \quad (27)$$

where

$$(n|m) = \int_{-\frac{1}{2}}^{\frac{1}{2}} W_m(z) \cos[(2n+1)\pi z]\, dz \quad (28)$$

Equation (27) is a system of N (or M) equations and N (or M) unknowns (one should take N and M equal in order to solve the system). To ensure that non-zero solution exists, the determinant of the system has to be zero. Hence,

$$\left\| \frac{\delta_{mn}}{\alpha^2 R \gamma_{2m+1}} + 2(n|m) \right\| = 0 \qquad (n = 0, 1, \ldots, N,\ m = 0, 1, \ldots, M) \quad (29)$$

Equation (29) is an implicit relation between R, l^2/K and a. One may assume a value for l^2/K and using equation (29) plot R vs.a. If a suitable range for a is selected, a minimum in R should be observed which is the critical Rayleigh number. The corresponding value of a is the critical wave number that gives the dimension of the convection cells formed (Appendix). It should be noted that, as larger N and M are selected, the accuracy of calculated R_c will be higher. However, the calculations become more complicated.

Rigid - Free Boundaries

Chandrasekhar [15] stated that in a layer of fluid (non-porous media), the critical where Dd/dz, and Θ and W are z functionalities of θ and vertical velocity of fluid w, respectively

The right hand side of equation (12) is defined as F and this definition is used in equation (13) to obtain:

$$(D^2 - \alpha^2)(D^2 - \alpha^2 - l^2/K)\, W = F \tag{14}$$

$$(D^2 - \alpha^2)\, F = -(l^2/K)\, R\alpha^2 W \tag{15}$$

where $R = \frac{g\alpha\beta}{D_e v} K l^2$ is the Rayleigh number.

4 - Boundary Conditions

Equations (14) and (15) should be solved with the appropriate boundary conditions. Two different cases are considered:

1. Rigid surfaces at top and bottom of the liquid layer
2. Free surface at top and rigid surface at bottom of the liquid layer

Rigid-Rigid Boundaries

For this case, the origin of coordinates system is placed in the center of the fluid layer. This choice causes the problem to be simplified, as will be discussed later.

Apparently, u =0, v = 0, and w=0 on rigid surfaces , and in addition, from equation (4) it follows that $\partial w/\partial z = 0$. Or, in non-dimensional form

$$W = DW = 0 \quad \text{for} \quad z = \pm\frac{1}{2} \tag{16}$$

Moreover, since the concentrations on boundaries are maintained at certain values, they can suffer no changes and $\theta = 0$. Therefore, from the definition of F it follows that

$$F = 0 \quad \text{for} \quad z = \pm\frac{1}{2} \tag{17}$$

Rigid-Free Boundaries

For this case, the origin of coordinates system is placed on the lower surface. Again W = 0, and since the shear stresses are zero on the free surface, it may be shown that [15]:

$$W = D^2 W = 0 \quad \text{for} \quad z = 1 \tag{18}$$

On the rigid surface, equation (14) is rewritten as:

$$W = DW = 0 \quad \text{for} \quad z = 0 \tag{19}$$

equation (15) is still satisfied by the same reason as Rigid-Rigid boundaries.

5 - Variational Solution

The perturbation equations (14) and (15) are to be solved for rigid-rigid and rigid-free boundaries using variational methods without eliminating $\mu\nabla^2 u_i$ term and imposing no restriction on λ^2/k. The solution strategy is analogous to non-porous system by Chandrasekhar [15].

Rigid-Rigid Boundaries

Consider equation (14) for rigid-rigid boundaries, according to equation (14), W should be zero on either boundaries. Assume that W can be represented in series as

$$W = \sum_m A_m W_m(z), \tag{20}$$

where Wm (z) is to satisfy the equation

$$(D^2 - \alpha^2)(D^2 - \alpha^2 - l^2/K) W_m = \cos[(2m+1)\pi z] \tag{21}$$

Note that right hand side of equation (21) becomes zero on the boundaries. Mulporous medium that is finite in height and infinite in two other directions. The variational method is used to solve the perturbation equations resulted from instability analysis. Values for critical Rayleigh numbers as well as wave number vs. l^2/K corresponding to two various top and bottom surfaces are obtained, i.e., rigid top-rigid bottom surfaces, and free top-rigid bottom surfaces.

2 - Governing Equations

Consider a fluid in which the density is a function of position x_j (j = 1, 2, 3), confined in a porous medium. The density variation is due to dissolution of gas with a concentration of C (z). Let u_j (j = 1, 2, 3) denote the components of the velocity. In writing the various equtions, we shall use the notation of Cartesian tensors with the usual summation convention. The governing equations are formulated as follows:

(i) Continuity equation

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad (4)$$

(ii) Equations of motion

$$\frac{1}{E}\frac{\partial u_i}{\partial t} + \frac{\mu}{\rho_0 K} u_i + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \frac{\rho}{\rho_0} X_i + \nu \nabla^2 u_i \qquad (5)$$

(iii) Equation of diffusion

$$E\frac{\partial C}{\partial t} + u_j \frac{\partial C}{\partial x_j} = D_e \nabla^2 C \qquad (6)$$

(iv) Equation of state

$$\rho = \rho_0 [1 - \alpha (C_0 - C)] \qquad (7)$$

where α, E, and x_i denote coefficient of expansion, porosity, and external force in i direction, respectively.

In writing equations (4) and (5), Boussinesq assumption was accepted (see Nadolin [12]). Therefore, variation of ρ and μ was neglected in most of the terms except in the term ρx_i in the equation of motion.

3 - The Perturbation Equations

Consider an infinite horizontal layer of fluid in which a steady inverse concentration gradient is maintained. Since there is no motion, the initial state is one in which

$$u_j \equiv 0 \quad \text{and} \quad C \equiv C \qquad (8)$$

Moreover , equation (6) reduces to

$$\nabla^2 C = 0 \qquad (9)$$

which upon integration yields

$$C = C_0 - \beta z \qquad (10)$$

where β is the inverse concentration gradient and C_0 is the concentration of gas at the upper surface. Let the initial state described by equations (8)-(10) be slightly perturbed and the altered concentration distribution be

$$C' = C_0 - \beta z + \theta \qquad (11)$$

The corresponding perturbation equations to equations of motion and diffusion derived in a way similar to that of Pellew and Southwell [5]:

$$(D^2 - \alpha^2)(D^2 - \alpha^2 - l^2) W = \left(\frac{ga}{\nu} l^2\right) \alpha^2 \Theta \qquad (12)$$

$$(D^2 - \alpha^2) \Theta = -\left(\frac{\beta}{D_e} l^2\right) W \qquad (13)$$

cosity and thermal diffusivity of the liquid, stable equilibrium is still possible when the density increases positively upwards, provided that the density gradient does not exceed a certain critical value. The appropriate dimensionless expression for the density gradient is the Rayleigh number

$$R_{thermal} = \frac{d\rho}{dz}\frac{gd^4}{\mu\kappa}, \tag{1}$$

where d denotes a typical linear dimension, such as the radius , and κ denotes thermal diffusivity.

Pellew and Southwell [5] have studied the problem of stability of a layer of liquid heated from below. Their theory is represented by Chandrasekhar [15] where values of 657.5, 1707.8, and 1100.7 are derived for two free boundaries (no momentum crosses the boundaries), two rigid boundaries (no heat crosses the boundary), and one rigid and one free boundaries, respectively. More boundary condition types are discussed by Hashim and Wilson [6].

The onset of convection in porous media has been studied via mass as well as thermal diffusion. When regarding the diffusion of heat into the fluid and the media, one should take into account part of the heat transferred through the media itself. This problem does not appear when working on mass diffusion in porous media, because the media is usually impermeable to mass.

Wooding [4] has studied the stability of a viscous liquid, the density of which increases with height because of the presence of a dissolved substance, in a vertical impermeable tube containing porous material. He defined the Rayleigh number for a viscous liquid of variable density in a porous medium to be

$$R = \frac{d\rho}{dz}\frac{gKd^2}{\mu D_e}, \tag{2}$$

where K is the permeability of the porous medium, d is tube diameter, and D_e is the effective diffusivity of the dissolved material. The value of critical Rayleigh number was suggested to be 67.94. Saidi [16] reports a value of 26.4 for the critical Rayleigh number defined as

$$R = \frac{d\rho}{dz}\frac{\alpha\rho_0 gKL^2}{\mu D}, \tag{3}$$

conducting a gas-oil diffusion-convection stability analysis in a cylinder. Here L is the height of liquid column that retains the inverse density gradient and ρ_0 is density of fluid with a reference gas concentration. Zebib [7], and Bau and Torrance [8] take into account the rule of cylinder aspect raio. Bau and Torrance [8] suggest a critical Rayleigh number of 27.1 [as defined in eq. (3)] for long cylinders.

Lapwood [9] studied the same problem for various boundary conditions in porous medium ignoring the term $\mu\nabla^2 u_i$ in the momentum equation. He proved that $R_c.K/l^2=4\pi^2$, which states that in porous media the critcal Rayleigh number is a function of dimensionless term l^2/K, where l is the layer thickness and K is permeability. Katto and Masuoka [10], though limiting the problem to low values of l^2/K where a simplifying symbolic approximation is valid, did not ingnore the $\mu\nabla^2 u_i$ term, hence giving more accurate critical Rayleigh numbers. Kaviany [11] experimentally examines the dependency of onset time on critical Rayleigh number and permeability.

This study, is concerned with the stability analysis of a layer of fluid, the density of which increases linearly with height, in a

Instability Analysis in a Porous Layer Saturated with Oil and Gas

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Abstract

An instability analysis in a horizontal porous layer is made for a fluid with an inverse density gradient. The governing equations resulted when instability analysis is applied to this problem, are non-linear and therefore mathematically complex. In published literature, the problem is solved with some simplifications such as ignoring certain terms in the governing equations, or finding algebraic approximations that are valid in some specific range of physical parameters.

In this study, the problem is solved in its general form for two rigid, and one rigid and one free boundary conditions using variational methods. The resulting critical Rayleigh numbers and critical wave numbers vs. l^2/K curves are compared to previous works.

Keywords

Onset of Convection, Diffusion - Convection in Porous Media, Critical Rayleigh Number

1 - Introduction

In a novel EOR (Enhanced Oil Recovery) method, diffusion-convection may be employed as a drive to force heavy, viscous oil to move . A bed of gas is expanded beneath oil, and gas is allowed to diffuse upward. As oil dissolves gas, it becomes lighter. Therefore, an inverse density gradient is developed in oil layer gradually. When this density gradient exceeds a critical value, convection is started . It may be concluded that developing a theory, which predicts the onset of convection, should be of practical importance to reservoir engineering.

Early investigations on the instability analysis were conducted in non-porous media. The stability of a column/layer of liquid that maintains an inverse density gradient has been of concern as early as 1900. The earliest experiments to demonstrate in a definitive manner the onset of thermal instability in fluids are those of Benard [1]. Benard set up an inverse density gradient in a layer of fluid and showed that when instability prevails, a regular hexagonal pattern forms. On the theoretical side, the fundamental theory is that of Rayleigh [2].

Hales [3] has studied the instability of water in a long vertical tube heated from below. Owing to the damping effect of vis-