# *Finsler Metrics of Generalized Isotropic Mean Berwald Curvature* [1]

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*ABSTRACT*

Differentiating of isotropic mean Berwald curvature along a geodesic gives us a special non-Riemannian quantity. We call it generalized isotropic mean Berwald curvature. In this paper, we classify the scalar flag curvature isotropic mean Landsberg Finsler metric with generalized isotropic mean Berwald curvature. As an application of this classification, we find the Gauss curvature of isotropic mean Landsberg Randers surface with isotropic mean Berwald curvature.

*KEYWORDS*

Flag curvature, Isotropic mean Landsberg metric, Isotropic mean Berwald curvature, Randers metric

## 1. INTRODUCTION

Finsler metrics arise naturally from many areas of mathematics as well as natural science. For example, the navigation problem in a Riemannian space gives rise to lots of interesting Finsler metrics with special geometric properties [13] [14]. In Finsler geometry, we study not only the shape of a space, but also the "color" of the space on an infinitesimal scale. The Riemannian quantity (such as the flag curvature) describes the shape of a space, while non-Riemannian quantities describe the "color" of the space.

For a Finsler manifold $(M,F)$, the flag curvature $\mathbf{K}=\mathbf{K}(P,y)$ is a function of tangent planes $P=\mathrm{span}\{y,v\}\subset T_x M$ and directions $y\in P\setminus\{0\}$. This quantity tells us how curved the space is at a point. If $F$ is Riemannian, $\mathbf{K}=\mathbf{K}(P)$ is independent of $y\in P\setminus\{0\}$, $\mathbf{K}$ being called the sectional curvature in Riemannian geometry. A Finsler metric $F$ is said to be *of scalar curvature* if the flag curvature $\mathbf{K}=\mathbf{K}(x,y)$ is a scalar function on the slit tangent bundle $TM_0$. Clearly, a Riemannian metric is of scalar curvature if and only if $\mathbf{K}=\mathbf{K}(x)$ is a scalar function on $M$ (which is a constant in dimension $n>2$ by the Schur lemma). There are lots of non-Riemannian Finsler metrics of scalar curvature. One of the important problems in Finsler geometry is to study and characterize Finsler metrics of scalar curvature. This problem has not been solved yet, even for Finsler metrics of constant flag curvature.

In Finsler geometry, there are several important non-Riemannian quantities: the Cartan tensor **C**, the Berwald curvature **B**, the mean Landsberg Berald curvature **E**, the **S**-curvature and **H**-curvature [8], etc. They all vanish for Riemannian metrics, hence they are said to be *non-Riemannian.* See Section 2 for more details about their definitions and geometric meanings.

The Berwald metrics are important special Finsler spaces. Finsler metrics with $E=0$ can be viewed as weakly Berwaldian metrics. The relationship between E-curvature and Berwald curvature is similar to Ricci curvature and the Riemannian curvature.

In [1], Arkar-Zadeh considered a non-Riemannian quantity $H$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle. Akbar-Zadeh proved that for a Finsler metric of scalar flag curvature, the flag curvature is a scalar function on the manifold if and only if $H=0$. Thus the quantity deserves further investigation. Recently in paper [8], Najafi, Shen and the first author introduce a new non-Riemannian curvature $H=\frac{(n+1)}{2}c(x)Fh$ and extent the Akbar-Zadeh's theorem for this quantity. Here, we call it *generalized isotropic mean Berwald curvature.* In this

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paper, we are going to classify Finsler metrics of scalar flag curvature flag curvature with generalized isotropic mean Berwald curvature (see Theorem 1).

Then as an application of this classification, we find the Gauss curvature of isotropic mean Landsberg Randers surface with isotropic mean Berwald curvature (see Theorem 2) One of the reasons why we would like to study Randers metrics for the above problem is because that Randers metrics are computable.

Throughout this paper, we make use of Einstein convention. One is referred to [4] and [15] for some of these connections. Throughout this paper, we set the Chern connection on Finsler manifolds. The h- and v-covariant derivatives of a Finsler tensor field are denoted by "|" and "," respectively.

## 2. Preliminaries

In this section, we are going to give a brief description on several geometric quantities in Finsler geometry.

A *Finsler structure* on a manifold $M$ is a function $F: TM \to [0,\infty)$ with the following properties:

(i) $F$ is $C^\infty$ on $TM_0 = TM/\{0\}$.

(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$:

$$\forall \lambda > 0 \quad F(x,\lambda y) = \lambda F(x,y).$$

(iii) The Hessian of $F^2$ with elements $g_{ij}(x,y) := \frac{1}{2}[F^2(x,y)]_{y^i y^j}$ is positively defined on $TM_0$.

Then the pair $(M,F)$ is called a *Finsler manifold*. $F$ is Riemannian if $g_{ij}(x,y)$ are independent of $y \neq 0$.

The Cartan tensor $C = C_{ijk}\, dx^i \otimes dx^j \otimes dx^k$ is defined by

$$C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}.$$

Clearly, $C_{ijk}$ is symmetric with respect to $i,j,k$. It is well-known that the Finsler metric $F$ is Riemannian if $C_{ijk} = 0$.

Let $F$ be a Finsler metric on an $n$-dimensional manifold $M$. The geodesics of $F$ are characterized by the following equations

$$\ddot{c}^i(t) + 2G^i(c(t),\dot{c}(t)) = 0,$$

where $G^i = G^i(x,y)$ are given by

$$G^i = \frac{1}{4} g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}.$$

where $(g^{ij}(x,y)) := (g_{ij}(x,y))^{-1}$. When $F$ is Riemannian, i.e., $g_{ij}(x,y) = g_{ij}(x)$ depend only on $x \in M$, $G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$. There are many non-Riemannian Finsler metrics with this property. Such Finsler metrics are called *Berwald metrics*. By definition, $F$ is called a *Berwald metric* if $G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y \in T_x M$ for any $x \in M$. Every Berwald metric $F$ is affinely equivalent to a Riemannian metric, namely, $F$ and $g$ have the same spray [10].

The Landsberg tensor $L = L_{ijk}\, dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{jkl} := -\frac{1}{2} F F_{y^i} [G^i]_{y^j y^k y^l}.$$

A Finsler metric is called a *Landsberg metric* if $L_{jkl} = 0$. One can easily see that every Berwald metric is Landsbergian. A natural problem is whether or not every Landsberg metric is Berwaldian.

Let

$$\tau(x,y) := \ln[\frac{\sqrt{\det(g_{ij}(x,y))}}{\mathrm{Vol}(\mathrm{B}^n(1))} \cdot \alpha],$$

where

$$\alpha := \mathrm{Vol}\{(y^i) \in \mathrm{R}^n \mid F(y^i \frac{\partial}{\partial x^i}|_x) < 1\}.$$

$\tau = \tau(x,y)$ is a scalar function on $TM_0$, which is called the *distortion* [11].

Define *mean Cartan torsion* $\mathbf{I}_y := I_i(x,y) dx^i$, where

$$I_i(x,y) := g^{jk}(x,y) C_{ijk}(x,y) = \frac{\partial \tau}{\partial y^j}(x,y).$$

According to Deicke's theorem, $F_x$ is Euclidean at $x \in M$ if and only if $\mathbf{I}_y = 0$, or equivalently, $\tau = \tau(x)$ at $x \in M$ [3].

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y := J_i(x,y) dx^i$, where $J_i = J_i(x,y)$ are given by

$$J_i := g^{jk} L_{ijk} = y^m \frac{\partial I_i}{\partial x^m} - I_m \frac{\partial G^m}{\partial y^i} - 2G^m \frac{\partial I_i}{\partial y^m}.$$

A Finsler metric $F$ is said to be *weakly Landsbergian* if $\mathbf{J} = 0$. $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. The generalized Funk metrics on the unit ball $\mathrm{B}^n \subset \mathrm{R}^n$ satisfy $\mathbf{J} + cF\mathbf{I} = 0$ for some constant $c \neq 0$ [12]. A Finsler metric $F$ is said to be isotropic mean Landsberg metric if $\mathbf{J} + cF\mathbf{I} = 0$ is hold for some scalar function $c(x)$ on $M$ ( see [5] and [7]).

The Riemann curvature $\mathbf{K}_y = K^i_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_x M \to T_x M$ is a family of linear maps on tangent spaces, defined by

$$K^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a flag $P = \mathrm{span}\{y, u\} \subset T_x M$ with flagpole $y$, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{K}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2},$$

where $\mathbf{g}_y = g_{ij}(x,y)dx^i \otimes dx^j$. When $F$ is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$, which is just the sectional curvature of $P$ in Riemannian geometry. We say that a Finsler metric $F$ is of scalar curvature if for any $y \in T_x M$, the flag curvature $K = K(x, y)$ is a scalar function on the slit tangent bundle $TM_0$. If $K$ =constant, then $F$ is said to be of constant flag curvature.

Let $M$ be an $n$-dimensional manifold. Suppose $\pi^*TM$ denote the pull-back tangent bundle by $\pi : TM_0 \to M$ and $(x, y, v)$ denote the elements of $\pi^*TM$, where $y \in T_x M_0$ and $v \in T_x M$. Let $\pi^*T^*M$ denote the horizontal cotangent bundle of $TM_0$, consisting of $\pi^*\theta$, where $\theta \in T^*M$. There is a natural duality between $\pi^*TM$ and $\pi^*T^*M$. Let $\{\mathbf{e}_i := (x, y, \frac{\partial}{\partial x^i})\}$ be a natural local frame for $\pi^*TM$. Then $\{\omega^i := \pi^* dx^i\}$ is the dual local coframe for $\pi^*T^*M$. $\pi^*TM$ has a canonical section, $Y := (x, y, y) = y^i \mathbf{e}_i$, where $y = y^i \frac{\partial}{\partial x^i}$. The Chern connection is a linear connection on $\pi^*TM$, which are characterized by

$$d\omega^i = \omega^j \wedge \omega^i_j,$$

$$dg_{ij} = g_{ik}\omega^k_j + g_{kj}\omega^k_i + 2C_{ijk}\{dy^k + y^j\omega^k_j\}.$$

See [3][13].

Let

$$\omega^{n+k} := dy^k + y^j \omega^k_j.$$

We obtain a local coframe $\{\omega^i, \omega^{n+i}\}$ for $T^*(TM_0)$.

Let

$$\Omega^i := d\omega^{n+i} - \omega^{n+j} \wedge \omega^i_j.$$

We can express $\Omega^i$ in the following form

$$\Omega^i = \frac{1}{2} K^i_{kl}\omega^k \wedge \omega^l - L^i_{kl}\omega^k \wedge \omega^{n+l},$$

where $K^i_{kl} + K^i_{lk} = 0$. Let

$$K^i_k := K^i_{kl} y^l.$$

We obtain the Riemann curvature $\mathbf{K} = K^i_k \omega^k \otimes \omega^l$ and $\mathbf{L} = L^i_{kl}\omega^k \otimes \omega^l \otimes \mathbf{e}_i$. In a standard local coordinate system $(x^i, y^i)$, $K^i_k$ are given by (2.1). Without much difficulty, one can show that

$$K^i_{kl} = \frac{1}{3}\{K^i_{k\cdot l} - K^i_{l\cdot k}\}$$

and

$$I_i = \tau_{\cdot i},\ S := \tau_{|m} y^m,\ J_i = I_{i|m} y^m.$$

Moreover, $L_{ijk} := g_{im} L^m_{kl} = C_{ijk|m} y^m$.

Let $(M, F)$ be a spray space. In a standard local coordinate system $(x^i, y^i)$ in $TM$, $G$ is expressed in the following form

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}.$$

Let

$$N^i_j(y) := \frac{\partial G^i}{\partial y^j}(y), \quad \Gamma^i_{jk} := \frac{\partial^2 G^i}{\partial y^j \partial y^k}(y).$$

$N^i_j$ and $\Gamma^i_{jk}$ are called the connection coefficient and the Christoffel symbols of $G$, respectively.

Set

$$B^i_{jkl} := \frac{\partial \Gamma^i_{jk}}{\partial y^l}(y) = \frac{\partial^2 N^i_j}{\partial y^k \partial y^l}(y) = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y).$$

This leads to an important quantity. For a tangent vector $y \in T_x M_0$, define

$$B_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$$

by

$$B_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. $B_y(u, v, w)$ is symmetric in $u$, $v$ and $w$. $\mathbf{B}$ is called the Berwald curvature. The Finsler metric $F$ is said to be Berwald metric if $\mathbf{B}$=0.

Recently, Chen and Shen introduce a new class of non-Riemannian Finsler metrics which is called the *isotropic Berwald metrics* [5]. $F$ is said to be isotropic Berwald metric if its Berwald curvature satisfies the following

$$B^i_{jkl} = \sigma(x)\{F_{y^j y^k}\delta^i_l + F_{y^k y^l}\delta^i_j + F_{y^l y^j}\delta^i_k + F_{y^j y^k y^l} y^i\},$$

where $\sigma$ is scalar function on $M$. It is obvious that $F$ is a Berwald metric if $\sigma(x) = 0$.

**Example 1.** Let $\Omega$ be a strongly convex domain in $R^n$. By definition, there is a Minkowski norm $\varphi$ on

$R^n$ and a point $x_o \in \Omega$ such that $\partial\Omega - \{x_o\} = \varphi^{-1}(1)$. Let $F$ be the Funk metric on $\Omega$. For any $y \in T_x\Omega = R^n$, $F = F(x,y)$ is determined by

$$\varphi(x - x_o + \frac{y}{F}) = 1.$$

Differentiating aboveequation yields a system of PDEs,

$$F_{x^i} = FF_{y^i}, \quad i = 1, \cdots, n.$$

The above system is proved in [9]. The Funk metric $F$ on the unit ball $B^n \subset R^n$ is given by

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y \rangle^2)} + \langle x,y \rangle}{1 - |x|^2}.$$

Funk metrics are of isotropic Berwald curvature [6]. Let

$$E_{jk}(y) := \frac{1}{2} B^m_{jkm}(y),$$

This set of local functions give rise to a tensor on $TM_0$. For a tangent vector $y \in T_xM_0$, define

$$E_y : T_xM \otimes T_xM \to R$$

by $E_y(u,v) := E_{jk}(y)u^jv^k$, where $u = u^i \frac{\partial}{\partial x^i}|_x$ and $v = v^i \frac{\partial}{\partial x^i}|_x$. $E$ is called the mean Berwald curvature. The Finsler metric $F$ is said to be weakly Berwald metric if $E = 0$. We have

$$E_{jk}(y) := \frac{1}{2}\frac{\partial^2 N^m_m}{\partial y^k \partial y^l}(y) = \frac{1}{2}\frac{\partial^3 G^m}{\partial y^j \partial y^k \partial y^m}(y).$$

The E-curvature is closely related to the flag curvature. For a two-dimensional plane $P \subset T_pM$ and a non-zero vector $y \in T_pM$, the *flag E-curvature* $E(P,y)$ is defined by

$$E(P,y) := \frac{F^3(y)E_y(u,u)}{g_y(y,y)g_y(u,u) - g_y(y,u)^2}$$

where $P = span(y,u)$. We say that $F$ has constant flag E-curvature if for any flag $(P,y)$, $E(P,y) = (n+1)c$, that is equivalent to the following system of equation,

$$E_{ij} = \frac{n+1}{2} cF^{-1}h_{ij},$$

where $h_{ij} = g_{ij} - F^{-2}y_iy_j$ is the angular metric.

The quantity $H_y = H_{ij}\,dx^i \otimes dx^j$ is defined as the covariant derivative of **E** along geodesics. More precisely,

$$H_{ij} := E_{ij|m}\, y^m.$$

Clearly, $H_{ij}$ have the following property $H_{ij}\, y^i = 0$. We can take further averaging on **H** as follows:

$$H := g^{ij} H_{ij}.$$

**H** is scalar functions on the slit tangent bundle $TM_0$. The important of the quantity **H**=0 lies in the following:

**Theorem A.** ([2]) Let $F$ be a Finsler metric of scalar flag curvature on an n-dimensional manifold ($n \geq 3$). Then the flag curvature **K** =constant if and only if **H**=0.

A Finsler metric $F$ is said to be *generalized isotropic mean Berwald metric* if its mean Berwald curvature satisfies the following:

$$H = \frac{(n+1)}{2} c(x) F^{-1} h,$$

where $c(x)$ is scalar function on $M$ [8].

Recently, Shen, Najafi with first author, generalized the above theorem and prove the following:

**Theorem B.** ([8]) Let $F$ be a Finsler metric of scalar curvature on an $n$-dimensional manifold ($n > 2$). Let $\theta$ be an arbitrary 1-form on $M$. Then

$$H = \frac{(n+1)}{2} c(x) F^{-1} h \tag{2.1}$$

if and only if

$$\mathbf{K} = \frac{3\theta}{F} + \sigma, \tag{2.2}$$

where $\sigma = \sigma(x)$ is a $C^\infty$ scalar function on $M$.

## 3. Classification of Isotropic Mean Landsberg Metrics

In this section, we study the Finsler metrics of scalar flag with generalized isotropic mean Berwald curvature.

**Theorem C.** ([5]) Let $(M,F)$ be an $n$-dimensional Finsler manifold of scalar curvature. Suppose that **J/I** is isotropic, $\mathbf{J} + c(x)F\mathbf{I} = 0$ where $c = c(x)$ is a $C^\infty$ scalar function on $M$. Then the flag curvature $\mathbf{K} = \mathbf{K}(x,y)$ and the distortion $\tau = \tau(x,y)$ satisfy

$$\frac{n+1}{3}\mathbf{K}_{y^k} + (\mathbf{K} + c(x)^2 - \frac{c_{x^m}(x)y^m}{F(x,y)})\tau_{y^k} = 0. \tag{3.1}$$

***Theorem 1.*** *If $(M,F)$ be an n-dimensional Finsler manifold of scalar flag curvature. Suppose that the mean Berwald curvature and the mean Landsberg curvature satisfy*

$$E_{ij|s}y^s = \frac{n+1}{2}c_{x^m}y^m F^{-1}h_{ij}, J + c(x)FI = 0,$$

where $c = c(x)$ is a scalar function on $M$. Then the flag curvature is given by

$$K = 3\frac{c_{x^m}(x)y^m}{F(x,y)} + \sigma(x)$$
$$= -\frac{3c(x)^2 + \sigma(x)}{2} + v(x)e^{-2\tau(x,y)/(n+1)},$$

*where $\sigma(x)$ and $v(x)$ are scalar functions on $M$.*

**(a)** *Suppose that $F$ is not Riemannian on any open subset in $M$. If $c(x) = c$ is a constant, then $K = -c^2$,*

$\sigma(x) = -c^2$ *and* $v(x) = 0$.

**(b)** *If* $c(x) \neq constant$, *then the distortion is given by*

$$\tau = ln\{\frac{2vF}{6c_{x^m}y^m + 3[\sigma + c^2]F}\}^{(n+1)/2}. \quad (3.2)$$

**Proof.** By the above argument, $\mathbf{K}$ is given by (2.2) and it satisfies (3.1). It follows from (2.2) that

$$\frac{c_{x^m}(x)y^m}{F(x,y)} = \frac{1}{3}(\mathbf{K} - \sigma(x)).$$

Plugging it into (3.1) yields

$$\frac{n+1}{3}\mathbf{K}_{y^k} + (\frac{2}{3}\mathbf{K} + c(x)^2 + \frac{1}{3}\sigma(x))\tau_{y^k} = 0.$$

We obtain

$$[(2\mathbf{K} + 3c(x)^2 + \sigma(x))^{\frac{n+1}{2}} e^{\tau}]_{y^k} = 0.$$

Thus there is a scalar function $v(x)$ on $M$ such that

$$\mathbf{K} = -\frac{3c(x)^2 + \sigma(x)}{2} + v(x)e^{-\frac{2\tau(x,y)}{n+1}}. \quad (3.3)$$

Comparing (2.2) with (3.3), we obtain

$$\frac{c_{x^m}(x)y^m}{F(x,y)} = -\frac{c(x)^2 + \sigma(x)}{2} + \frac{v(x)}{3}e^{-\frac{2\tau(x,y)}{n+1}}. \quad (3.4)$$

**Case (a).** Suppose that $c(x) = c$ is a constant. We claim that $v(x) = 0$. If it false, then $U := \{x \in M, v(x) \neq 0\} \neq \varnothing$. From (3.4), one can see that $\tau = \tau(x)$ is a scalar function on $U$, hence $F$ is Riemannian on $U$ by Deicke's theorem [3]. This contradicts the assumption in (a). Now (2.4) is reduced to that $\sigma(x) = c(x)^2$ and (3.3) is reduced to that $\mathbf{K} = -c^2$.

**Case (b).** If $c \neq contant$, then by (2.4) we have $v \neq 0$. In this case, we can solve (3.4) for $\tau$ and obtain (3.2). □

**Corollary 1.** *If* $(M,F)$ *be an n-dimensional isotropic mean Landsberg Finsler manifold* $(J + c(x)FI = 0)$ *of scalar flag curvature with isotropic mean Berwald curvature* $(E_{ij} = \frac{n+1}{2}c(x)F^{-1}h_{ij})$ *where* $c = c(x)$ *is a scalar function on* $M$. *Then the flag curvature is given by*

$$K = 3\frac{c_{x^m}(x)y^m}{F(x,y)} + \sigma(x)$$
$$= -\frac{3c(x)^2 + \sigma(x)}{2} + v(x)e^{-2\tau(x,y)/(n+1)},$$

*where* $\sigma(x)$ *and* $v(x)$ *are scalar functions on* $M$.

**(a)** *Suppose that* $F$ *is not Riemannian on any open subset in* $M$. *If* $c(x) = c$ *is a constant, then* $K = -c^2$, $\sigma(x) = -c^2$ *and* $v(x) = 0$.

**(b)** *If* $c(x) \neq constant$, *then the distortion is given by*

$$\tau = ln\{\frac{2v(x)F(x,y)}{6c_{x^m}(x)y^m + 3[\sigma(x) + c(x)^2]F(x,y)}\}^{(n+1)/2}.$$

**Corollary 2.** *If* $(M,F)$ *be an n-dimensional isotropic mean Landsberg Finsler metric of scalar flag curvature. If* $F$ *is of isotropic mean Berwald curvature then the* **S**-*curvature of* $F$ *is given by*

$$\mathrm{S} = \frac{n+1}{F}\{\frac{v^{\bullet}}{v} - \frac{F\{\sigma^{\bullet} + 2cc^{\bullet}\} - 2c^{\bullet\bullet}}{2c^{\bullet} + (\sigma + c^2\}F}\}$$

*where* $c^{\bullet} = c_{|m}y^m$, $c^{\bullet\bullet} = (c^{\bullet})_{|m}y^m$, $v^{\bullet} = v_{|m}y^m$, $\sigma^{\bullet} = \sigma_{|m}y^m$.

**Proof.** From Theorem 1, the distortion $\tau$ satisfy in (3.2).

First, we simplify $\tau$ :

$$\tau = \ln(\frac{2vF}{6c_{x^m}y^m + 3[\sigma + c^2]F})^{(n+1)/2}$$

which is equal to

$$\tau = \frac{n+1}{2}(\ln(2vF) - \ln(6c_{x^m}y^m + 3[\sigma + c^2]F))$$

Then we get

$$\tau = \frac{n+1}{2}(\ln 2F + \ln v - \ln(6c_{x^m}y^m + 3[\sigma + c^2]F)).$$

Given horizontally differentiating of the above relation along geodesic, we get the proof. □

**Corollary 3.** *If* $(M,F)$ *be an n-dimensional isotropic Berwald Finsler manifold. Suppose that* $F$ *is a isotropic mean Landsberg Finsler metric of scalar flag curvature. If* $F$ *is of isotropic mean Berwald curvature then the flag curvature is given by*

$$K = 3\frac{c_{x^m}(x)y^m}{F(x,y)} + \sigma(x),$$

where $\sigma(x)$ and $v(x)$ are scalar functions on $M$.

## 4. Randers Surface of Isotropic Mean Berwald Curvature

In this section, by using Theorem 1, we study the Randers surface of isotropic mean Berwald curvature.

**Theorem 2.** *If* $(M,F)$ *be an isotropic mean Landsberg Randers surface. Suppose that* $F$ *is of isotropic mean Berwald curvature. Then the Gauss curvature of* $F$ *is given by*

$$K = 3c_{x^m}(x)y^m + \sigma(x) \quad (4.1)$$
$$= -\frac{3c^2 + \sigma(x)}{2} + v(x)e^{-2\tau(x,y)/(n+1)}$$

where $\sigma(x)$ and $v(x)$ are scalar functions on $M$. Moreover $c(x)$ is satisfy in following equations:

$$dc = \frac{-1}{2}(\sigma + c^2)\beta = \frac{2\tau}{3}(1 - \|\beta\|_{\alpha}^2)\beta. \quad (4.2)$$

**Proof.** From Theorem 1, it is sufficient that we prove the second part of Theorem 2. The distorsion function of $F$ satisfy in following equation:

$$\tau = \ln[\frac{2\mu}{6\frac{c_{x^i}y^i}{F} + 3(c^2+\sigma)}]^{\frac{3}{2}}. \qquad (4.3)$$

Since $c \neq constant$, then we have:

$$c_{x^i} = \frac{\partial c}{\partial x^i}.$$

On the other hand, for a Randser metric $F = \alpha + \beta$ the distorsion function $\tau$ is given by:

$$\tau = \ln[\frac{F}{\alpha(1-\|\beta\|_\alpha^2)}]^{\frac{3}{2}}. \qquad (4.4)$$

From the relations (4.3) and (4.4) we have:

$$\frac{2\mu}{6\frac{c_{x^i}y^i}{F} + 3(c^2+\sigma)} = \frac{F}{\alpha(1-\|\beta\|_\alpha^2)}. \qquad (4.5)$$

By (4.5) we get:

$$\alpha[3(c^2+\sigma) - 2\mu(1-\|\beta\|_\alpha^2)] = 6c_{x^i}y^i + \nu, \qquad (1.6)$$

where

$$\nu = 3(c^2+\sigma)b_i y^i \text{ and } b_i = \beta(\tfrac{\partial}{\partial x^i}).$$

Let

$$\zeta(x) := 3(c^2+\sigma) - 2\mu(1-\|\beta\|_\alpha^2), \qquad (4.7)$$

*and*

$$\eta_i(x) := 3[2c_{x^i} + (c^2+\sigma)b_i]. \qquad (4.8)$$

By using of (4.7) and (4.8), relation (4.6) is written in following form:

$$\zeta(x)\alpha = \eta_i(x)y^i, \qquad (4.9)$$

*or*

$$\zeta(x)a_{ij}(x)y^i y^j = \alpha\eta_i(x)y^i, \qquad (4.10)$$

*where* $\alpha = \sqrt{a_{ij}(x)y^i y^j}$. *By relation (4.10),we conclude:*

$$\zeta(x)a_{ij} = 0, \eta_i(x) = 0. \qquad (4.11)$$

$\alpha$ is positive then:

$$\zeta(x) = 0. \qquad (4.12)$$

*Therefore, we have:*

$$3(c^2+\sigma) = 2\mu(1-\|\beta\|_\alpha^2), \qquad (4.13)$$

and

$$2c_{x^i} = -(c^2+\sigma)b_i, \qquad (4.14)$$

which proves relation (4.2). □

## 5. REFERENCES


[1] H. Akbar-Zadeh, Sur Les Espaces De Finsler A Courbures Sectionnelles Constantes, Acad. Roy. Belg. Bull. Cl. Sci. (5), (1988), 271-322.

[2] H. Akbar-Zadeh, Initiation to global Finslerian Geometry, North-Holland Mathematical Library, 2006.

[3] D. Bao, S. S. Chern and Z. Shen, An Introduction to Riemann-Finsler Geometry, Springer-Verlage, 2000.

[4] B. Bidabad and A. Tayebi, A classification of some Finsler connections, Publ. Math. Debrecen, **71**. (2007), 253-260.

[5] X. Chen, X. Mo and Z. Shen, On the flag curvature of Finsler metrics of scalar flag curvature, J. London Mathematical Society. **68**. (2003), 762-780.

[6] X. Chen and Z. Shen, On Douglas metrics, Publ. Math. Debrecen, **66**. (2005), 503-512.

[7] B. Najafi, A. Tayebi and M. M. Rezaei, General Relatively Isotropic Mean Landsberg Finsler manifolds, Iranian Journal of Science and Technology, Transaction A, **29**. Number A3, (2005), 497-505.

[8] B. Najafi, Z. Shen and A. Tayebi , Finsler Metrics of Scalar Flag Curvature with Special Non-Riemannian Curvature Properties, Geometrie Dedicata, Preprint.

[9] T. Okada, On models of projectively flat Finsler spaces of constant negative curvature, Tensor, N. S. (1983), 117-123.

[10] Z. Shen, Differential Geometry of Spray and Finsler Spaces, Kluwer Academic Publishers, Dordrecht, 2001.

[11] Z. Shen, Lectures on Finsler Geometry, World Scientific, Sngapore (2001).

[12] Z. Shen, Landsberg Curvature, S-Curvature and Riemann Curvature, MSRI Publication Series, Cambridge University Press, 2004.

[13] Z. Shen, Finsler metrics with K=0 and S=0, Canadian J. Math. (2003), 112-132.

[14] Z. Shen, Two-dimensional Finsler metrics of constant curvature, Manuscripta Mathematica (2002), 349-366.

[15] A. Tayebi, E. Azizpour and E. Esrafilian, on a family of connections in Finsler geometry, Publ. Math. Debrecen, (preprint).