Elastic Post-buckling Stiffness of Rectangular Frames Using Perturbation Technique

Mohammad Z. Kabir [i]; Afshin Moslehi Tabar [ii]

ABSTRACT

The post-buckling behavior of rectangular frames in elastic domain is studied in depth. In analysis, unsymmetrical geometry, sway possibility and support conditions are considered in order to find their influences on load- deflection path and non-linear deformations. The static perturbation technique is used for analysis and discussion. The first, second and third order perturbation problem, as an accuracy measurement, for the frame are solved and the solutions are compared with previously published papers. The results reveal that the symmetric frame with sway movement, due to non-axial force in beam, has symmetric bifurcation point in the first order perturbation analysis. However, the post-buckling behavior of un-symmetric frames with or without sway is bifurcated in an asymmetric manner.

KEYWORDS

Post-Buckling Behavior, Perturbation Technique, Nonlinear Deformation, Bifurcation.

1. INTRODUCTION

The analysis of elastic post-buckling behavior of frames is rather complicated since it necessitates a geometrically non-linear bending. The post-critical analysis of elastic structures invariably requires the solution of a set of non-linear differential equations and is based solely on equilibrium equations. Koiter [1] analyzed the post critical behavior of asymmetric bifurcation by power series expansion approach. Roorda [2] conducted model experiments, which confirmed Koiter's calculations, particularly the reduction of a maximum load as a function of the initial imperfection in the form of load eccentricity. Roorda and Chilver [3] employed the perturbation method with power series expansions for analysis of non-linear equations. A nonlinear finite element analysis for frames is proposed by Care et al. [4]. The post-buckling behavior of perfect framed structures as well as the nonlinear geometric analysis of imperfect frames can be handled by this method. Since the nonlinear formulation is based on the incremental finite element procedure, frame elements have to be subdivided into a number of smaller segments to attain the required convergence. Bazant and Cedolin [5] adopted the stiffness matrix method and stability functions from linear theory and enhancing by additional terms to deal with the second order deformation in non-linear behavior. Ekhande et al. [6] studied the stability functions for three-dimensional beam-columns. They considered the effect of flexure on axial stiffness and the effect of axial force on flexural stiffness. The non-linearity of geometric stiffness matrix in their manipulation is limited to the first order. The present work employs the suggested static perturbation method, Roorda and Chilver [3], based on equilibrium and flexibility approach and by simple manipulations for higher order non–linear stability solutions for utilizing the behavior of rectangular frames with symmetric and non-symmetric geometry and applied loading. The main assumptions of coming analysis are based on frame continuity and straightness of slender members. The degree of instabilities at the onset of buckling in a frame is generally not high, and the purpose of the paper is not to suggest that this instability is important for practical purposes, but rather to show the application of non- linearity is extensively dependent on geometry of frame and the applied load configurations.

2. ANALYSIS

Consider a straight member ij of length L_{ij} and flexural stiffness EI_{ij} of a continuous, rigidly jointed plane frame.

[i] M. Z. Kabir is with the Department of Civil and Environmental Engineering, Amirkabir University of Technology, Tehran, Iran (e-mail: mzkabir@aut.ac.ir).
[ii] A. Moslehi Tabar is Ph.D. Candidate of Civil and Environmental Engineering, Amirkabir University of Technology, Tehran, Iran (e-mail: m7824929@aut.ac.ir).

The unbuckled states of the member, which is supposed to be axially rigid but flexible in bending, are sketched in Figure 1 along with the positive directions of the rotations (θ_{ij}, θ_{ji}), chord rotation (φ_{ij}), end moments (M_{ij} ,M_{ji}), end shears (Q_{ij}) and axial force (P_{ij}).

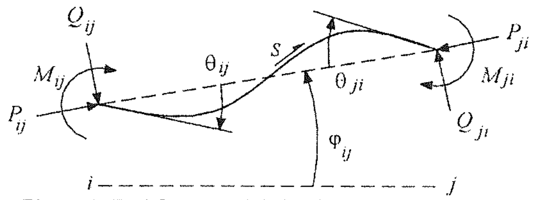

Figure 1: End forces and deflections of member ij.

Now, for a portion of the beam of length ds from end i along with the deflected member as shown in Figure 2, the equilibrium conditions require:

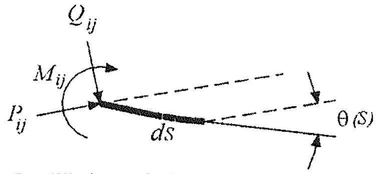

Figure 2: Equilibrium of a beam portion with length of ds.

$$EI_{ij}\frac{d\theta}{ds} + P_{ij}\int_0^s \sin\theta(s)\,ds + Q_{ij}\int_0^s \cos\theta(s)\,ds + M_{ij} = 0 \tag{1}$$

By differentiation of above equation with respect to s

$$EI_{ij}\frac{d^2\theta}{ds^2} + P_{ij}\sin\theta(s) + Q_{ij}\cos\theta(s) = 0 \tag{2}$$

From equilibrium of Figure 1, the following relations can be written

$$Q_{ij} = \frac{M_{ij} + M_{ji}}{L_{ij} - e_{ij}};\quad Q_{ij} = Q_{ji};\quad P_{ij} = P_{ji} \tag{3}$$

e_{ij} represents shortening of the member length due to buckling, and its value is obtained as follow

$$e_{ij} = L_{ij} - \int_0^L \cos\theta(s)\,ds \tag{4}$$

The boundary conditions of nonlinear differential equation are as follow:

$$\left.\frac{d\theta}{ds}\right|_{s=0} = -\frac{M_{ij}}{EI_{ij}};\qquad \left.\frac{d\theta}{ds}\right|_{s=L_{ij}} = \frac{M_{ji}}{EI_{ij}} \tag{5}$$

For simplicity, the dimensionless quantities, $p_i, p_{ij}, q_{ij}, m_{ij}, n_{ij}, x, s_{ij}, A_{ij}, \lambda_{ij}$ are utilized in the following as defined in Appendix.

By appropriate substituting of normalized relations in (A-1a) to (A-1i), see Appendix, the dimensionless form of (2) is rewritten

$$\theta''(x) + p_{ij}\sin\theta(x) - q_{ij}\cos\theta(x) = 0 \tag{6}$$

with the boundary conditions:

$$\theta'(0) = -m_{ij};\qquad \theta'(1) = m_{ji} \tag{7}$$

and equilibrium equations as follow:

$$q_{ij} = \frac{m_{ij} + m_{ji}}{1 - \lambda_{ij}};\quad q_{ij} = q_{ji};\quad p_{ij} = p_{ji} \tag{8}$$

In this study the static perturbation technique is applied for the solution of nonlinear differential (6). For this respect, $\theta, p_{ij}, q_{ij}, m_{ij}$ and m_{ij} are expressed in power series form as functions of some parameter ε which increases from zero along an equilibrium path leaving a known equilibrium point, see details in Appendix..

Before deriving the perturbation equations, it is convenient to eliminate q_{ij} using the equilibrium equation

$$q_{ij}(\varepsilon) = \frac{m_{ij}(\varepsilon) + m_{ji}(\varepsilon)}{1 - \lambda_{ij}(\varepsilon)} \tag{9}$$

where λ_{ij} is the dimensionless flexural shortening of the member which can be expressed as a series function of ε as

$$\lambda_{ij}(\varepsilon) = \frac{\varepsilon^2}{2!}\int_0^1 \dot{\theta}^2 dx + \frac{\varepsilon^3}{2}\int_0^1 \dot{\theta}\ddot{\theta} dx + \cdots \tag{10}$$

By substituting (A-2) and (10) into (9), and equating terms of the same power in ε, yields

$$\dot{q}_{ij} = \dot{m}_{ij} + \dot{m}_{ji} \tag{11a}$$

$$\ddot{q}_{ij} = \ddot{m}_{ij} + \ddot{m}_{ji} \tag{11b}$$

$$\dddot{q}_{ij} = \dddot{m}_{ij} + \dddot{m}_{ji} + 3(\dot{m}_{ij} + \dot{m}_{ji})\int_0^1 \dot{\theta}^2 dx \tag{11c}$$

Finally, upon substitution of (A-2) and (9) in (6) and equating coefficients of same power of ε, the following infinite system of perturbation equations are obtained

$$\dot{\theta}'' + p_{ij}\dot{\theta} = \dot{m}_{ij} + \dot{m}_{ji} \tag{12a}$$

$$\ddot{\theta}'' + p_{ij}\ddot{\theta} = \ddot{m}_{ij} + \ddot{m}_{ji} - 2\dot{p}_{ij}\dot{\theta} \tag{12b}$$

$$\ddot{\theta}'' + p_{ij}\ddot{\theta} = \dddot{m}_{ij} + \dddot{m}_{ji} + 3(\dot{m}_{ij} + \dot{m}_{ji})\int_0^1 \dot{\theta}^2 dx -$$
$$3[\dot{p}_{ij}\ddot{\theta} + \ddot{p}_{ij}\dot{\theta} + (\dot{m}_{ij} + \dot{m}_{ji})\ddot{\theta}^2] + p_{ij}\dot{\theta}^3 \quad (12c)$$

whose respective boundary conditions become

At x=0 :

$$\dot{\theta}' = -\dot{m}_{ij};\quad \ddot{\theta}' = -\ddot{m}_{ij};\quad \dddot{\theta}' = -\dddot{m}_{ij} \quad (13a)$$

At x=1 :

$$\dot{\theta}' = -\dot{m}_{ji};\quad \ddot{\theta}' = -\ddot{m}_{ji};\quad \dddot{\theta}' = -\dddot{m}_{ji} \quad (13b)$$

The important point to note is that this infinite system can be solved recursively. For example, the determination of the nth order solution requires a knowledge of the solution of the 1st up to, and including, the (n-1)th order problems. At each step in the analysis, the solution of a non-homogeneous differential equation of the type

$$\phi''(x) + p\phi(x) = v(x) \quad (14)$$

is subjected to the boundary conditions

$$\phi'(0) = -\mu_{ij};\qquad \phi'(1) = \mu_{ji} \quad (15)$$

is required. Using common methods for solution of the differential (14), differential (12) can be solved.

A. First order solution

The solution of (12-a) is as follow, [7]:

$$\dot{\theta} = \frac{1}{p_{ij}}(\dot{m}_{ij} + \dot{m}_{ji}) - \frac{1}{\sqrt{p_{ij}}}\dot{m}_{ij}\cot\sqrt{p_{ij}} +$$
$$\dot{m}_{ji}\csc\sqrt{p_{ij}}\cos\sqrt{p_{ij}}x - \frac{1}{\sqrt{p_{ij}}}\dot{m}_{ij}\sin\sqrt{p_{ij}}x \quad (16)$$

By substituting x=0, x=1 in (16), $\dot{\theta}_{ij}$ and $\dot{\theta}_{ji}$ are determined, as

$$\dot{\theta}_{ij} = f_{ij}\dot{m}_{ij} + g_{ij}\dot{m}_{ji} \quad (17a)$$

$$\dot{\theta}_{ji} = g_{ij}\dot{m}_{ij} + f_{ij}\dot{m}_{ji} \quad (17b)$$

Coefficients of f_{ij} and g_{ij} are given in (18)

$$f_{ij} = \frac{1}{p_{ij}}[1 - \sqrt{p_{ij}}\cot\sqrt{p_{ij}}] \quad (18a)$$

$$g_{ij} = \frac{1}{p_{ij}}[1 - \sqrt{p_{ij}}\csc\sqrt{p_{ij}}] \quad (18b)$$

When axial force is zero (such as beams) extreme values of f_{ij} and g_{ij} are equal to

$$f_{ij} = \frac{1}{3};\qquad g_{ij} = -\frac{1}{6} \quad (19)$$

B. Second order solution

The solution of (12b) is as follow:

$$\ddot{\theta} = \frac{1}{p_{ij}}(\ddot{m}_{ij} + \ddot{m}_{ji}) - \frac{1}{\sqrt{p_{ij}}}[\ddot{m}_{ij}\cot\sqrt{p_{ij}} +$$
$$\ddot{m}_{ji}\csc\sqrt{p_{ij}}]\cos\sqrt{p_{ij}}x - \frac{1}{\sqrt{p_{ij}}}\ddot{m}_{ij}\sin\sqrt{p_{ij}}x -$$
$$\frac{\dot{p}_{ij}}{p_{ij}}\{\frac{2}{p_{ij}}(\dot{m}_{ij} + \dot{m}_{ji}) - [\dot{m}_{ij}\cot\sqrt{p_{ij}} +$$
$$\dot{m}_{ji}\csc\sqrt{p_{ij}}]\times[x\sin\sqrt{p_{ij}}x + (\cot\sqrt{p_{ij}} +$$
$$\frac{1}{\sqrt{p_{ij}}})\cos\sqrt{p_{ij}}x] - \frac{1}{\sqrt{p_{ij}}}\dot{m}_{ij}[\sin\sqrt{p_{ij}}x -$$
$$\sqrt{p_{ij}}x\cos\sqrt{p_{ij}}x + \sqrt{p_{ij}}\cos\sqrt{p_{ij}}x]\} \quad (20)$$

Substituting x=0 and x=1 in (20), $\ddot{\theta}_{ij}$ and $\ddot{\theta}_{ji}$ are determined, respectively

$$\ddot{\theta}_{ij} = f_{ij}\ddot{m}_{ij} + g_{ij}\ddot{m}_{ji} + 2\dot{p}_{ij}(F_{ij}\dot{m}_{ij} + G_{ij}\dot{m}_{ji}) \quad (21a)$$

$$\ddot{\theta}_{ji} = g_{ij}\ddot{m}_{ij} + f_{ij}\ddot{m}_{ji} + 2\dot{p}_{ij}(G_{ij}\dot{m}_{ij} + F_{ij}\dot{m}_{ji}) \quad (21b)$$

where

$$F_{ij} = -\frac{1}{2p_{ij}}[\frac{2}{p_{ij}} - \cot\sqrt{p_{ij}}(\cot\sqrt{p_{ij}} + \frac{1}{\sqrt{p_{ij}}}) - 1]$$

$$G_{ij} = -\frac{1}{2p_{ij}}[\frac{2}{p_{ij}} - \csc\sqrt{p_{ij}}(\cot\sqrt{p_{ij}} + \frac{1}{\sqrt{p_{ij}}})] \quad (22)$$

in the absence of axial force

$$F_{ij} = \frac{1}{45};\qquad G_{ij} = -\frac{7}{360} \quad (23)$$

Finally, considering (A-2h) and (10), one can write

$$\ddot{\lambda}_{ij} = (\dot{m}_{ij} + \dot{m}_{ji})F_{ij} + 2\dot{m}_{ij}\dot{m}_{ji}G_{ij} \quad (24)$$

3. Stability of Frames

Frames generally consist of a number of beam-columns. As a result, to study buckling of frames, beam-column relations are applied, however, boundary conditions are no longer as simple as those of beam-columns. To analyze frames, a series of extra equations are needed. Followings are short reviewing of these equations.

A. Equilibrium of a joint

Consider two transverse members which are joined at the point (i) as shown in Figure (3).

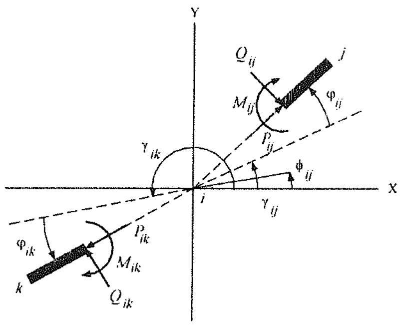

Figure 3. Positive forces and deformations in a joint

Two equilibrium equations in x and y directions are written as

$$-c_{ij}\cos\phi_i - \sum n_{ij}p_{ij}\cos(\gamma_j+\phi_{ij}) - \sum n_{ij}q_{ij}\sin(\gamma_j+\phi_{ij}) = 0 \qquad (25a)$$

$$-c_{ij}\sin\phi_i - \sum n_{ij}p_{ij}\sin(\gamma_j+\phi_{ij}) - \sum n_{ij}q_{ij}\cos(\gamma_j+\phi_{ij}) = 0 \qquad (25b)$$

$$s_{ij}m_{ij} + s_{ik}m_{ik} + \cdots = 0 \qquad (25c)$$

B. Continuity of joint rotation

Continuity of joint rotation, as shown in Figure 4, is written as:

$$\theta_{ij} - \varphi_{ij} = \theta_{ik} - \varphi_{ik} = \theta_{il} - \varphi_{il} \qquad (26)$$

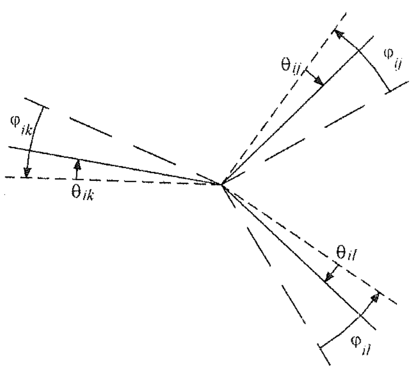

Figure 4. Rotation in a joint

C. Deformation compatibility equation in a closed loop

Consider a closed loop as shown in Figure 5 solid and dashed lines show the undeformed and deformed shape, respectively. Geometry of loops indicate

$$\sum A_i(1-\lambda_i)\cos(\gamma_i+\varphi_i) = 0 \qquad (27a)$$

$$\sum A_i(1-\lambda_i)\sin(\gamma_i+\varphi_i) = 0 \qquad (27b)$$

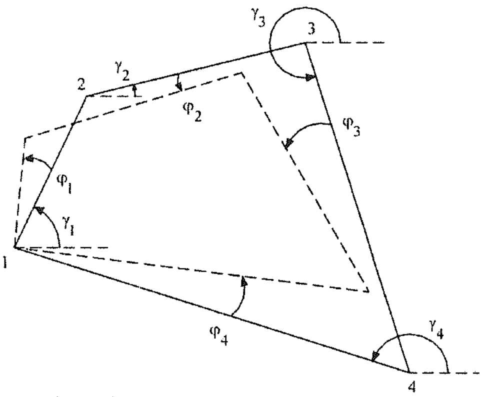

Figure 5. Deformed and undeformed configuration of a closed frame.

In the following, the post-buckling behavior of a number of frames with typical specifications including: sway possibility, boundary conditions, frame symmetry or asymmetry and relative stiffness of beams and columns is investigated by means of the method described earlier.

4. A FRAME WITH SIDE-SWAY

Consider a side-sway permitted frame that is generally asymmetric, as shown in Figure 6. The asymmetry was induced by taking different columns length. Conventional positive deformations are shown in the figure.

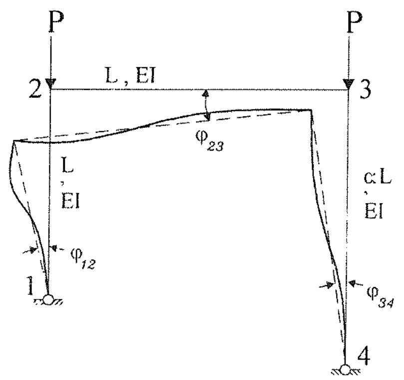

Figure 6: Frame with side-sway

By making use of compatibility equations of the closed loop as well as equilibrium conditions of the joints 2 and 3, following equation is resulted in,

$$\dot{\varphi}_{12} = \alpha\dot{\varphi}_{34} = \frac{\alpha}{(1+\alpha)k}[\dot{m}_{21} + \frac{1}{\alpha^2}\dot{m}_{34}] \qquad (28a)$$

$$\ddot{\varphi}_{12}+\ddot{\varphi}_{34}=\frac{1}{k}[\ddot{m}_{21}+\frac{1}{\alpha^2}\ddot{m}_{34}-2\dot{p}_{12}\dot{\varphi}_{12}$$

$$-\frac{2}{\alpha^2}\dot{p}_{34}\dot{\varphi}_{34}] \quad (28b)$$

In addition, from continuity conditions of the joints 2 and 3, two series of key equations can be readily achieved, as follows,

$$\dot{\theta}_{12}-\dot{\varphi}_{12}=\dot{\theta}_{23}\,; \qquad \ddot{\theta}_{12}-\ddot{\varphi}_{12}=\ddot{\theta}_{23} \quad (29a)$$

$$\dot{\theta}_{34}-\dot{\varphi}_{34}=\dot{\theta}_{32}\,; \qquad \ddot{\theta}_{34}-\ddot{\varphi}_{34}=\ddot{\theta}_{32} \quad (29b)$$

A. First order analysis

End moment-end rotation relationships corresponding to the first order analysis are

$$\dot{\theta}_{21}=f_{12}\dot{m}_{21} \quad (30a)$$

$$\dot{\theta}_{23}=f_{23}\dot{m}_{23}+g_{23}\dot{m}_{32}=-\frac{1}{3}\dot{m}_{21}+\frac{1}{6\alpha}\dot{m}_{34} \quad (30b)$$

$$\dot{\theta}_{32}=g_{23}\dot{m}_{23}+f_{23}\dot{m}_{32}=\frac{1}{6}\dot{m}_{21}-\frac{1}{3\alpha}\dot{m}_{34} \quad (30c)$$

$$\dot{\theta}_{34}=f_{34}\dot{m}_{34} \quad (30d)$$

By substituting (30), (28) into (29)

$$[f_{12}+\frac{1}{3}-\frac{\alpha}{(1+\alpha)k}]\dot{m}_{21}-[\frac{1}{6\alpha}+\frac{1}{\alpha(1+\alpha)k}]\dot{m}_{34}=0 \quad (31a)$$

$$-[\frac{1}{6}+\frac{1}{(1+\alpha)k}]\dot{m}_{21}+$$

$$[f_{34}+\frac{1}{3\alpha}-\frac{1}{\alpha^2(1+\alpha)k}]\dot{m}_{34}=0 \quad (31b)$$

in which coefficients f_{12},f_{34} are functions of load factor k. Non-trivial solutions are possible when the determinant of the coefficients in (31) vanishes.

$$[f_{12}+\frac{1}{3}-\frac{\alpha}{(1+\alpha)k}]\,[f_{34}+\frac{1}{3\alpha}-\frac{1}{\alpha^2(1+\alpha)k}]-$$

$$[\frac{1}{6\alpha}+\frac{1}{\alpha(1+\alpha)k}]\,[\frac{1}{6}+\frac{1}{(1+\alpha)k}]=0 \quad (32)$$

The above equation is a characteristic equation of critical load which its smallest root is critical load of the frame,. The buckling load parameters (k) of the frame for various values of α, i.e. 0,1 and 2 are equal to 13.88, 1.82 and 1.22, respectively.

B. Second order analysis

End moment-end rotation relationships corresponding to the second order analysis are

$$\ddot{\theta}_{21}=f_{12}\ddot{m}_{21}+2\dot{p}_{12}(F_{12}\dot{m}_{21}) \quad (33a)$$

$$\ddot{\theta}_{23}=-\frac{1}{3}\ddot{m}_{21}+\frac{1}{6\alpha}\ddot{m}_{34}+2\dot{p}_{23}(-\frac{1}{45}\dot{m}_{21}+\frac{7}{360\alpha}\dot{m}_{34}) \quad (33b)$$

$$\ddot{\theta}_{32}=\frac{1}{6}\ddot{m}_{21}-\frac{1}{3\alpha}\ddot{m}_{34}+2\dot{p}_{23}(\frac{7}{360}\dot{m}_{21}-\frac{1}{45\alpha}\dot{m}_{34}) \quad (33c)$$

$$\ddot{\theta}_{34}=f_{34}\ddot{m}_{34}+2\dot{p}_{34}(F_{34}\dot{m}_{34}) \quad (33d)$$

Upon substitution of (33) and (28) in (29) results in:

$$[f_{12}+\frac{1}{3}-\frac{\alpha}{(1+\alpha)k}]\ddot{m}_{21}-[\frac{1}{6\alpha}+\frac{1}{\alpha(1+\alpha)k}]\ddot{m}_{34}=U(\dot{k},\dot{m}_{21}) \quad (34a)$$

$$-[\frac{1}{6}+\frac{1}{(1+\alpha)k}]\ddot{m}_{21}+[f_{34}+\frac{1}{3\alpha}-\frac{1}{\alpha^2(1+\alpha)k}]\ddot{m}_{34}=V(\dot{k},\dot{m}_{21}) \quad (34b)$$

Coefficients of (34) are exactly similar to those of (31) that the determinant of coefficients is zero in critical load. If the solutions $\{\ddot{m}_{21},\ddot{m}_{34}\}$ are to be finite, then by Cramer's rule, the determinant obtained by replacing any column of coefficient by $\{U,V\}$ must also be zero.

$$\begin{vmatrix} f_{12}+\frac{1}{3}-\frac{\alpha}{(1+\alpha)k} & U \\ -(\frac{1}{6}+\frac{1}{(1+\alpha)k}) & V \end{vmatrix}=0 \quad (35)$$

To illustrate the implementation of the static perturbation method, the post-buckling analysis of the frame shown in Figure 6 has been involved in Appendix and in the following just the results are mentioned.

Numerical results of (32) and (35) for various values of α are given in Table 1,and also they are depicted in Figure 7.

TABLE I

CRITICAL LOADS AND INITIAL SLOPE OF POST-BUCKLING PATHS FOR DIFFERENT VALUES OF α

α	k_{cr}	$\dot{k}$	$\dot{k}/k_{cr}$
0	13.88	5.238 $\dot{\theta}_{21}$	0.381 $\dot{\theta}_{21}$
1	1.82	0	0
2	1.22	-1.57 $\dot{\theta}_{21}$	- 1.29 $\dot{\theta}_{21}$

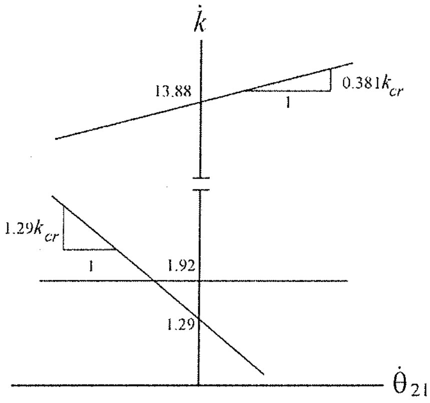

Figure 7. Post-buckling paths for different values of α

The results, in the case of α =0, coincide to the results obtained by Roorda (1965). Because the buckling directions of the frame for α =0, 2 are different, their post-buckling slopes are on the contrary. It should be noted that in the case of α =1, the first order axial load in beam becomes zero, consequently, the post-buckling path in this case is symmetric bifurcation, whereas in the other cases, the first order axial load in the beam 2-3 is exist, so post-buckling paths are asymmetric.

5. A FRAME WITHOUT SIDE-SWAY

A. Hinged supports

Consider a symmetric side-sway prevented frame in which the length and flexural stiffness of beam are α , β times those of columns, respectively. Simply supports are considered for the frame. Initial frame and conventional positive deformations are shown in Figure 8.

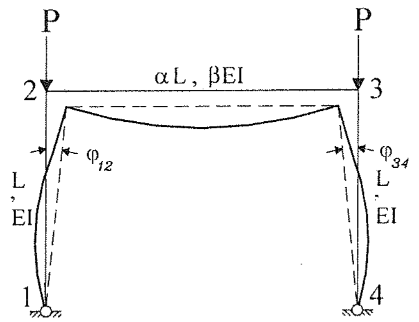

Figure 8. One story portal frame without side-sway (hinged supports)

The procedure of analysis is similar to previous example, except that symmetry properties can be used. In the following, merely final results are mentioned. Characteristic equation is obtained as, [7]:

$$f_{12} + \frac{\alpha}{2\beta} = 0 \qquad (36)$$

and, the equation of post-buckling path becomes

$$\dot{k} = \frac{-\alpha^3}{16\beta^2 f_{12} F_{12}} \dot{\theta}_{21}; \quad (k = k_{cr}) \qquad (37)$$

TABLE 2
CRITICAL LOADS AND INITIAL SLOPE OF POST-BUCKLING CURVES (CASE II-A)

	α	k_{cr}	$\dot{k}/\dot{\theta}_{21}$
β =1	1	12.85	0.54
	1.5	12.10	0.705
	2	11.60	0.75
β =2	1	14.70	0.70
	1.5	13.60	0.96
	2	12.85	1.07

Numerical results of (36) and (37) for various values of α and β are given in Tables 2 and they are also depicted in Figure 9.

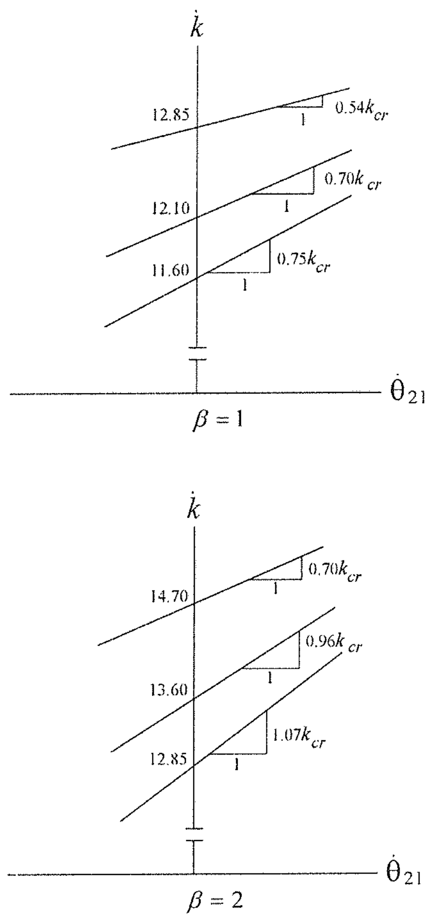

Figure 9. Post-buckling paths for different values of α and β (Case II-A)

B. Fixed supports

Now consider the previous example (case A) with fixed supports, Figure 10, the only difference between these two cases is that rotations of supports are equal to zero.

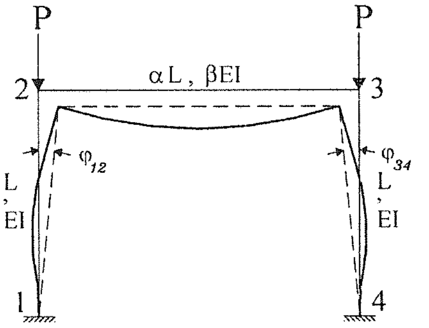

Figure 10. One story portal frame without side-sway (fixed supports)

Characteristic equation of critical load is

$$f^2_{12} - g^2_{12} + \frac{\alpha}{2\beta} f_{12} = 0 \tag{38}$$

and the of post-buckling path becomes

$$\dot{k} = \frac{1}{24}\frac{\alpha^3}{\beta^2}\frac{f_{12}}{f^2_{12} - g^2_{12}} \times \frac{\frac{3}{2} - \frac{g_{12}}{f_{12}}}{\frac{g_{12}}{f^2_{12}}(G_{12}f_{12} - F_{12}g_{12}) - \frac{1}{f_{12}}(F_{12}f_{12} - G_{12}g_{12})}\dot{\theta}_{21} \tag{39}$$

Numerical results of (38), (39) for various values of α and β are given in Table 3, and also they are depicted in Figure 11.

TABLE 3
CRITICAL LOADS AND INITIAL SLOPE OF POST-BUCKLING CURVES (CASE II-B)

	α	k_{cr}	$\dot{k}/\dot{\theta}_{21}$
$\beta=1$	1	25.2	-0.60
	1.5	23.8	-1.40
	2	23.0	-2.26
$\beta=2$	1	28.4	-0.07
	1.5	26.4	-0.56
	2	25.2	-1.22

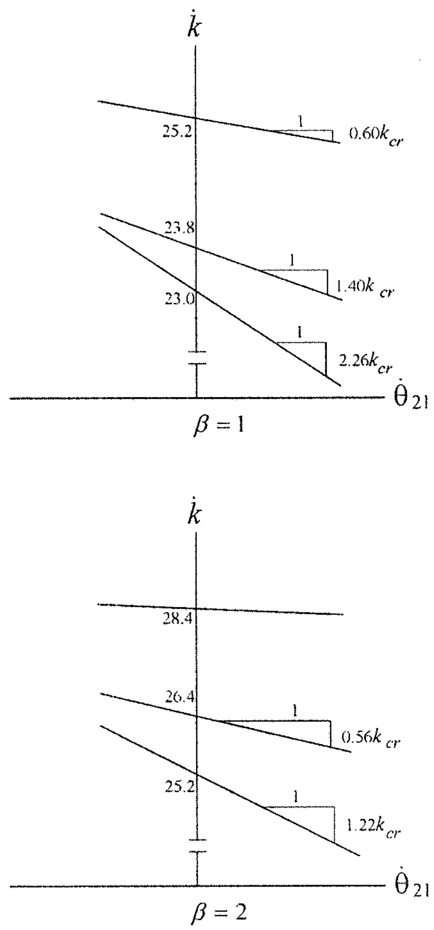

Figure 11. Post-buckling paths for different values of α and β (Case II-B)

6. CONCLUSIONS

In this paper, the non-linear post-buckling behavior of rectangular plane frames is analyzed using the standard procedure of the static perturbation technique. The solution of a highly non-linear problem is reduced to the recursive solution of an infinite set of linear problems. Based on parametric studies made in this study, the following remarkable conclusions are pointed out:

1. In the stability of symmetrical portal frames with permitted sway, due to zeroes of axial force in the beam resulted from the first order non-linear analysis, the post buckling of frame becomes unstable. But, at the second order analysis of the beam, axial force is exist and make stable post-buckling of the frame.
2. The post-buckling behavior of non-symmetrical frames is stable, due to tensile force in the beam.
3. In portal frames, when sway is not permitted, the potential loading of frames beyond buckling is reduced due to compressive force in the beam and decreasing of total frame stiffness.
4. Reducing ratio of S_b/S_c, S_b and S_c are beam and column stiffness, respectively, improves the post-buckling

behavior in sway permitted frames and weakens this capacity in non-permitted sway frames.

5. The influence of support conditions on the post buckling stiffness of frames is revealed that in sway-permitted frames, the simple supports reactions improve post-buckling stiffness rather than fixed supports.

7. APPENDIX

The dimensionless quantities, $p_i, p_{ij}, q_{ij}, m_{ij}, n_{ij}, x, s_{ij}, A_{ij}, \lambda_{ij}$, are defined as follows,

$$P_i = k_i(\frac{L^2}{EI})_{REF}$$; (P_i is external force and k_i is load factor) (A-1a)

$$p_{ij} = P_{ij}(\frac{L^2}{EI})_{ij} \quad \text{(A-1b)}$$

$$q_{ij} = Q_{ij}(\frac{L^2}{EI})_{ij} \quad \text{(A-1c)}$$

$$m_{ij} = M_{ij}(\frac{L}{EI})_{ij} \quad \text{(A-1d)}$$

$$n_{ij} = \frac{(EI/L^2)_{ij}}{(EI/L^2)_{REF}} \quad \text{(A-1e)}$$

$$x = s / L_{ij} \quad \text{(A-1f)}$$

$$s_{ij} = \frac{(EI/L)_{ij}}{(EI/L\ )_{REF}} \quad \text{(A-1g)}$$

$$A_{ij} = \frac{L_{ij}}{L_{REF}} \quad \text{(A-1h)}$$

$$\lambda_{ij} = \frac{e_{ij}}{L_{ij}} \quad \text{(A-1i)}$$

To apply the static perturbation technique for the solution of (6), suppose that $\theta, p_{ij}, q_{ij}, m_{ij}$, and m_{ji} can be expressed in power series form as functions of some parameter ε which increases from zero along an equilibrium path leaving a known equilibrium point. Starting from the un-deflected configuration, and allowing the possibility of an unbuckled configuration at finite values of p_{ij}, one can set:

$$p_{ij}(\varepsilon) = p_{ij} + \varepsilon\,\dot{p}_{ij} + \frac{\varepsilon^2}{2!}\ddot{p}_{ij} + \cdots \quad \text{(A-2a)}$$

$$\theta_{ij}(\varepsilon) = \varepsilon\dot{\theta}_{ij} + \frac{\varepsilon^2}{2!}\ddot{\theta}_{ij} + \cdots \quad \text{(A-2b)}$$

$$q_{ij}(\varepsilon) = \varepsilon\dot{q}_{ij} + \frac{\varepsilon^2}{2!}\ddot{q}_{ij} + \cdots \quad \text{(A-2c)}$$

$$m_{ij}(\varepsilon) = \varepsilon\dot{m}_{ij} + \frac{\varepsilon^2}{2!}\ddot{m}_{ij} + \cdots \quad \text{(A-2d)}$$

$$m_{ji}(\varepsilon) = \varepsilon\,\dot{m}_{ji} + \frac{\varepsilon^2}{2!}\ddot{m}_{ji} \cdots \quad \text{(A-2e)}$$

$$w_{ij}(\varepsilon) = k + \varepsilon\,\dot{k}_{ij} + \frac{\varepsilon^2}{2!}\ddot{k}_{ij} + \cdots \quad \text{(A-2f)}$$

$$\varphi_{ij}(\varepsilon) = \varepsilon\,\dot{\varphi}_{ij} + \frac{\varepsilon^2}{2!}\ddot{\varphi}_{ij} + \cdots \quad \text{(A-2g)}$$

$$\lambda_{ij}(\varepsilon) = \varepsilon\,\dot{\lambda}_{ij} + \frac{\varepsilon^2}{2!}\ddot{\lambda}_{ij} + \cdots \quad \text{(A-2h)}$$

ε is a small parameter and becomes zero at the unbuckled configuration. It is increased along load-deflection path. In above equations, dots indicate differentiation with respect to ε.

8. REFERENCES

[1] Koiter W.T., " Post-buckling analysis of a simple two-bar frame", *Recent progress in Applied Mechanics*, edited by Bretram Brobergs et al, Folke Odqvist Volume, p.337, 1967.

[2] Roorda J., "Stability of structures with small imperfections," *J. Engng. Mech. Div. ASCE 91 EMD*, 1965.

[3] Roorda J., and Chilver A.H, " Frame buckling: an illustration of the perturbation technique," *Int. J. Non-Linear Mechanics*, Vol. 5, pp. 235-246, 1970.

[4] Care, R. F., Lawther, R. E., and Kabaila, A. P., "Finite element post-buckling analysis for frames" *Int. J. for Numerical Methods in Engng*, Vol. 11, pp. 833-849, 1977.

[5] Bazant Z.P., and Cedolin L., "Initial post critical analysis of asymmetric bifurcation in Frames," *J. Struct. Engng*, Vol. 113, No. 7, pp. 1501-1517, 1989.

[6] Ekhande S. G., Selvappalam M., and Madugula K. S., "Stability functions for three-dimensional beam-columns," *J. Struct. Engng*, Vol. 115, No. 2, pp. 467-479, 1989.

[7] Moslehi Tabar A., "Elastic post-buckling stability of rectangular frames," M. Sc. Thesis, Dept. of Civil Eng., Amirkabir Univ. of Technology, Tehran, Iran (in Persian), 1999.