Fixed Point Properties For Non-expansive Representations Of Topological Semigroups

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ABSTRACT

In this paper, first we extend theorem of Goebel-Schoneberg for non-expansive representations of a left amenable semitopological semigroup on a nonempty subset D of a Hilbert space H. Then we state and prove a common absolute fixed point theorem for these semigroups.

KEYWORDS

Nonexpansive mapping, fixed point, semitopological semigroup, invariant mean, Hilbert space.

1. INTRODUCTION

Let S be a semitopological semigroup with identity, that is, S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \rightarrow a.s$ and $s \rightarrow s.a$ from S into S are continuous. Let D be a non-empty subset of a Hilbert Space H and $\Im = \{T_t : t \in S\}$ be a continuous representation of S as mappings from D into D, that is, (i) $T_e = I$; (ii) $T_{st} = T_s T_t$ for all $s, t \in S$; (iii) The map $S \times C \rightarrow S$ defined by $(t, s) \rightarrow T_t x$, $t \in S$, $x \in C$, is continuous when $S \times C$ has the product topology. Let $\Im$ be a continuous representation of the semigroup S and each T_s, $s \in S$, be a non-expansive self-map of D, i.e, $\|T_s x - T_s y\| \leq \|x - y\|$, for all $x, y \in D$ and $s \in S$. Then $\Im$ is called a continuous nonexpansive semigroup on D. In this paper, first we extend theorem of Goebel-Schoneberg [3] for the continuous non-expansive representations of a left amenable semitopological semigroup on a nonempty subset D of a Hilbert space H and then we consider the concept of absolute fixed point, and prove a common absolute fixed point theorem for this semigroup.

2. PRELIMINARIES

All topologies in this paper are assumed to be Hausdorff. Given a non-empty set S, we denote by $\ell^\infty(S)$ the Banach space of all bounded complex valued functions on S with supremum norm. Let S be a semigroup. A subspace X of $\ell^\infty(S)$ is left (resp. right) translation invariant if $\ell_a(X) \subset X$ (resp. $r_a(X) \subset X$) for all $a \in S$, where $\ell_a(f)(s) = f(as)$ and $r_a(f)(s) = f(sa)$, $s \in S$. If S is a semitopological semigroup, we denote by $CB(S)$ the closed subalgebra of $\ell^\infty(S)$ consisting of continuous functions. Let $LUC(S)$ (resp. $RUC(S)$) be the space of left (resp. right) uniformly continuous functions on S, that is, all $f \in CB(S)$ such that the mapping from S into $CB(S)$ defined by $s \rightarrow \ell_s f$ (resp. $s \rightarrow r_s f$) is continuous when $CB(S)$ has the supremum norm topology. It is well-known that $LUC(S)$ and $RUC(S)$ are left and right translation

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invariant closed subalgebras of $CB(S)$ respectively, containing constants [1]. Note that when S is a topological group, then $LUC(S)$ is precisely the space of left uniformly continuous functions on S defined in [4]. Now suppose X be a subspace of $\ell^{\infty}(S)$ containing constants. Then $\mu \in X^*$ is called a mean on X if $\|\mu\| = \mu(1) = 1$. Moreover, let X be right invariant. Then, a mean μ on X is right invariant if $\mu(r_s f) = \mu(f)$ for all $s \in S$ and $f \in X$. Similarly, we can define left invariant means. μ is called an invariant mean if it is left and right invariant. The value of a mean μ at $f \in X$ will also be denoted by $\mu(f)$ or $\langle \mu, f \rangle$. A net $\{\mu_\alpha\}$ on $RUC(S)$ is called asymptotically invariant [10] if for each $f \in RUC(S)$ and $a \in S$, $\mu_\alpha(r_a f) - \mu_\alpha(f) \to 0$ and $\mu_\alpha(\ell_a f) - \mu_\alpha(f) \to 0$.

Let μ be a mean on X and E be a Banach space, $\phi : S \to E$ be a bounded function, and K be a closed convex subset of E. Suppose for each $x \in K$, the real-valued function f_x on S defined by

$f_x(t) = \|\phi(t) - x\|^2$, for all $t \in S$ belongs to X. Then set, $r(x) = \langle \mu, f_x \rangle$, for all $x \in K$ and define $r = \inf_{x \in K} r(x)$ and $M_\mu = \{y \in K : r(y) = r\}$.

Lemma 2.1. The non-negative real-valued function r on K defined as above is continuous, convex and $r(x_n) \to \infty$ as $x_n \to \infty$. If E is reflexive or K is weakly compact, then M_μ is a non-empty closed convex subset of K. Furthermore, if E is a Hilbert Space, then M_μ contains a unique element y such that $r + \|y - x\| \le r(x)$ for all $x \in K$ [8].

Definition 2.2. Let D be a nonempty subset of a Hilbert space H and $\Im = \{T_t : t \in S\}$ be a non-expansive semigroup on S. We call a point $p \in D$ a common absolute fixed point for semigroup $\Im$ if there exists a non-expansive extension semigroup $\tilde{\Im} = \{\tilde{T}_t : t \in S\}$ on $D \cup \{p\}$ such that $\tilde{T}_t p = p$ for all $t \in S$ and if p is a fixed point for every non-expansive extension of $\Im$ to the union of D and a subset of H containing p. We denote by $AF(\Im)$ the set of all common absolute fixed points of semigroup $\Im$ in p [2].

Remark 2.3. Let D be a subset of a real Hilbert space H, $x \in H$ and let $\Im = \{T_t : t \in S\}$ be a non-expansive semigroup on D. Suppose $\{T_t z : t \in S\}$ is bounded for some $z \in D$. Then, the functions $f_x(t) = \|T_t z - x\|^2$ and $g_x(t) = \langle T_t x, x \rangle$ are in $RUC(S)$, [7] (see also [6]). Now if μ is a mean on $RUC(S)$, by the Riesz representation theorem, there exists $z_\mu \in H$ such that $\langle \mu, g_x(.) \rangle = \langle z_\mu, x \rangle$ for each $x \in H$ [9].

Lemma 2.4. Let S be a semitopological semigroup, D be a non-empty subset of a Hilbert space H and $\Im = \{T_t : t \in S\}$ be a non-expansive continuous representation of a semitopological semigroup S on D. Suppose $\{T_t z : t \in S\}$ is bounded for some $z \in D$ and $RUC(S)$ has a left invariant mean μ. Then $z_\mu \in H$, and $\|T_{st} x - z_\mu\| \le \|T_t x - z_\mu\|$ for all $x \in D$ and $s, t \in S$.

Proof: Let μ be a LIM on $RUC(S)$. By Remark 2.3 for $x \in H$, the function $f_x(t) = \|T_t z - x\|^2$ and $g_x(t) = \langle T_t z, x \rangle$, $t \in S$ are in $RUC(S)$. Let M_μ, $r(x)$ and r be as above for the convex set H. By Lemma 2.1, M_μ contains a unique element y such that $r + \|y - x\| \le r(x)$ for all $x \in H$. Now let $x \in H$ and $t \in S$. Then,

$$\|z_\mu - x\|^2 = \|T_t z - x\|^2 - \|T_t z - z_\mu\|^2 - 2\langle T_t z - z_\mu, z_\mu - x \rangle$$

So

$$0 \le \|z_\mu - x\|^2 = \langle \mu, f_x(.) \rangle - \langle \mu, f_{z_\mu}(.) \rangle - 2\langle \mu, g_{z_\mu - x}(.) \rangle + 2\langle z_\mu, z_\mu - x \rangle$$

$$= \langle \mu, f_x(.) \rangle - \langle \mu, f_{z_\mu}(.) \rangle - 2\langle z_\mu, z_\mu - x \rangle + 2\langle z_\mu, z_\mu - x \rangle$$

$$= \langle \mu, f_x(.) \rangle - \langle \mu, f_{z_\mu}(.) \rangle$$

This implies that $z_\mu \in M_\mu$, so by lemma 2.1 M_μ consists of a single point z_μ. To prove the second part, let $x \in D$. Then $\{T_t x : t \in S\}$ is bounded because $\|T_t x - T_t z\| \le \|x - z\|$ for $t \in S$. For $p = z_\mu \in M_\mu$ and $s, t, \theta \in S$ we have,

$$2\langle T_t x - T_{st} x, T_\theta z - p\rangle = \|T_t x - p\|^2 - \|T_{st} x - p\|^2$$
$$+ \|T_{st} x - T_\theta z\|^2 - \|T_t x - T_\theta z\|^2$$

Now, applying μ to both sides of the above equality with respect to θ we have

$$0 = 2\langle T_t x - T_{st} x, p - p\rangle = \|T_t x - p\|^2 - \|T_{st} x - p\|^2$$
$$+ \langle \mu, f_{T_{st}x}(.)\rangle - \langle \mu, f_{T_t x}(.)\rangle.$$

On the other hand, since

$${}_s f_{T_{st}x}(\theta) = f_{T_{st}x}(s\theta) = \|T_{st} x - T_{s\theta} z\|^2$$
$$\le \|T_t x - T_\theta z\|^2 = f_{T_t x}(\theta).$$

So, by left invariance of μ we have,

$$\langle \mu, f_{T_{st}x}(.)\rangle = \langle \mu, {}_s f_{T_{st}x}(.)\rangle$$
$$\le \langle \mu, f_{T_t x}(.)\rangle.$$

Therefore,

$$\|T_{st} x - p\|^2 \le \|T_x x - p\|^2 + \langle \mu, f_{T_t x}(.)\rangle$$
$$- \langle \mu, f_{T_t x}(.)\rangle$$

Hence

$$\|T_{st} x - p\| \le \|T_t x - p\|.$$

This competes the proof.

3. FIXED POINT THEOREMS

We now state and prove our main fixed point theorems. First, we extend Goebel-Schoneberg's theorem [3] for non-expansive continuous representations of a left amenable semitopological semigroup on a subset D of a Hilbert space.

Theorem 3.1. Let S be a semitopological semigroup with identity, D be a non-empty subset of a Hilbert space H and let $\Im = \{T_t : t \in S\}$ be a non-expansive continuous representation of S on D. Suppose $RUC(S)$ has a left invarinat mean μ. Then $\Im$ has a common fixed point in D, if and only if $\{T_t x : t \in S\}$ is bounded for some $x \in D$ and for any $y \in \overline{co}\{T_t x : t \in S\}$, there is a unique $p \in D$ such that $\|y - p\| = \inf_{z \in D} \|y - z\|$.

Proof: Necessity is obvious. Let us prove the sufficiency. Assume $\{T_t x : t \in S\}$ is bounded for some $x \in D$, M_μ is as in the proof of Lemma 2.4 and $c = z_\mu \in M_\mu$. Then using [5] we have $c \in \overline{co}\{T_t x : t \in S\}$. Therefore there exists a unique $p \in D$ such that $\|c - p\| \le \|c - z\|$, for all $z \in D$. On the other hand, by Lemma 2.4 we know that $\|T_t p - c\| = \|T_{te} p - c\| \le \|T_e p - c\| = \|p - c\|$. Hence, we have $\|c - p\| = \inf_{z \in D} \|c - z\| \le \|c - T_t p\| \le \|c - p\|$, $t \in S$, and the uniqueness of p implies that $T_t p = p$ for all $t \in S$, that is, p is a common fixed point for $\Im$.

Corollary 3.2. (Goebel-Schoneberg [3]). Let T be a non-expansive self-mapping of a nonempty subset D of a Hilbert space H. Then T has a fixed point in D if and only if $\{T^n x\}$ is bounded for some $x \in D$ and for any $y \in \overline{co}\{T^n x : n > 0\}$ there is a unique $p \in D$ such that $\|y - p\| = \inf_{z \in D} \|y - z\|$.

Proof: Let $S = (N, +)$ and $\Im = \{T^n : n \in N\}$. Since S is amenable, then by Theorem 3.1 the proof is complete.

Now we prove a necessary and suffcient condition (namely, boundedness of orbits) for the existence of an absolute common fixed point of a non-expansive continuous representation of a left amenable semitopological semigroup S.

Theorem 3.3. Let S be a semitopological semigroup with identity and let $\Im = \{T_t : t \in S\}$ be a non-expansive strongly continuous representation of S on a non-empty subset D of a Hilbert space H. Suppose $RUC(S)$ has a left invariant mean μ. Then $\Im$ has a common absolute fixed point in H, if and only if $\{T_t z : t \in S\}$ is bounded for some $z \in D$. In this case z_μ is a absolute fixed point of $\Im$.

Proof: By assumption there exists $z \in D$ such that $\{T_s z : s \in S\}$ is bounded. Let $f_x(t) = \|T_t z - x\|^2$ and $p \in M_\mu$. For each $s \in S$, let $\tilde{T}_s : D \cup \{p\} \to D \cup \{\tilde{T}_s p\}$ be a non-expansive extension of T_s. As a result by Lemma 2.1 we have,

$$\|p - y\| \le \langle \mu, f_y(.) \rangle - \langle \mu, f_p(.) \rangle$$

for every $y \in H$. Now for each $s \in S$ let $y = \tilde{T}_s p$. Therefore,

$$\|p - \tilde{T}_s p\| \le \langle \mu, f_{\tilde{T}_s p}(.) \rangle - \langle \mu, f_p(.) \rangle.$$

since

$$
\begin{aligned}
{}_s f_{\tilde{T}_s p}(t) &= f_{\tilde{T}_s p}(st) = \|T_{st} z - \tilde{T}_s p\|^2 \\
&= \|\tilde{T}_{st} z - \tilde{T}_s p\|^2 \le \|\tilde{T}_t z - p\|^2 = f_p(t).
\end{aligned}
$$

So

$$
\begin{aligned}
\|p - \tilde{T}_s p\| &\le \langle \mu, f_{\tilde{T}_s p}(.) \rangle - \langle \mu, f_p(.) \rangle \\
&= \langle \mu, {}_s f_{\tilde{T}_s p}(.) \rangle - \langle \mu, f_p(.) \rangle \\
&\le \langle \mu, f_p(.) \rangle - \langle \mu, f_p(.) \rangle \\
&= 0.
\end{aligned}
$$

Therefore, $\tilde{T}_s p = p$ for each $s \in S$. Thus p must be a fixed point for every non-expansive semigroup extending $\Im$ to the union of D and a subset of H containing p. To prove that p is an absolute fixed point of $\Im$, we have to show that there exists a semigroup $\tilde{\Im} = \{\tilde{T}_s : s \in S\}$ of non-expansive strongly continuous representation $\tilde{T}_s$ of S on $D \cup \{p\}$ extending and leaving p fixed. To do this, we define the semigroup $\tilde{\Im} = \{\tilde{T}_s : s \in S\}$ on $D \cup \{p\}$ by $\tilde{T}_s x = T_s x$ for $x \in D$ and $s \in S$ and $\tilde{T}_s p = p$ for all $s \in S$. By Lemma 2.1 we know that for any $y \in D$ we have $\|T_s y - p\| \le \|y - p\|$ (take $t = e$). This shows that $\tilde{T}_s$ is non-expansive on $D \cup \{p\}$ for every $s \in S$. So $\tilde{\Im} = \{\tilde{T}_s : s \in S\}$ is a semigroup of nonexpansive strongly continuous representation $\tilde{T}$ of S on $D \cup \{p\}$. Therefore p is an absolute fixed point of $\Im$ in H. Since by Lemma 2.1, $z_\mu \in M_\mu$, then z_μ is an absolute fixed point of $\Im$. For the converse, let p be an absolute fixed point for $\Im$. Then, by definition we can extend the semigroup $\Im$ to the semigroup $\tilde{\Im}$ of non-expansive mappings on $D \cup \{p\}$ such that p is a fixed point of $\tilde{\Im}$. So for each $x \in D$ we have,

$$
\begin{aligned}
\|T_s x\| &\le \|\tilde{T}_s x - \tilde{T}_s p\| + \|\tilde{T}_s p\| \\
&\le \|x - p\| + \|p\|
\end{aligned}
$$

This shows that $\{T_t x : t \in S\}$ is bounded and the proof is complete.

Theorem 3.4. Let H, D, S, $\Im$ be as in Theorem 3.3. If μ is an invariant mean on $RUC(S)$, and $\{T_t z : t \in S\}$ is bounded for some $z \in D$, then for any asymptotically invariant net $\{\mu_\alpha\}$ of means on $RUC(S)$, the net z_{μ_α} converges weakly to z_μ. In particular, if υ is another invariant mean on $RUC(S)$, then $z_\mu = z_\upsilon$.

Proof: By Theorem 3.3, we can extend the semigroup $\Im$ to the semigroup $\tilde{\Im}$ of non-expansive strongly continuous representation of S on $D \cup \{p\}$ such that z_μ is a common fixed point of the semigroup $\tilde{\Im}$. The rest of the proof is similar to [6, Theorem 4.8]. Note that since $z \in D$, then we have $\tilde{T}_t z = T_t z$.

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