# *Almost Product Structures Ontangent Manifold of a Space Form*

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***ABSTRACT***

A set of locally product structures on tangent manifold TM of a space form M is pointed out. This is found in a study of a type of Sasaki metric whose second term is a special deformation of the first one. Introducing an adequate almost product structure we find at first a large class of locally almost product structures on TM for a (pseudo)- Riemannian manifold M. When M is a space form, a subset of it is made of locally product structures.

***KEYWORDS***

Almost product structure, Constant curvature, Nijenhuis tensor, Sasaki metric.

## 1. INTRODUCTION

Let *(M, g)* be a (pseudo)- Riemannian manifold and $\nabla$ its Levi-Civita connection. In a local chart $(U,(x^i))$ we set $g_{ij}=g(\partial_i,\partial_j)$, where $\partial_i:=\frac{\partial}{\partial x^i}$ and we denote the Christoffel symbols by $\Gamma^i_{jk}(x)$. Let $(x^i,y^i)\equiv(x,y)$ be the local coordinates on the manifold *TM* projected on *M* by $\tau$. The indices $i,j,k,...$ will run from *1* to *n = dimM.*

The functions $N^i_j(x,y):=\Gamma^i_{jk}(x)y^k$ are the local coefficients of a nonlinear connection, that is the local vector fields $\delta_i=\partial_i-N^k_i(x,y)\partial_{\bar{k}}$, where $\partial_{\bar{k}}:=\frac{\partial}{\partial y^k}$ span a distribution on *TM* called horizontal, which is supplementary to the vertical distribution $u\to V_uTM=\ker\tau_{*u}$, $u\in TM$. Let us denote by $u\to H_uTM$ the horizontal distribution and let $(\delta_i,\partial_{\bar{i}})$ be the basis adapted to the decomposition $T_uTM=H_uTM\oplus V_uTM$, $u\in TM$. The dual basis are $(dx^i,\delta y^i)$ with $\delta y^i=dy^i+N^i_k(x,y)dx^i$.

The Sasaki metric on *TM* is as follows:

$$G_S=g_{ij}(x)dx^i\otimes dx^j+g_{ij}(x)\delta y^i\otimes\delta y^j \tag{1.1}$$

If in the second term of $G_S$ one replaces $g_{ij}(x)$ with the components $h_{ij}(x,y)$ of a generalized Lagrange metric (see Ch.X in *[4]*) one gets a type of Sasaki metric

$$G(x,y)=g_{ij}(x)dx^i\otimes dx^j+h_{ij}(x,y)\delta y^i\otimes\delta y^j \tag{1.2}$$

In particular, $h_{ij}(x,y)$ could be a deformation of $g_{ij}(x)$, a case studied by M. Anastasiei and H. Shimada in [1].

In this paper, we study the metrical structure (1.2) in the case when $h_{ij}(x,y)$ is the following special deformation of $g_{ij}(x)$

$$h_{ij}(x,y)=a(L^2)g_{ij}(x)+b(L^2)y_iy_i, \tag{1.3}$$

where $L^2=g_{ij}(x)y^iy^j$, $y_i=g_{ij}(x)y^i$ and

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$a,b := \mathrm{Im}(L^2) \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ with $a>0, b\geq 0$ .[4]

For $b=0$ and $a=\frac{c^2}{L^2}$ for any constant $c$, the metrical structure (1.2), (1.3) was studied by R. Miron in [3] as a homogeneous lift of $g_{ij}(x)$ to *TM*.

In the following section, we introduce an almost product structure which paired with $G$ given by (1.2), (1.3) which provides a large set of almost product structures on *TM*.

Finally, we find in section 3 that, when *(M, g)* is of constant curvature, some of them are locally product structures.

Let $P$ be an endomorphism of the tangent bundle *TM* satisfying $P^2 = I$ , where *I= identity*. Then P defines an almost product structure on *M*. If $g$ is metric on M such that $g(PX, PY) = g(X, Y)$ for arbitrary vector fields $X$ and $Y$ on *M*, then the triple *(M, g, P)* will introduce the natural almost product structure. If $P$ is an almost product structure and the Nijenhuis tensor field $N_P$ of $P$ vanishes then $P$ is called a product structure on *M*.

## 2. Some almost Product structures on *TM*

Let $P_S$ be the almost product structure on *TM* given in the adapted basis $(\delta_i, \partial_{\bar{i}})$ by

$$P_S(\delta_i) = \partial_{\bar{i}},\ P_S(\partial_{\bar{i}}) = \delta_i \tag{2.1}$$

It is well known that the pair $(G_S, P_S)$ is an almost product structure on *TM*, thatis

$$G_S(P_S X, P_S Y) = G_S(X, Y).$$

We look for a new almost product structure which paired with $G$ to provide a product structure. We modify $P_S$ to a linear map $P$ given in the basis $(\delta_i, \partial_{\bar{j}})$ as follows:

$$P(\delta_i) = (\alpha\delta_i^k + \beta y_i y^k)\partial_{\bar{k}}$$
$$P(\partial_{\bar{j}}) = (\gamma\delta_j^h + \delta y_j y^h)\delta_h \ , \tag{2.2}$$

where $\alpha, \beta, \gamma, \delta$ are functions on *TM* to be determined. The condition $P^2 = I$ leads to

$$\alpha\gamma = 1, \alpha\delta + \beta\gamma + \beta\delta L^2 = 0. \tag{2.3}$$

Then the condition $G(P(X), P(Y)) = G(X, Y)$ gives

$$a\alpha^2 = 1, \gamma^2 = a, 2\gamma\delta + \delta^2 L^2 = b,$$
$$(2\alpha\beta + \beta^2 L^2)(a + bL^2) + b\alpha^2 = 0. \tag{2.4}$$

The solution of the system of equation (2.3), (2.4) is

$$\alpha = -\frac{1}{\sqrt{a}}, \beta = \frac{\sqrt{a} + \sqrt{a+bL^2}}{L^2\sqrt{a(a+bL^2)}}, \gamma = -\sqrt{a},$$
$$\delta = \frac{\sqrt{a} + \sqrt{a+bL^2}}{L^2} \tag{2.5}$$

We notice that for $b=0$, besides the solution provided by *(2.5)*, that is

$$\alpha = -\frac{1}{\sqrt{a}}, \gamma = -\sqrt{a}, \beta = \frac{2}{L^2\sqrt{a}}, \delta = \frac{2\sqrt{a}}{L^2} \tag{2.6}$$

There exists also the solution

$$\alpha = -\frac{1}{\sqrt{a}}, \gamma = -\sqrt{a}, \beta = 0, \delta = 0 \tag{2.7}$$

Let us make the substitution

$$a \to \frac{a^2}{L^2},\ a \to \frac{b^2 - a^2}{L^4}$$

Then (2.5) and (2.6) are unified to

$$\alpha = -\frac{L}{a}, \beta = \frac{a+b}{abL}, \gamma = -\frac{a}{L}, \delta = \frac{a+b}{L^3},$$
$$b \geq a > 0 \tag{2.8}$$

and (2.7) modifies to

$$\alpha = -\frac{L}{a}, \gamma = -\frac{a}{L}, \beta = \delta = 0. \tag{2.9}$$

The metric $G$ takes the form

$$G_{a,b}(x,y) = g_{ij}(x)dx^i \otimes dx^j$$
$$+(\frac{a^2}{L^2}g_{ij}(x) + \frac{b^2 - a^2}{L^4}y_i y_j)\delta y^i \otimes \delta y^j$$
$$b \geq a > 0. \tag{2.10}$$

Let $P_{a,b}$ be the almost product structures given by (2.2), (2.8) and $P_a$ those given by (2.2), (2.9). Then the pairs $(G_{a,b}, P_{a,b})$ and $(G_{a,a}, P_a)$ are almost product structures on TM.

For $a^2 = \frac{L^2}{1+L^2}$ , $b = L^2$, the metric $G_{a,b}(x,y)$ is the Cheeger- Gromoll metric, [5], [6]

$$G_{CG}(x,y) = g_{ij}(x)dx^i \otimes dx^j$$
$$+\frac{1}{1+L^2}(g_{ij}(x) + y_i y_j)\delta y^i \otimes \delta y^j \tag{2.11}$$

If $a^2 = \varphi' L^2, b^2 = L^2(\varphi' + 2\varphi'' L^2)$ for $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi'(t) \neq 0, t \in \mathrm{Im}(L^2)$, one obtains the Antonelli – Hirimiuc metrical structure ,[2]

$$G_{AH}(x,y)=g_{i,j}(x)dx^i\otimes dx^j$$
$$+(\varphi' g_{i,j}(x)+2\varphi'' y_i y_j)\delta y^i\otimes\delta y^j \quad (2.12)$$

## 3. PRODUCT STRUCTURES ON *TM*

We know that a Riemannian manifold $(M,g)$ has a constant curvature $k$ if

$$\forall i,j,l,s \quad K_{ijls}=k(g_{is}g_{jl}-g_{js}g_{il})$$

where $K_{ijls}=K_{ijl}{}^{r}g_{rs}$ and $K_{ijl}{}^{r}$ denote the components of the curvature tensor of $M$.

***Lemma3.1.*** The lie brackets satisfy the following:

$$[\delta_i,\delta_j]=y^r K_{jir}{}^{m}\partial_{\bar m},$$
$$[\delta_i,\partial_{\bar j}]=\Gamma_{ji}{}^{m}\partial_{\bar m},$$
$$[\partial_{\bar i},\partial_{\bar j}]=0$$

In order to find conditions that $(G_{a,b},P_{a,b})$ be a locally product structure we have to put zero for the Nijenhuis tensor field of $P:=P_{a,b}$,

$$N_P=[PX,PY]-P[PX,Y]-P[X,PY]+[X,Y], \qquad X,Y\in\chi(TM) \quad (3.1)$$

As the evaluation on the basis $(\delta_i,\partial_{\bar i})$ is in general very complicated, we confine ourselves to the structures $(G_{a,a},P_a)$. In this case, we have:

$$N_P(\delta_i,\delta_j)=\gamma(\delta_j(\alpha)\delta_i^r-\delta_i(\alpha)\delta_j^r)\delta_r$$
$$+\left(\left(\frac{2a'L^2-a}{L^3}\right)(y_j\delta_i^r-y_i\delta_j^r)+y^aK_{jia}{}^{r}\right)\partial_{\bar r}$$
$$N_P(\delta_i,\partial_{\bar j})=\frac{a^2}{L^2}\left[\left(\frac{2a'L^2-a}{L^3}\right)(y_i\delta_j^r-y_j\delta_i^r)-y^aK_{jia}{}^{r}\right]\delta_r$$
$$-(\gamma\delta_j(\alpha)\delta_i^r+\alpha\delta_i(\gamma)\delta_j^r)\partial_{\bar r}$$
$$N_P(\partial_{\bar i},\partial_{\bar j})=\gamma(\delta_i(\gamma)\delta_j^r-\delta_j(\gamma)\delta_i^r)\delta_r$$
$$+\frac{a^2}{L^2}\left[\left(\frac{2a'L^2-a}{L^3}\right)(y_j\delta_i^r-y_i\delta_j^r)+y^aK_{jia}{}^{r}\right]\partial_{\bar r}$$

From above equation, the conditions

$$N_P(\delta_i,\delta_j)=0,\ N_P(\delta_i,\partial_{\bar j})=0,\ N_P(\partial_{\bar i},\partial_{\bar j})=0 \quad (3.2)$$

are equivalent with six equations. Three of them are identities because of $\delta_i\alpha=\delta_i\gamma=0$ and the other three are equivalent with

$$K_{ji}{}^{k}=\frac{2a'L^2-a}{a^3}(y_i\delta_j^k-y_j\delta_i^k), \quad (3.3)$$

where $K_{ji}{}^{k}=K_{jis}{}^{k}(x)y^s$ and $K_{jis}{}^{k}$ is the curvature tensor of $\nabla$. By a contraction with $g_{rk}$ the Eq. (3.3) reduces to

$$K_{jisr}(x)y^s=\frac{2a'L^2-a}{a^3}(g_{jr}g_{si}-g_{ir}g_{sj})y^s. \quad (3.4)$$

The Eq. (3.4) reminds us of the condition that $(M,g)$ is of constant curvature (space form). It suggests that we look for functions such that, $\frac{2a'L^2-a}{a^3}=k$, where $k$ is a constant. For $t=L^2$, solving the Bernoulli equation $a'=\frac{1}{2t}a+\frac{k}{2t}a^3$ one gets $a(L^2)=\sqrt{\frac{L^2}{c-kL^2}}$ for $c-kL^2>0$, where $c$ is a constant of integration. So Eq. (3.4) becomes

$$K_{jisr}(x)y^s=k(g_{jr}g_{si}-g_{ir}g_{sj})y^s, \quad (3.5)$$

which means that $(M,g)$ is of constant curvature $k$. Then we have proved.

***Theorem 3.2.*** *If the (pseudo)-Riemannian manifold $(M,g)$ is of constant curvature $k\in\mathbb{R}$, for $a(L^2)=\sqrt{\frac{L^2}{c+kL^2}}$ with $c$ a constant such that $c+kL^2>0$, the structures $(G_{a,a},P_a)$ are locally product structures on TM.*

The explicit form of these structures are as follows:

$$G_{a,a}(x,y)=g_{i,j}(x)dx^i\otimes dx^j$$
$$+\left(\frac{1}{c+kL^2}(g_{ij}(x))\delta y^i\otimes\delta y^j\right. \quad (3.6)$$

$$P_a(\delta_i)=\sqrt{c+kL^2}\,\partial_{\bar i},\quad P_a(\partial_{\bar i})=\frac{1}{\sqrt{c+kL^2}}\delta_i \quad (3.7)$$

***Corollary 3.3.*** For $a(L^2)=c_0\sqrt{L^2}$, with $c_0$ a strict positive constant, the pairs $(G_{a,a},P_a)$ are product structures on *TM* if and only if *(M, g)* is flat.

**Proof.** Since $a(t)=c_0\sqrt{t}$ we have

$$a'(t) = \frac{c_0}{2\sqrt{t}} \Rightarrow a'(L^2) = \frac{c_0}{2L}$$

Therefore, Eq. (3.3) gives $K_{ji}{}^{k} = 0$, equivalently $K_{jisr}(x) = 0$. By Theorem 3.2. the structures $(G_{a,a}, P_a)$ are product structures on *TM* if and only if *(M, g)* is flat.

Looking at (3.6) and (3.7), we see that the structures $(G_{a,a}, P_a)$ from Corollary *3.3* are very close to $(G_S, P_S)$ which is obtained for $c=1$. Thus the Corollary *3.3.* covers a well-known result: $(G_S, P_S)$ is a product structure if and oly if *(M, g)* is flat.

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