A Methodology to Analyze the Transient Behavior of Exponential Queuing Systems

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ABSTRACT

This paper applies the Markov models,optimal control theory, eigen vectors and eigen values concepts to propose a methodology to analyze the transient behavior of Exponential queuing systems. We propose a procedure to calculate the transient probability, the average queue length and the duration of the system to reach the steady state. The method is implemented through an algorithm which is developed and tested in MATLAB software environment. The new method enjoys a stronger mathematical foundation and more flexibility to analyze the transient behavior of Exponential queuing systems.

KEY WORDS:

Markov models, Eigen - vectors, Eigen - values, Transient state, Queuing system, Optimal control theory.

1. INTRODUCTION

Queuing theory and reliability are the most important applications of stochastic processes. We can use queuing and reliability models in some systems such as transportation, airport traffic control, repair and maintenance and refineries. As is known, a queuing system experiences a transient state and then enters its steady state. Most of the authors, have studied queuing systems in its steady state while ignoring the transient state effects could increase the error in a queuing system [1].

Bhat [2] studied the queuing model of M/M/∞ in transient state using differential equations and Z transform. Shanthikumar et al. [3], [4], [5] studied the queuing model of M/M/1 in transient state approximately using transition rate matrix and it's inverse. Azaron et al. [6],[7] introduced a new methodology, using continuous–time Markov processes and shortest path technique, for the reliability evaluation of an L-dissimilar unit non-repairable cold-standby redundant system which was considered as a queuing model.

Amiri et al. [8] introduced a methodology to analyze system transient survivability and availability with identical components and identical repairmen. They employed the Markov models, eigen vectors and eigen values concepts to develop the methodology for the transient reliability of such systems.

This paper presents a method for transient analysis of a queuing system using Markov models,optimal control theory, eigen-values and eigen-vectors. The considered system consists of capacities of n customers and k servers and the service time and the inter–arrival time are exponentially distributed. A procedure is proposed to calculate the transient probability, the average queue length and the duration of the system to reach the steady state.

The paper is organized is as follows. Section 2 presents nomenclature and definitions. Section 3 deals with the model and the proposed methodology . Numerical example is given in section 4. Finally section 5 is devoted to conclusions and recommendations for future studies.

2. NOMENCLATURE AND DEFINITIONS

A(t): Cumulative arrivals time from 0 to t

D(t): Cumulative departures time from 0 to t

X(t): Number of customers in the system at t

X(t)=A(t)-D(t)

$p_n(t)$: Probability of having n customers in the system at time t

$p_n(t)=P(X(t)=n)$

$L_q(t)$: The average queue length in the time interval (0,t)

$$L_q(t) = \sum_{n=k}^{+\infty}(n-k)p_n(t)$$

$L_s(t)$: The average number of customers in the system in the time interval (0,t)

$$L_s(t) = \sum_{n=0}^{\infty} np_n(t)$$

L_q: The average queue length in the steady state

$$L_q = \lim_{t\to\infty} L_q(t)$$

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L_s: The average number of customers in the system in the steady state

$$L_s = \lim_{t\to\infty} L_s(t)$$

λ: The average arrival rate to the system

μ: The average service rate

k: Number of server

n: The capacity of the system

Definition 1. If Q be an n×n matrix then λ is an eigen value if Q.X=λ.X, where X is a non-zero vector and eigenvector.

Definition 2. Let {X(t): t≥0} be a continuous-time stochastic process with finite or countable state space R; usually R is {0,1,2, . . .}, or a subset thereof.

We say {X(t)} is a continuous-time Markov chain if the transition probabilities have the fallowing property: For every t, s≥0 and j∈R,

$P(X(s+t)=j \mid X(s)=i, X(u)=x(s), u\le s)=P(X(s+t)=j \mid X(s)=i)$

and

$p_{ij}(t)=P(X(t+s)=j \mid X(s)=i)=P(X(t)=j \mid X(0)=i)$.

Definition 3. Matrix Q is transition rate matrix if we have:

$$Q=(q_{ij}) \quad q_{ii} = -\sum_j q_{ij}$$

where q_{ij} is transition rate from i to j.

3. THE MODEL AND THE PROPOSED METHODOLOGY

In this paper our aim is the determining of transient probability, the average queue length and the average number of customers, for a queuing system with the following assumptions:

1. The capacity queuing system consists of n identical and independent arrivals.
2. The system consists of k identical servers.
3. The inter-arrival time is exponentially distributed with the parameter λ.
4. The service time of each customer by each server is exponentially distributed with the parameter μ.

Consider X(t) as the number of customers in the system at time t, we will have the Markov model shown in Figure 1:

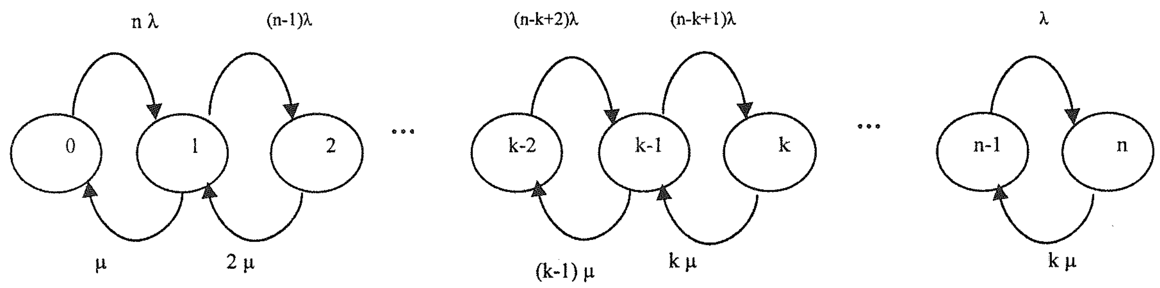

Figure 1. State transition diagram of the system with n customers and k servers

Example 1. If we let n=4 and k=3 the Markov model is represented as follows:

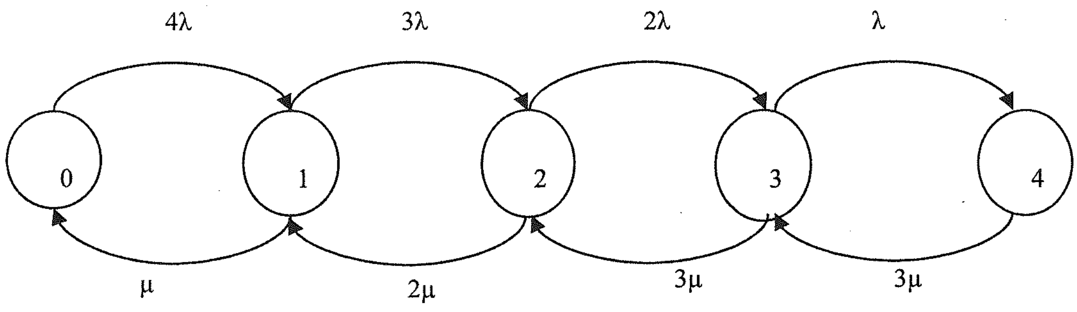

Figure 2. State transition diagram of the system with 4 customers and 3 servers

Lemma 1. [9] If we consider Q as the state transient rate matrix and P(t) as the state transient probability in the exponential Markov chain with the continuous time, then we have:

$$P'(t) = P(t) \cdot Q$$
$$P_n'(t) = P_n(0) \cdot Q$$
$$P_n(t) = P_n(0) \cdot P(t)$$

in which Q and P(t) are square matrices and, $P_n(t)$ and $P_n(0)$ are row vectors.

Theorem 1. Let us consider a continuous time exponential Markov chain in which $P'(t) = P(t).Q$, then we have:

$$P(t) = e^{Q.t}$$
$$P_n(t) = P_n(0) \cdot e^{Q.t}$$

Proof:

$$P'(t) = P(t).Q \Rightarrow \frac{dP(t)}{dt} = P(t).Q \Rightarrow \frac{dP(t)}{P(t)} = Q.dt$$

$$\int \frac{dP(t)}{P(t)} = \int Q.dt \Rightarrow \ln P(t) = Q.t + C.I \Rightarrow e^{\ln P(t)} = e^{Q.t}.e^{C.I} \Rightarrow \ln P(t) = Q.t + C.I \Rightarrow$$

$$e^{\ln P(t)} = e^{Q.t}.e^{C.I} \Rightarrow P(t) = e^{Q.t}.e^{C.I}$$

in which I is an identity matrix. Since P(0)=I then we have $P(t) = e^{Q.t}$. By Lemma1 we will have:

$$P_n(t) = P_n(0) \cdot P(t) = P_n(0) \cdot e^{Q.t}$$

Theorem 2. [10] Let us consider Q as an n×n square matrix which has n non-repeating eigen values, then we have

$$e^{Q.t} = V \cdot e^{d.t} \cdot V^{-1}$$

where t represents time, V is a matrix of eigen vectors of Q, V^{-1} is the inverse of V, and d is a diagonal matrix of eigen values of Q defined as follows:

$$d = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Theorem 3. Let us consider Q as an $n \times n$ square matrix which has repeating eigen values(for example eigen value λ_k has been repeated R times), then we have:

$$e^{Q.t} = \sum_{i=0}^{n-1} \alpha_i Q^i$$

That α_i's are obtained through solving the following equations:

$$e^{\lambda.t} = \sum_{i=0}^{n-1} \alpha_i \lambda^i \qquad \lambda = \lambda_i \quad i = 1,2,\ldots,n$$

$$\frac{d}{d\lambda}(e^{\lambda.t}) = \frac{d}{d\lambda}(\sum_{i=0}^{n-1} \alpha_i \lambda^i) \qquad \lambda = \lambda_k$$

$$\frac{d^2}{d\lambda^2}(e^{\lambda.t}) = \frac{d^2}{d\lambda^2}(\sum_{i=0}^{n-1} \alpha_i \lambda^i) \qquad \lambda = \lambda_k$$

:

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$$\frac{d^{R-1}}{d\lambda^{R-1}}(e^{\lambda.t}) = \frac{d^{R-1}}{d\lambda^{R-1}}(\sum_{i=0}^{n-1} \alpha_i \lambda^i) \qquad \lambda = \lambda_k$$

Theorem 4. Consider Q as the transition rate matrix.In matrix Q one of the eigen -values is zero and the remaining eigen values are complex numbers with the negative real part.

Proof:

Since in every row of transition matrix the summation of row elements is zero, we can deduce that one of its eigen value of matrix Q is zero. By Theorem 2 we have

$$P(t) = V \cdot e^{d.t} \cdot V^{-1} = (p_{ij}(t))$$

$$p_{ij}(t) = \pi_j + \sum_{k=1}^{n} \alpha_{ijk} \cdot e^{\lambda_k.t}$$

in which λ_k is the k^th eigen value, α_{ijk}'s are constant values, and π_j is the limiting probability. Using the contradictory concept, if we assume that one of the eigen values of Q is a complex number with positive real part then we have:

$$\lim_{t \to \infty} e^{\lambda_k t} = \infty$$

Therefore $\lim_{t \to \infty} p_{ij}(t) = \infty$

which contradicts the fact that

$$\lim_{t \to +\infty} p_{ij}(t) = \pi_j$$

and therefore the eigen values of Q are complex numbers with the negative real part.

Theorem 5. Consider $P(t) = e^{Q.t}$ in which Q is the transition matrix.The time until system reaches the steady state $p(t) = \Pi$ can be calculated by the following formula:

$$t = \frac{\ln \varepsilon}{S_r}$$

in which ε is a very small positive number (i.e. ε≤0.0001), S_r is the largest real part of the eigen- values excluding the zero element of matrix Q and Π is a square matrix representing the limiting probabilities. The elements of matrix P(t) and Π are shown as follows:

$$P(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t) & \cdots & p_{0n}(t) \\ p_{10}(t) & p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n0}(t) & p_{n1(t)} & \cdots & p_{nn}(t) \end{bmatrix}$$

$$\Pi = \begin{bmatrix} \pi_0 & \pi_1 & \cdots & \pi_n \\ \pi_0 & \pi_1 & \cdots & \pi_n \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0 & \pi_1 & \cdots & \pi_n \end{bmatrix}$$

Proof:

$$p_{kj}(t) = \pi_j + \sum_{m=1}^{n} \alpha_{kjm} \cdot e^{\lambda_m \cdot t} = \pi_j + \sum_{m=1}^{n} \alpha_{kjm} \cdot e^{(S_m + C_m \cdot i) \cdot t}$$

By Theorem 4 all S_m are negative, and $i = \sqrt{-1}$ (π_j, α_{kjm}, S_m, and C_m are constant numbers). Now suppose S_r is greater then S_m, then for large values of t we have:

$$p_{kj}(t) = \pi_j + \varepsilon'$$

where ε' is a very small positive number.Therefore we have:

$$p_{kj}(t) \approx \pi_j$$

$$\varepsilon = e^{S_r \cdot t}$$

$$S_r \cdot t = \ln \varepsilon$$

$$t = \frac{\ln \varepsilon}{S_r}$$

Based on the proof of these theorems, we now propose an algorithm for calculating the transient probabilities and the average number of customers in system and queue at time t.

ALGORITHM:

1.Determine the transition matrix Q.

2.Determine the eigen- values and eigen -vectors of matrix Q.

3.Determine.

$$P(t) = V \cdot e^{d \cdot t} \cdot V^{-1}$$

4.Determine

$$P_n(t) = P_n(0) \cdot P(t)$$

5.Determine $L_s(t)$ and $L_q(t)$ as follows:

$$L_s(t) = \sum_{n=0}^{\infty} n p_n(t)$$

$$L_q(t) = \sum_{n=c}^{\infty} (n-c) p_n(t)$$

Note that if matrix Q has repeating eigen values, determine $P(t) = e^{Q.t}$ according to Theorem 3.We should note that the complexity of above algorithm is $O(n^3)$.

4. A NUMERICAL EXAMPLE

Consider a system having five identical components. There are two identical repairmen for repairing this system. It is assumed that the time to failure of repaired component is a random variable with exponential distribution function with the mean of 1/2 hour. The repair time is also considered to be a random variable distributed exponentially with the mean of 1/10 of hour. We want to calculate the transient probabilities, the average number of failed components at time t , the average queue length at time t and the elapsed time until the system reaches the steady state.

SOLUTION:

The graphical Markov model and the transition rate matrix can be represented as follows:

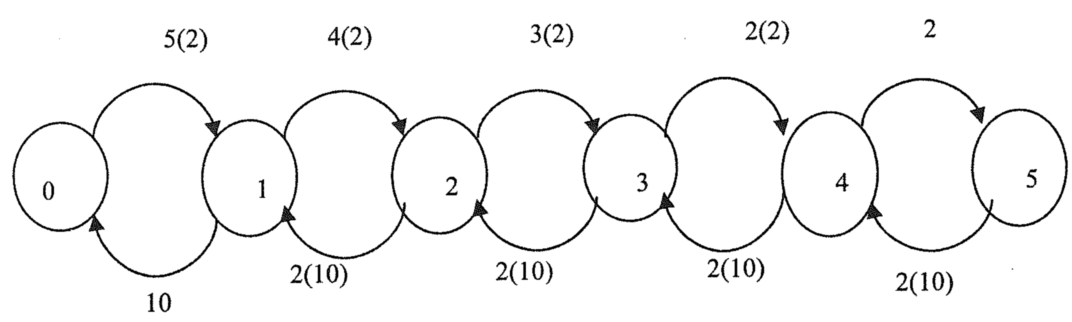

Figure 3.State transition diagram of the system with 2 repairmen

$$Q=\begin{bmatrix} -10 & 10 & 0 & 0 & 0 & 0 \\ 10 & -18 & 8 & 0 & 0 & 0 \\ 0 & 20 & -26 & 6 & 0 & 0 \\ 0 & 0 & 20 & -24 & 4 & 0 \\ 0 & 0 & 0 & 20 & -22 & 2 \\ 0 & 0 & 0 & 0 & 20 & -20 \end{bmatrix}$$

According to the algorithm we have:

$$P_n(0)=(1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

$$P_n(t)=P_n(0)\cdot V\cdot e^{d\cdot t}\cdot V^{-1}=(p_0(t) \quad p_1(t) \quad p_2(t) \quad p_3(t) \quad p_4(t) \quad p_5(t))$$

$$p_0(t)=0.0159e^{-42.2t}+0.0435e^{-30.9t}+0.204e^{-9.02t}+0.12e^{-22.3t}+0.224e^{-15.5t}+0.392$$

$$p_1(t)=-0.0512e^{-42.2t}-0.0912e^{-30.9t}+0.0201e^{-9.02t}-0.148e^{-22.3t}-0.123e^{-15.5t}+0.392$$

$$p_2(t)=0.0541e^{-42.2t}+0.0372e^{-30.9t}-0.0931e^{-9.02t}-0.0279e^{-22.3t}-0.127e^{-15.5t}+0.1571$$

$$p_3(t)=-0.0234e^{-42.2t}+0.0273e^{-30.9t}-0.0871e^{-9.02t}+0.054e^{-22.3t}-0.0179e^{-15.5t}+0.0471$$

$$p_4(t)=0.00507e^{-42.2t}-0.0206e^{-30.9t}-0.0373e^{-9.02t}+0.0129e^{-22.3t}+0.0306e^{-15.5t}+0.00942$$

$$p_5(t)=-0.000457e^{-42.2t}+0.00377e^{-30.9t}-0.00679e^{-9.02t}-0.011e^{-22.3t}+0.0135e^{-15.5t}+0.000942$$

$$L_s(t)=\sum_{n=0}^{d} np_n(t)=0.0048e^{-42.2t}+0.00144e^{-30.9t}-0.611e^{-9.02t}-0.0452e^{-22.3t}-0.241e^{-15.5t}+0.89$$

and in the steady state we have Ls=0.89

$$L_q(t)=\sum_{n=2}^{d}(n-2)p_n(t)=p_3(t)+2p_4(t)+3p_5(t)=-0.0146e^{-42.2t}-0.00268e^{-30.9t}-0.182e^{-9.02t}+0.0466e^{-22.3t}+0.084e^{-15.5t}+0.069$$

and in the steady state we have L_q=0.0687

According to the Theorem 5 we can also calculate the elapsed time until the system reaches the steady state.

$$t=\frac{\ln\varepsilon}{s_r}=\frac{\ln 0.0001}{-9.02}=1$$

Table 1 represents $p_n(t)$, $L_s(t)$, $L_q(t)$, for different values of t. The system reaches the steady state after one unit time.

The limiting probability can also be calculated as follows:

π_0=0.3927 π_1=0.3927 π_2=0.1571 π_3=0.0471 π_4=0.0094 π_5=0.0009

Table 1. $p_n(t)$, $L_s(t)$, $L_q(t)$, for different values of t and elapsed time until system reaches the steady state

t	$p_0(t)$	$p_1(t)$	$p_2(t)$	$p_3(t)$	$p_4(t)$	$p_5(t)$	$L_s(t)$	$L_q(t)$
0.05	0.6763	0.2748	0.0444	0.00402	0.000192	0.0000037	0.37645	0.004415
0.1	0.5381	0.3539	0.0917	0.0146	0.0013	0.00005	0.5867	0.0174
0.15	0.472	0.3796	0.12	0.025	0.00304	0.00016	0.7074	0.0315
0.2	0.4378	0.3885	0.1358	0.0326	0.00477	0.000317	0.7787	0.04313
0.25	0.4192	0.3916	0.1446	0.0378	0.00619	0.000472	0.8214	0.0516
0.4	0.3986	0.3929	0.1543	0.0447	0.00848	0.000784	0.8735	0.064035
0.5	0.3949	0.3928	0.156	0.0462	0.00903	0.00087	0.8837	0.066824
0.7	0.3930	0.3926	0.1569	0.047	0.00936	0.00093	0.8894	0.06847
1	0.3926	0.3926	0.1571	0.0471	0.00942	0.000942	0.89	0.069
1.2	0.3926	0.3926	0.1571	0.0471	0.00942	0.000942	0.89	0.069
1.5	0.3926	0.3926	0.1571	0.0471	0.00942	0.000942	0.89	0.069
2	0.3926	0.3926	0.1571	0.0471	0.00942	0.000942	0.89	0.069

5. CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE STUDIES

In this paper we proposed a methodology to analyze the transient behavior of Exponential queuing systems.

We employed the Markov models, optimal control theory,eigen-vectors and eigen-values concepts to develop the methodology to analyze the transient behavior of Exponential queuing systems.

We proposed a procedure to calculate the transient probability, the average queue length and the duration of the system to reach the steady state.

The following topics are recommended for future studies:

1- Analyzing the transient behavior of M/G/C model.
2- Analyzing the transient behavior of G/M/C model.
3- Analyzing the transient behavior of G/G/C model.

6. ACKNOWLEDGMENT

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