Almost structures: product and anti-Hermitian

E. Peyghan and A. Razavi

ABSTRACT

Noting that the complete lift of a Riemannian metric g defined on a differentiable manifold M is not 0-homogeneous on the fibers of the tangent bundle TM, in this paper, we introduce a new lift $\tilde{g}_2$ which is 0-homogeneous. It determines on $\widetilde{TM} = TM \setminus \{0\}$ a pseudo-Riemannian metric, which depends only on the metric g. We study some of the geometrical properties of this pseudo-Riemannian space and define the natural almost complex structure $\tilde{J}$ and natural almost product structure $\tilde{Q}$ which preserve the property of homogeneity and find some new results.

KEYWORDS

Almost complex structure, Almost anti-Hermitian structure, Almost product structure, Complete lift metric.

1. INTRODUCTION

The importance of the complete lift g_2, (2.5), of a Riemannian metric g is well known in Riemannian geometry, Finsler geometry and Physics, and has many applications in Biology too (see [1]). The tensor field g_2 determines a pseudo-Riemannian structure on $\widetilde{TM} = TM \setminus \{0\}$, but g_2 is not 0-homogeneous on the fibers of the tangent bundle TM. Therefore, we cannot study some global properties of the pseudo-Riemannian space $(\widetilde{TM}, g_2)$. For instance, we cannot prove a theorem of Gauss-Bonnet type for this space.

In this paper, by means of (3.1), we define a new kind of lift $\tilde{g}_2$ to TM of the Riemannian metric g. Thus $\tilde{g}_2$ determines on $\widetilde{TM}$ a pseudo-Riemannian structure, which is 0-homogeneous on the fibers of TM and depends only on g. Some geometrical properties of $\tilde{g}_2$ such as the Levi-Civita connection, Riemannian curvature, are studied.

Almost complex and almost product structures are among the most important geometrical structures which can be considered on a manifold [12], [13]. We introduce the natural almost complex and product structures $\tilde{J}$ and $\tilde{Q}$, respectively by (5.1) and (6.1), they depend only on g and preserve the property of homogeneity, then we get almost anti-Hermitian structure $(\tilde{g}_2, \tilde{J})$ and almost product structure $(\tilde{g}_2, \tilde{Q})$.

Let Q be an endomorphism of the tangent bundle TM satisfying $Q^2 = I$, where $I=$ identity. Then Q defines an *almost product structure* on M. If g is a metric on M such that $g(QX, QY) = g(X, Y)$ for arbitrary vector fields X and Y on M, then the triple (M, g, Q) defines a (pseudo-) Riemannian almost product structure. Geometric properties of (pseudo-) Riemannian almost product structure have been studied in [2] to [6]. If, moreover, g is an Einstein metric (i.e., $Ric(g) = \lambda g$ holds, where $Ric(g)$ is the Ricci tensor defined by $R_{ij} = K_{ijk}{}^{k}$ and λ is a constant) then the triple (M, g, Q) is called an *almost product Einstein* manifold. Analogously, if J is an endomorphism of the tangent bundle TM satisfying $J^2 = -I$, then J defines an *almost complex structure* on M. An almost complex structure is *integrable* if and only if it comes from a

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complex structure. If g is a metric on M such that $g(JX,JY)=-g(X,Y)$ for arbitrary vector fields X and Y on M then the triple (M,g,J) defines an *almost anti-Hermitian* structure.

2. THE COMPLETE LIFT

Let $\Gamma_{ij}^{\ \ k}$ be the coefficients of the Riemannian connection of M, then $N_j^h=\Gamma_{0j}^{\ \ h}=y^a\Gamma_{aj}^{\ \ h}(x)$ can be regarded as coefficients of the canonical nonlinear connection N of TM, where (x^h,y^h) are the induced coordinates in TM.

N determines a horizontal distribution on $\widetilde{TM}$, which is supplementary to the vertical distribution V, such that, we have:

$$T_u\widetilde{TM}=N_u\oplus V_u,\quad \forall u\in\widetilde{TM}. \qquad (2.1)$$

The adapted basis to N and V is given by $\{\frac{\delta}{\delta x^h},\frac{\partial}{\partial y^h}\}$ where

$$\frac{\delta}{\delta x^h}=\frac{\partial}{\partial x^h}-y^a\Gamma_{ah}^{\ \ m}\frac{\partial}{\partial y^m}, \qquad (2.2)$$

and its dual basis is $\{dx^i,\delta y^i\}$ where

$$\delta y^i=dy^i+y^a\,\Gamma_{aj}^{\ \ i}\,dx^j. \qquad (2.3)$$

The indices $a,b,\ldots,\bar{a},\bar{b},\ldots,$ run over the range $\{1,2,\ldots,n\}$. The summation convention will be used in relation to this system of indices. By straightforward calculations, we have the following lemma:

Lemma 1. *The Lie bracket of the adapted frame of TM satisfies the following:*

(1) $[X_i,X_j]=y^aK_{jia}^{\ \ \ m}\frac{\partial}{\partial y^m}$,

(2) $[X_i,X_{\bar{j}}]=\Gamma_{ji}^{\ \ m}\frac{\partial}{\partial y^m}$,

(3) $[X_{\bar{i}},X_{\bar{j}}]=0$,

where $K_{jia}^{\ \ \ m}$ denote the components of the curvature tensor of M.

Let (M,g) be a Riemannian space, M being a real n-dimensional manifold and (TM,π,M) its tangent bundle. On a domain $U\subset M$ of a local chart, g has the components $g_{ij}(x)$, $(i,j,\ldots=1,\ldots,n)$. Then on the domain of chart $\pi^{-1}(U)\subset TM$ we consider the functions $g_{ij}(x,y)=g_{ij}(x),\forall(x,y)\in\pi^{-1}(U)$ and put

$$\|y\|=\sqrt{g_{ij}(x)y^iy^j}. \qquad (2.4)$$

Then, $\|y\|$ is globally defined on TM, differentiable on $\widetilde{TM}$ and continuous on the null section.

The complete lift of g to TM is defined by

$$g_2(x,y)=2g_{ij}(x)\,dx^i\delta y^j,\ \ \forall(x,y)\in\widetilde{TM}. \qquad (2.5)$$

The following properties hold:

1. g_2 is globally defined on $\widetilde{TM}$.
2. g_2 is a pseudo-Riemannian metric on $\widetilde{TM}$.
3. g_2 is not 0-homogeneous on the fibers of TM.

Namely, for the homothety $h_t:(x,y)\rightarrow(x,ty)$ for all $t\in R^+$ we get

$$(g_2\circ h_t)(x,y)=2tg_{ij}(x)dx^i\delta y^j$$
$$=tg_2(x,y)\neq g_2(x,y).$$

Let us consider the $F(\widetilde{TM})$-linear mapping $J:\chi(\widetilde{TM})\rightarrow\chi(\widetilde{TM})$, given in the adapted basis by

$$J(\frac{\delta}{\delta x^i})=-\frac{\partial}{\partial y^i},\quad J(\frac{\partial}{\partial y^i})=\frac{\delta}{\delta x^i} \qquad (2.6)$$

for $i=1,\ldots,n$. It follows that:

4. J is globally defined on $\widetilde{TM}$ and it is a tensor field of type $(1,1)$.
5. J is an almost complex structure on TM, i.e., $J\circ J=-I$
6. J depends only on g.
7. J is a complex structure on $\widetilde{TM}$ if and only if the Riemannian space M^n is locally flat.
8. The pair (g_2,J) is an anti-Hermitian structure on $\widetilde{TM}$.

Let us consider the $F(\widetilde{TM})$-linear mapping $Q:\chi(\widetilde{TM})\rightarrow\chi(\widetilde{TM})$, given in the adapted basis by

$$Q(\frac{\delta}{\delta x^i})=\frac{\partial}{\partial y^i},\quad Q(\frac{\partial}{\partial y^i})=\frac{\delta}{\delta x^i} \qquad (2.7)$$

for $i=1,\ldots,n$. Then if N_Q is Nijenhuis tensor for Q, we have

$$N_Q(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}) = y^a K_{jia}{}^{s}\frac{\partial}{\partial y^s}$$

$$N_Q(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}) = -y^a K_{jia}{}^{s}\frac{\delta}{\delta x^s}$$

$$N_Q(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}) = y^a K_{jia}{}^{s}\frac{\partial}{\partial y^s}$$

where $K_{jia}{}^{s}$ are components of the curvature tensor of manifold M.

It follows that:

9. Q is globally defined on $\widetilde{TM}$ and it is a tensor field of type $(1,1)$.

10. Q is an almost product structure on TM, i.e., $Q \circ Q = I$

11. Q depends only on g.

12. $N_Q = 0$ if and only if the Riemannian space M^n is locally flat.

The previous space, called "the geometrical model on TM of the Riemannian space (M,g)", is important in the study of the geometry of initial Riemannian space (M,g) ([5],[6]).

3. The 0-Homogeneous Lift of the Riemannian Metric g

We can eliminate the inconvenience of the complete lift g_2 given by the property "3" introducing a new kind of lift to TM of the Riemannian metric g.

Definition 2. *Let $\tilde{g}_2$ be a tensor field on $\widetilde{TM}$ defined by*

$$\tilde{g}_2(x,y) = \frac{2}{\|y\|} g_{ij}(x) dx^i \delta y^i \tag{3.1}$$

where $\|y\|$ was defined in (2.4). Then $\tilde{g}_2$ is called the 0-homogeneous lift of the Riemannian metric g to $\widetilde{TM}$.

We get, evidently:

Theorem 3. *The following properties hold:*

1. *The pair $(\widetilde{TM},\tilde{g}_2)$ is a pseudo-Riemannian space, depending only on the metric g.*

2. *$\tilde{g}_2$ is 0-homogeneous on the fibers of the tangent bundle TM.*

In order to study the geometry of the pseudo-Riemannian space $(\widetilde{TM},\tilde{g}_2)$ we can apply the theory of the (h,v)-Riemannian metric on TM given in the books [4], [6] and [8]. Looking at the relations (2.5) and (3.1) we can assert:

Proposition 4. *The lifts g_2 and $\tilde{g}_2$ coincide on the hyper sphere $g_{ij}(x_0)y^i y^j = 1$, for every point $x_0 \in M$.*

4. Riemannian Connections of TM

Let $\bar{\bar{\nabla}}$ be the Riemannian connection of TM with respect to $\tilde{g}_2$, that is:

$$\bar{\bar{\nabla}}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \bar{\bar{\Gamma}}_{ji}{}^{m}\frac{\delta}{\delta x^m} + \bar{\bar{\Gamma}}_{ji}{}^{m}\frac{\partial}{\partial y^m}, \tag{4.1}$$

$$\bar{\bar{\nabla}}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \bar{\bar{\Gamma}}_{\bar{j}i}{}^{m}\frac{\delta}{\delta x^m} + \bar{\bar{\Gamma}}_{\bar{j}i}{}^{\bar{m}}\frac{\partial}{\partial y^m},$$

$$\bar{\bar{\nabla}}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = \bar{\bar{\Gamma}}_{j\bar{i}}{}^{m}\frac{\delta}{\delta x^m} + \bar{\bar{\Gamma}}_{j\bar{i}}{}^{\bar{m}}\frac{\partial}{\partial y^m},$$

$$\bar{\bar{\nabla}}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = \bar{\bar{\Gamma}}_{\bar{j}\bar{i}}{}^{m}\frac{\delta}{\delta x^m} + \bar{\bar{\Gamma}}_{\bar{j}\bar{i}}{}^{\bar{m}}\frac{\partial}{\partial y^m}$$

Then, we have

$$\bar{\bar{\nabla}}_{\frac{\delta}{\delta x^i}} dx^h = -\bar{\bar{\Gamma}}_{mi}{}^{h} dx^m - \bar{\bar{\Gamma}}_{\bar{m}i}{}^{h}\delta y^m, \tag{4.2}$$

$$\bar{\bar{\nabla}}_{\frac{\delta}{\delta x^i}} \delta y^h = -\bar{\bar{\Gamma}}_{mi}{}^{\bar{h}} dx^m - \bar{\bar{\Gamma}}_{\bar{m}i}{}^{\bar{h}}\delta y^m,$$

$$\bar{\bar{\nabla}}_{\frac{\partial}{\partial y^i}} dx^h = -\bar{\bar{\Gamma}}_{m\bar{i}}{}^{h} dx^m - \bar{\bar{\Gamma}}_{\bar{m}\bar{i}}{}^{h}\delta y^m,$$

$$\bar{\bar{\nabla}}_{\frac{\partial}{\partial y^i}} \delta y^h = -\bar{\bar{\Gamma}}_{m\bar{i}}{}^{\bar{h}} dx^m - \bar{\bar{\Gamma}}_{\bar{m}\bar{i}}{}^{\bar{h}}\delta y^m,$$

Since the torsion tensor $T(X,Y)$ of $\bar{\bar{\nabla}}$ defined by $T(X,Y) = \bar{\bar{\nabla}}_X Y - \bar{\bar{\nabla}}_Y X - [X,Y]$ vanishes, we have the following relations by means of Lemma 1 and (4.1).

$$\begin{array}{ll} (1)\ \bar{\bar{\Gamma}}_{ji}{}^{h} = \bar{\bar{\Gamma}}_{ij}{}^{h} & (2)\ \bar{\bar{\Gamma}}_{ji}{}^{\bar{h}} = \bar{\bar{\Gamma}}_{ij}{}^{\bar{h}} + y^a K_{jia}{}^{h} \\ (3)\ \bar{\bar{\Gamma}}_{\bar{j}i}{}^{h} = \bar{\bar{\Gamma}}_{i\bar{j}}{}^{h} & (4)\ \bar{\bar{\Gamma}}_{\bar{j}i}{}^{\bar{h}} = \bar{\bar{\Gamma}}_{i\bar{j}}{}^{\bar{h}} + \Gamma_{ji}{}^{h} \\ (5)\ \bar{\bar{\Gamma}}_{\bar{j}\bar{i}}{}^{h} = \bar{\bar{\Gamma}}_{\bar{i}\bar{j}}{}^{h} & (6)\ \bar{\bar{\Gamma}}_{\bar{j}\bar{i}}{}^{\bar{h}} = \bar{\bar{\Gamma}}_{\bar{i}\bar{j}}{}^{\bar{h}} \end{array} \tag{4.3}$$

Furthermore, we have the following lemma:

Lemma 5. The connection coefficients $\bar{\Gamma}_{BC}{}^{A}$ of $\bar{\nabla}$ of the complete metric $\tilde{g}_2$ satisfy the following relations

(1) $\bar{\Gamma}_{ji}{}^{h} = \Gamma_{ji}{}^{h}$

(2) $\bar{\Gamma}_{ji}{}^{\bar{h}} = y^{a} K_{aij}{}^{h}$

(3) $\bar{\Gamma}_{\bar{j}i}{}^{h} = \frac{1}{2\|y\|^2}(g_{ij}y^h - \delta_i^h y_j)$

(4) $\bar{\Gamma}_{j\bar{i}}{}^{h} = \frac{1}{2\|y\|^2}(g_{ij}y^h - \delta_j^h y_i)$

(5) $\bar{\Gamma}_{\bar{j}i}{}^{\bar{h}} = \Gamma_{ji}{}^{h}$

(6) $\bar{\Gamma}_{j\bar{i}}{}^{\bar{h}} = 0$

(7) $\bar{\Gamma}_{\bar{j}\bar{i}}{}^{h} = 0$

(8) $\bar{\Gamma}_{\bar{j}\bar{i}}{}^{\bar{h}} = -\frac{1}{2\|y\|^2}(\delta_i^h y_j + \delta_j^h y_i)$

Proof. By virtue of (2.2) and the connection $\bar{\nabla}$ being metrical, that is $\bar{\nabla}\tilde{g}_2 = 0$, we have:

$$
\begin{aligned}
0 &= \bar{\nabla}_{\frac{\delta}{\delta x^m}} \tilde{g}_2 \\
&= \bar{\nabla}_{\frac{\delta}{\delta x^m}} \left(\frac{2}{\|y\|} g_{ij} dx^i \delta y^j\right) \\
&= \frac{\delta}{\delta x^m}\left(\frac{2}{\|y\|} g_{ij}\right) dx^i \delta y^j + \frac{2}{\|y\|} g_{ij} (\bar{\nabla}_{\frac{\delta}{\delta x^m}} dx^i) \delta y^j \\
&+ \frac{2}{\|y\|} g_{ij} dx^i \bar{\nabla}_{\frac{\delta}{\delta x^m}} \delta y^j \\
&= \frac{-2}{\|y\|} g_{ir} \bar{\Gamma}_{jm}{}^{r} dx^i dx^j \\
&+ \frac{2}{\|y\|} (g_{ir}\Gamma_{jm}{}^{r} + g_{jr}\Gamma_{im}{}^{r} - g_{jr}\bar{\Gamma}_{im}{}^{r} - g_{ir}\bar{\Gamma}_{jm}{}^{\bar{r}}) dx^i \delta y^j \\
&- \frac{2}{\|y\|} g_{jr} \bar{\Gamma}_{\bar{i}m}{}^{r} \delta y^i \delta y^j
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \bar{\nabla}_{\frac{\partial}{\partial y^m}} \tilde{g}_2 \\
&= \bar{\nabla}_{\frac{\partial}{\partial y^m}} \left(\frac{2}{\|y\|} g_{ij} dx^i \delta y^j\right) \\
&= 2\frac{\partial}{\partial y^m}\left(\frac{1}{\|y\|}\right) g_{ij} dx^i \delta y^j \\
&+ \frac{2}{\|y\|} g_{ij} (\bar{\nabla}_{X_{\bar{m}}} dx^i) \delta y^j \\
&+ \frac{2}{\|y\|} g_{ij} dx^i \bar{\nabla}_{X_{\bar{m}}} \delta y^j \\
&= \frac{-2}{\|y\|} g_{ir} \bar{\Gamma}_{j\bar{m}}{}^{r} dx^i dx^j \\
&+ \frac{2}{\|y\|} \left(-g_{jr}\bar{\Gamma}_{i\bar{m}}{}^{r} - g_{ir}\bar{\Gamma}_{j\bar{m}}{}^{\bar{r}} - \frac{g_{ij}y_m}{\|y\|^2}\right) dx^i \delta y^j \\
&+ \frac{2}{\|y\|} g_{jr} \bar{\Gamma}_{\bar{i}\bar{m}}{}^{r} \delta y^i \delta y^j
\end{aligned}
$$

It follows that

$$g_{ir}\bar{\Gamma}_{jm}{}^{\bar{r}} + g_{jr}\bar{\Gamma}_{im}{}^{\bar{r}} = 0, \quad (4.4)$$

$$g_{ir}(\bar{\Gamma}_{jm}{}^{r} - \bar{\Gamma}_{\bar{j}m}{}^{\bar{r}}) + g_{jr}(\Gamma_{im}{}^{r} - \bar{\Gamma}_{im}{}^{r}) = 0, \quad (4.5)$$

$$g_{ir}\bar{\Gamma}_{\bar{j}m}{}^{r} + g_{jr}\bar{\Gamma}_{\bar{i}m}{}^{r} = 0 \quad (4.6)$$

$$g_{ir}\bar{\Gamma}_{j\bar{m}}{}^{\bar{r}} + g_{jr}\bar{\Gamma}_{i\bar{m}}{}^{\bar{r}} = 0 \quad (4.7)$$

$$g_{ir}\bar{\Gamma}_{j\bar{m}}{}^{\bar{r}} + g_{jr}\bar{\Gamma}_{i\bar{m}}{}^{r} + \frac{1}{\|y\|^2} g_{ij} y_m = 0 \quad (4.8)$$

$$g_{ir}\bar{\Gamma}_{\bar{j}\bar{m}}{}^{r} + g_{jr}\bar{\Gamma}_{\bar{i}\bar{m}}{}^{r} = 0 \quad (4.9)$$

From (4.9) we have $\bar{\Gamma}_{\bar{j}\bar{i}}{}^{h} = 0$, thus we get (7). From (4.3), (4.8) and (4.6), we have

$$
\begin{aligned}
g_{ir}\bar{\Gamma}_{\bar{j}m}{}^{r} &= -g_{ir}\bar{\Gamma}_{\bar{i}m}{}^{r} = -g_{ir}\bar{\Gamma}_{m\bar{i}}{}^{r} = g_{mr}\bar{\Gamma}_{\bar{i}\bar{j}}{}^{\bar{r}} + \frac{g_{mj}y_i}{\|y\|^2} \\
&= -g_{ir}\bar{\Gamma}_{m\bar{j}}{}^{r} - \frac{g_{im}y_j}{\|y\|^2} + \frac{g_{mj}y_i}{\|y\|^2} \\
&= -g_{ir}\bar{\Gamma}_{\bar{j}m}{}^{r} - \frac{g_{im}y_j}{\|y\|^2} + \frac{g_{mj}y_i}{\|y\|^2}
\end{aligned}
$$

thus we get (3). From (3) and (4.3), we have (4). From $(4.8), (4)$ and (4.3), we have

$$g_{ir}\overline{\Gamma}_{\bar{j}\bar{m}}{}^{\bar{r}}+\frac{1}{2\|y\|^2}g_{im}y_j-\frac{1}{2\|y\|^2}g_{ji}y_m$$
$$+\frac{g_{ij}y_m}{\|y\|^2}=0,$$

Thus we obtain (8).

From (4.3) and (4.4) we have

$$\begin{aligned} g_{ir}\overline{\Gamma}_{jm}{}^{\bar{r}} &= -g_{jr}\overline{\Gamma}_{im}{}^{\bar{r}} = -g_{jr}(\overline{\Gamma}_{mi}{}^{\bar{r}}+y^aK_{ima}{}^{r}) \\ &= g_{mr}\overline{\Gamma}_{ji}{}^{\bar{r}}-y^aK_{imaj} \\ &= g_{mr}\overline{\Gamma}_{ij}{}^{\bar{r}}+y^a(K_{jiam}-K_{imaj}) \\ &= -g_{ir}\overline{\Gamma}_{mj}{}^{\bar{r}}+y^a(K_{jiam}-K_{imaj}) \\ &= -g_{ir}(\overline{\Gamma}_{jm}{}^{\bar{r}}+y^aK_{mja}{}^{r})+y^a(K_{jiam}-K_{imaj}), \end{aligned}$$

thus we get (2).

From $(4.3),(4.5)$ and (4.7), we have

$$\begin{aligned} g_{ir}\overline{\Gamma}_{j\bar{m}}{}^{\bar{r}} &= -g_{jr}\overline{\Gamma}_{i\bar{m}}{}^{\bar{r}} = -g_{jr}(\overline{\Gamma}_{\bar{m}i}{}^{\bar{r}}-\Gamma_{mi}{}^{r}) \\ &= g_{jr}(\Gamma_{mi}{}^{r}-\overline{\Gamma}_{\bar{m}i}{}^{\bar{r}}) = -g_{mr}(\Gamma_{ji}{}^{r}-\overline{\Gamma}_{ji}{}^{r}) \\ &= -g_{mr}(\Gamma_{ij}{}^{r}-\overline{\Gamma}_{ij}{}^{r}) = g_{ir}(\Gamma_{mj}{}^{r}-\overline{\Gamma}_{\bar{m}j}{}^{\bar{r}}) \\ &= g_{ir}\Gamma_{mj}{}^{r}-g_{ir}\overline{\Gamma}_{\bar{m}j}{}^{\bar{r}} \\ &= g_{ir}\Gamma_{mj}{}^{r}-g_{ir}(\overline{\Gamma}_{j\bar{m}}{}^{\bar{r}}+\Gamma_{mj}{}^{r}) \end{aligned}$$

Thus we obtain (5) and (6). From (4.5) and (5), we have (1).

5. THE ALMOST ANTI-HERMITIAN STRUCTURE $(\tilde{g}_2,\tilde{J})$

The almost complex structure J defined in (2.6) has not the property of homogeneity. The $F(\widetilde{TM})$-linear mapping $J:\chi(\widetilde{TM})\to\chi(\widetilde{TM})$, applies the 1-homogeneous vector fields $\frac{\delta}{\delta x^i}$ into 0-homogeneous vector fields $\frac{\partial}{\partial y^i}$ $(i=1,\ldots,n)$. Therefore, we consider the $F(\widetilde{TM})$-linear mapping $\tilde{J}:\chi(\widetilde{TM})\to\chi(\widetilde{TM})$, given on the adapted basis by

$$\tilde{J}(\frac{\delta}{\delta x^i})=-\|y\|\frac{\partial}{\partial y^i},\quad \tilde{J}(\frac{\partial}{\partial y^i})=\frac{1}{\|y\|}\frac{\delta}{\delta x^i},\quad (5.1)$$

It is not difficult to prove:

Theorem 6. *$\tilde{J}$ has the following properties:*

1. *$\tilde{J}$ is a tensor field of type $(1,1)$ on $\widetilde{TM}$;*
2. *$\tilde{J}$ is an almost complex structure on $\widetilde{TM}$, i.e, $\tilde{J}\circ\tilde{J}=-I$*
3. *$\tilde{J}$ depends only on the metric g.*
4. *$\tilde{J}$ is homogeneous on the fibers of TM.*

Proposition 7. *In the adapted basis we have the unique decomposition*

$$N_{\tilde{J}}(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})=(N_{\tilde{J}})_{ij}{}^{k}\frac{\delta}{\delta x^k}+(N_{\tilde{J}})_{ij}{}^{\bar{k}}\frac{\partial}{\partial y^k}$$
$$N_{\tilde{J}}(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})=(N_{\tilde{J}})_{i\bar{j}}{}^{k}\frac{\delta}{\delta x^k}+(N_{\tilde{J}})_{i\bar{j}}{}^{\bar{k}}\frac{\partial}{\partial y^k}$$
$$N_{\tilde{J}}(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})=(N_{\tilde{J}})_{\bar{i}\bar{j}}{}^{k}\frac{\delta}{\delta x^k}+(N_{\tilde{J}})_{\bar{i}\bar{j}}{}^{\bar{k}}\frac{\partial}{\partial y^k}$$

with

$$(N_{\tilde{J}})_{ij}{}^{\bar{k}}=y_i\delta_j^s-y_j\delta_i^s-y^aK_{jia}{}^{s},$$
$$(N_{\tilde{J}})_{i\bar{j}}{}^{k}=\frac{1}{\|y\|^2}(y_i\delta_j^s-y_j\delta_i^s-y^aK_{jia}{}^{s}),$$
$$(N_{\tilde{J}})_{\bar{i}\bar{j}}{}^{\bar{k}}=\frac{1}{\|y\|^2}(y_j\delta_i^s-y_i\delta_j^s+y^aK_{jia}{}^{s}),$$
$$(N_{\tilde{J}})_{ij}{}^{k}=0,\ (N_{\tilde{J}})_{i\bar{j}}{}^{\bar{k}}=0,\ (N_{\tilde{J}})_{\bar{i}\bar{j}}{}^{k}=0.$$

Proof. *Recall that the Nijenhuis tensor field $N_{\tilde{J}}$ defined by $\tilde{J}$ is given by*

$$N_{\tilde{J}}(X,Y)=[\tilde{J}X,\tilde{J}Y]-\tilde{J}[\tilde{J}X,Y]-\tilde{J}[X,\tilde{J}Y]$$
$$-[X,Y],\qquad \forall X,Y\in\chi(\widetilde{TM}).$$

By the compatibility and direct computation we have:

$$\begin{aligned} \frac{\delta}{\delta x^i}(\|y\|) &= \frac{\delta}{\delta x^i}(\sqrt{g_{rs}y^ry^s}) \\ &= \frac{1}{2\|y\|}\frac{\delta}{\delta x^i}(g_{rs}y^ry^s)=0, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial y^i}(\|y\|) &= \frac{\partial}{\partial y^i}(\sqrt{g_{rs}y^ry^s}) \\ &= \frac{1}{2\|y\|}\frac{\partial}{\partial y^i}(g_{rs}y^ry^s) \\ &= \frac{1}{2\|y\|}(2g_{is}y^s)=\frac{y_i}{\|y\|}. \end{aligned}$$

Replacing the basis $(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i})$ in the $N_{\tilde{J}}$ and using above relation we get the proof.

Theorem 8. $\tilde{J}$ *is a complex structure on* $\widetilde{TM}$ *if and only if*

$$y^a K_{jia}{}^s = y_i \delta_j^s - y_j \delta_i^s . \quad (5.2)$$

Proof. Setting $N_{\tilde{J}} = 0$ in the previous proposition, the proof is completed.

Theorem 9. The almost complex structure $\tilde{J}$ *is a complex structure on* $\widetilde{TM}$ *if and only if the Riemannian space* (M, g) *is of constant curvature 1.*

Proof. From (5.2) and $y_i = g_{ia} y^a$ we obtain

$$K_{jia}{}^s = g_{ia}\delta_j^s - g_{ja}\delta_i^s . \quad (5.3)$$

Theorem 10. We have:

1. $(\tilde{g}_2, \tilde{J})$ *is an almost anti-Hermitian structure on* $\widetilde{TM}$ *;*
2. $(\tilde{g}_2, \tilde{J})$ *depends only on the metric* g *of the pseudo-Riemannian space* (M, g).

Proof.

1. Follows from the equation $\tilde{g}_2(\tilde{J}X, \tilde{J}Y) = -\tilde{g}_2(X, Y)$ on $\widetilde{TM}$.
2. $\tilde{g}_2$ and $\tilde{J}$ depending only on g, the anti-Hermitian structure $(\tilde{g}_2, \tilde{J})$ has the same property.

Corollary 11. The almost anti-Hermitian structure $(\tilde{g}_2, \tilde{J})$ *is an anti-Hermitian structure on* $\widetilde{TM}$ *if and only if the space* (M, g) *is of constant curvature 1.*

Proof. From the theorem 9 and the first part of theorem 10, we get the proof.

From (5.3) we have

$$R_{ij} = (n-1) g_{ij}, \quad (n > 1) \quad (5.4)$$

where R_{rk} is the Ricci tensor and

$$S = n(n-1), \quad (5.5)$$

where S is the scalar tensor defined by $S = R_k^k$.

Corollary 12. If the structure $(\tilde{g}_2, \tilde{J})$ *is a Hermitian structure on* $\tilde{T}M$ *then* (M, g) *is an Einstein space with positive scalar curvature.*

Since $R_{ij} = R_{ji}$ then from (5.4) we get:

Corollary 13. If $(\widetilde{TM}, \tilde{J})$ *is a complex manifold, then* $(M, R_{ij}(x))$ *is a Riemannian space.*

6. THE ALMOST PRODUCT STRUCTURE $(\tilde{g}_2, \tilde{Q})$

The almost product structure Q defined in (2.7) has not the property of homogeneity. The $F(\widetilde{TM})$-linear mapping $Q : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, applies the 1-homogeneous vector fields X_i into 0-homogeneous vector fields $\frac{\partial}{\partial y^i}$ $(i = 1, \dots, n)$. Therefore, we consider the $F(\widetilde{TM})$-linear mapping $\tilde{Q} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, given on the adapted basis by

$$\tilde{Q}(\frac{\delta}{\delta x^i}) = \| y \| \frac{\partial}{\partial y^i}, \quad \tilde{Q}(\frac{\partial}{\partial y^i}) = \frac{1}{\| y \|} \frac{\delta}{\delta x^i}, \quad (6.1)$$

It is not difficult to prove:

Theorem 14. $\tilde{Q}$ *has the following properties:*

1. $\tilde{Q}$ *is a tensor field of type* $(1,1)$ *on* $\widetilde{TM}$ *;*
2. $\tilde{Q}$ *is an almost product structure on* $\widetilde{TM}$; $\tilde{Q} \circ \tilde{Q} = I$
3. $\tilde{Q}$ *depends only on the metric* g *;*
4. $\tilde{Q}$ *is homogeneous on the fibres of* TM.

In order to find conditions that $\tilde{Q}$ be a product structure, we have to put zero for the Nijenhuis tensor field of $\tilde{Q}$,

$$N_{\tilde{Q}}(X, Y) = [\tilde{Q}X, \tilde{Q}Y] - \tilde{Q}[\tilde{Q}X, Y] - \tilde{Q}[X, \tilde{Q}Y] + [X, Y], \forall X, Y \in \chi(\widetilde{TM}).$$

Theorem 15. The almost product structure $\tilde{Q}$ *is a product structure on* $\widetilde{TM}$ *if and only if the Riemann space* (M, g) *is of constant curvature* -1.

Proof. In the adapted basis, the Nijenhuis tensor is as follows:

$$N_{\tilde{Q}}(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}) = (y_i\delta_j^s - y_j\delta_i^s + y^a K_{jia}{}^s)\frac{\partial}{\partial y^s}$$

$$N_{\tilde{Q}}(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}) = \frac{1}{\| y \|^2}(y_j\delta_i^s - y_i\delta_j^s - y^a K_{jia}{}^s)\frac{\delta}{\delta x^s}$$

$$N_{\tilde{Q}}(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}) = \frac{1}{\| y \|^2}(y_i\delta_j^s - y_j\delta_i^s + y^a K_{jia}{}^s)\frac{\partial}{\partial y^s}$$

then $N_{\tilde{Q}}$ vanishes if and only if we have:

$$y^a K_{jia}{}^s = -(y_i\delta_j^s - y_j\delta_i^s). \qquad (6.2)$$

According to $y_i = g_{ia}y^a$ and the above equation one obtains

$$K_{jia}{}^s = -(g_{ia}\delta_j^s - g_{ja}\delta_i^s). \qquad (6.3)$$

which completes the proof.

Theorem 16. *We have:*

1. $(\tilde{g}_2,\tilde{Q})$ *is an almost product structure on* $\widetilde{TM}$;

2. $(\tilde{g}_2,\tilde{Q})$ depends only on the metric g of the pseudo-Riemannian space (M,g).

Proof.

1. Follows from the equation $\tilde{g}_2(\tilde{Q}X,\tilde{Q}Y) = \tilde{g}_2(X,Y)$ on $\widetilde{TM}$.

2. $\tilde{g}_2$ and $\tilde{Q}$ depending only on g, the almost product structure $(\tilde{g}_2,\tilde{Q})$ has the same property.

Corollary 17. *The almost product structure* $(\tilde{g}_2,\tilde{Q})$ *is a product structure on* $\tilde{T}M$ *if and only if the space* (M,g) *is of constant curvature* -1.

Theorem 18. *If the structure* $(\tilde{g}_2,\tilde{Q})$ *is a product structure on* $\tilde{T}M$ *then* (M,g) *is an Einstein space with negative scalar curvature.*

Proof. *From* (6.2) *we have* $R_{ij} = (1-n)g_{ij}$, $S = n(1-n)$ *for* $n>1$.

Corollary 19. *If the almost product structure* $\tilde{Q}$ *is a product structure then* $(M, R_{ij}(x))$ *is a Riemannian space.*

Proof. Since $R_{ij} = R_{ji}$ then we get proof.

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7. References

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