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An Exact Analytical Solution for Convective Heat Transfer in Elliptical Pipes

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ABSTRACT: In this paper, an analytical solution for convective heat transfer in straight pipes with the elliptical cross section is presented. The solution is obtained for steady-state fluid flow and heat transfer under the constant heat flux at walls using the finite series expansion method. Here, the exact solution of Nusselt number as well as temperature distribution in terms of aspect ratio is presented as the correlation in the Cartesian coordinate system and validated with the previous investigations. It is shown that the minimum amount of Nusselt number, as well as the maximum absolute value of dimensionless temperature at the center of the cross section, are related to the aspect ratio equal to 1 (circular pipe). The solution indicated that the amount of Nusselt number is increased by changing the geometry of cross section from circular to an elliptical shape and it finally tends to 4356/833 at large enough aspect ratios. Our results also show that 95% of the increase in Nusselt number to the circular cylinder is related to aspect ratio equal to 18.36. The present method of solution could be used to obtain the exact solution of convective heat transfer in elliptical pipes for other thermal boundary conditions and fluid rheological behaviors.

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1- Introduction

The investigation of the convective heat transfer in straight pipes with various shapes of cross section has been an interesting subject for many researchers from the experimental, numerical and analytical point of view. This problem is very important in medical treatments, industrial processes, engineering applications and biological systems.

The most of the previous studies in this field are restricted to the investigation of heat transfer and fluid flow in straight and curved pipes with circular cross section. Shah [1] investigated numerically the effect of the cross-sectional shape on the forced convective heat transfer inside straight ducts by considering isosceles triangular, rounded corner equilateral triangular, sine, rhombic, and trapezoidal cross-sections. His results show that rounding the corners of the pipe cross-section increases the rate of the heat transfer. Shah and London [2] reviewed analytical solutions for laminar fluid flow and forced convective heat transfer in circular and noncircular pipes. They considered both H1 (axial uniform wall heat flux with peripherally uniform wall temperature) and H2 (The both circumferential and axial uniform heat flux) boundary conditions at walls in their study. Wibulswas [3] used a numerical method to solve force convection heat transfer in rectangular, right-angled isosceles triangular and equilateral triangular ducts. Lyczkowski et al. [4] numerically studied the forced convective heat transfer of Newtonian fluid flow in a straight pipe with rectangular cross section. They simplified boundary conditions by ignoring axial conduction in the heat transfer equations. Zhang et al. [5] in an analytical and numerical investigation, studied convective heat transfer in the inlet thermal zone of ducts with various shapes of the cross-section. Their solution is valid for thermally developing and hydrodynamically fully developed laminar flow. Barletta et al. [6] in a numerical investigation studied the mixed convective heat transfer in a vertical rectangular duct. They used Galerkin finite element method to solve a dimensionless form of energy and momentum equations. Their results show that heat transfer rate is affected by that aspect ratio as well as the ratio of the Grashof number to the Reynolds number. Nonino et al. [7] studied numerically the convective heat transfer of laminar flow in entrance thermal region of straight ducts under constant wall temperature boundary conditions. They considered temperature as a function of viscosity and showed that temperature dependence of the viscosity cannot be ignored. Rennie and Vijaya Raghavan [8] also investigated the effects of the temperature dependence of the viscosity on heat transfer in a horizontal cylinder for a double-pipe helical exchanger of non-Newtonian fluid. Iacovides et al. [9] studied experimentally and numerically the flow and heat transfer in straight ducts with ribs along two opposite walls. They found that the rate of heat transfer is greatly affected by the ribs. Some related studies have been also carried out which are generally focused on the heat transfer in rectangular ducts with ribs along the walls. ([10- 12]). Ray and Misra [13] studied convective heat transfer of fully developed laminar flow in straight ducts with square and triangular cross sections with rounded corners. Zhang and Chen [14] studied fluid flow and convective heat transfer in corrugated triangular cross section ducts under the uniform heat flux boundary condition. They showed that in contrast with parallel-plate ducts, these kinds of channels have the higher value of heat transfer rate. Shahmardan et al. [15] presented an analytical solution for temperature distribution and Nusselt number of convective heat transfer in straight ducts with the rectangular cross section under the constant heat flux at walls. Shahmardan et al. [16] in another investigation proposed an exact analytical solution for fully developed convective heat

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transfer in equilateral triangular with H1 boundary condition. Some studies have been carried out on convective heat transfer of non-Newtonian flows in rectangular ducts. Sayed-Ahmed and Kishk [17] in a numerical investigation studied the laminar flow and heat transfer of Herschel-Bulkley fluids in the entrance region of a rectangular duct. They investigated the effects of the aspect ratio, Prandtl number, velocity, and pressure gradient on the Nusselt number. The effect of corner vortices in viscoelastic flows in straight rectangular ducts is also studied by Norouzi et al [18]. Their results show that the rate of heat and mass transfer in straight rectangular channels is enhanced by intensifying the corner vortices.

A cursory inspection of the aforementioned surveys clearly reveals that the bulk of the literature relates to a circular cylinder, followed by that of rectangular cross section. However, the limited body of knowledge relates to the study of flow and heat transfer in the pipe with elliptical cross section. The fully developed laminar H1 and H2 heat transfer problem for elliptical ducts was first investigated by Claiborne [19]. He used Fourier series to solve the energy equation to obtain approximate temperature distribution and also the Nusslet number for various amounts of aspect ratios. Tao [20] by employing the method of complex variables studied the forced convection problems in few basic problems, including flows in equilateral triangular ducts and in elliptical tubes. This method [20] because of using a complex variable to solve the energy equation has some complexities to compute temperature distribution as well as Nusselt number. Cheng [21] studied natural convective heat transfer in a horizontal isothermal elliptical cylinder with temperature-dependent viscosity. This study [21] showed that the total heat transfer rate and the total skin friction of the elliptical cylinder with slender orientation are higher than those of the elliptical cylinder with blunt orientation. Moreover, increasing the viscosity-variation parameter enhances the heat transfer rates. Sakalis et al. [22] numerically studied the laminar fully developed and developing heat transfer in a straight pipe with the elliptical cross section under isothermal boundary condition. They showed that by increasing the aspect ratio of the ellipse, the friction factor is decreased. Their results also showed that in the thermally developing flow, Nusselt Number is increased by decreasing the aspect ratio. Velusamy et al. [23] numerically studied fully developed laminar flow and heat transfer in ducts of the semi-elliptical cross section for isothermal and a uniform axial heat flux condition on the duct walls. They showed that for ducts in which the baseplate is on the major axis, friction factor and Nusselt number for the uniform heat flux condition are increased as the aspect ratio decreases. They also represented that the ratio of the Nusselt number to friction factor is higher for semi-elliptical ducts in comparison to that for other ducts, such as sinusoidal, circular segmental, and isosceles triangular ducts. Velusamy and Garg [24] studied the same problem for vertical elliptical ducts, including buoyancy forces and H1 thermal boundary condition. Javeri [25] numerically analyzed the hydrodynamically developed and thermally developing flow into straight ducts of square, circular and elliptical cross-section, for the thermal boundary condition of linearly varying wall temperature in the axial direction (LAWT). In an experimental investigation, the simultaneously developing hydrodynamic and thermal flow in straight elliptical ducts with the LAWT boundary condition has been performed by Abdel-Wahed et al. [26].

According to the literature, most of the previous studies deal with fluid flow and heat transfer in ducts with circular and rectangular cross section and only a few numerical studies are available about convective heat transfer in pipes with elliptical cross sections. While the flows in elliptical ducts are of increasing importance in microfluidics, where lithographic methods typically produce channels with the noncircular shape of cross section. These channels are widely used in biological kits (like the kits for extraction the DNA, detection of cancers cells and bacteria, blood sample preparation and glucose monitoring), fuel cells and cooling systems for small scales. The flow in elliptical pipes is especially important in microfluidic, compact heat exchangers and biological flows (such as flow in vessels). In this study, a new accurate technique based on an analytical solution is used for convective heat transfer in straight pipes with an elliptical cross section. The present analytical solution is a step forward in the field of heat transfer through non-circular pipes as a benchmark problem which is tested and validated based on the previous analytical studies. The proposed technique of the solution for flow in elliptical pipes could be generalized to solve the other similar problems. The main novelty of this study is suggesting an analytical method to study the forced convection heat transfer in elliptical pipes which is characterized by simple computation, easy implementation of boundary conditions, easy to use in different geometries as well as using in forced convection of dissipative fluid in the channel. The solution is obtained for steady convective heat transfer under the constant heat flux at walls using the series expansion method. The geometry of the problem is shown in Fig. 1. The closed form of dimensionless temperature distribution is obtained in Cartesian coordinate system. Here, the exact solution of the Nusselt number in terms of aspect ratio is presented.

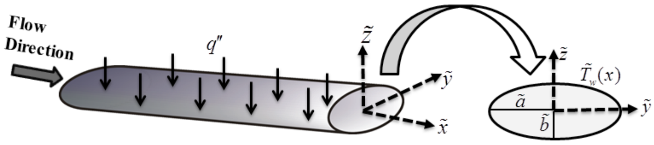


Fig. 1. The geometry of duct in the present study

2- Governing Equations

The governing equations of incompressible fluid flow and heat transfer in a duct with elliptical cross section consist of the continuity, momentum, and energy equations are presented as follows [2]:

$$\nabla.\tilde{V} = 0 \tag{1a}$$

$$\rho\left(\frac{\partial \tilde{V}}{\partial t} + \tilde{V}.\nabla\tilde{V}\right) = -\nabla\tilde{P} + \rho\tilde{g} + \mu\nabla^2\tilde{V} \tag{1b}$$

$$\rho c_p \tilde{V}.\nabla\tilde{T} = k\,\nabla^2\tilde{T} \tag{1c}$$

Where $\tilde{V}$ is the velocity vector, ρ is the density, $\tilde{P}$ is the static pressure, $\tilde{t}$ is the time, μ is the viscosity and $\tilde{T}$ is the temperature of the fluid flow. The non-dimensional parameters of this problem can be expressed as:

$$y=\frac{\tilde{y}}{d_h} \qquad z=\frac{\tilde{z}}{d_h} \qquad a=\frac{\tilde{a}}{d_h}$$
$$b=\frac{\tilde{b}}{d_h} \qquad \eta=\frac{\tilde{b}}{\tilde{a}}=\frac{b}{a} \qquad u=\frac{\tilde{u}}{u_b} \tag{2}$$
$$\alpha=\frac{k}{\rho c_p} \qquad T=\frac{\tilde{T}-\tilde{T}_w}{q''d_h/k}$$

where y and z are the coordinates in the Cartesian coordinate system (see Fig. 1), $\tilde{a}$ and $\tilde{b}$ are the major and minor axis of the elliptical cross-section, η is the aspect ratio, α is the thermal diffusivity coefficient and $\tilde{u}$ is the main flow velocity of the fluid flow. In addition, d_h is the hydraulic diameter, u_b is the bulk velocity, and T_m is the mean temperature of fluid flow. These parameters are defined as follows [27]:

$$\mathrm{d_h}=\frac{4\tilde{\mathrm{A}}}{\tilde{\mathrm{P}}}=\frac{\pi\tilde{\mathrm{b}}}{\mathrm{E(e)}} \tag{3a}$$

$$\mathrm{u_b}=\frac{1}{\tilde{\mathrm{A}}}\int_{\mathrm{A}}\tilde{\mathrm{u}}\,\mathrm{d}\tilde{\mathrm{A}} \tag{3b}$$

$$\tilde{\mathrm{T}}_{\mathrm{m}}=\frac{1}{\rho \mathrm{c_p u_b}\tilde{\mathrm{A}}}\int_{\mathrm{A}}\rho \mathrm{c_p}\tilde{\mathrm{u}}\tilde{\mathrm{T}}\mathrm{d}\tilde{\mathrm{A}} \tag{3c}$$

where $\tilde{A}$ and $\tilde{P}$ are the cross-sectional area and perimeter of the duct, respectively. In addition, e is the eccentricity of the ellipse which has the value of $e=\sqrt{1-(\tilde{a}/\tilde{b})^2}$ and the function E is the complete elliptic integral of the second kind that can be calculated as follows [27]:

$$E(e)=\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\;d\theta \tag{4}$$

Applying thermal energy balance on a differential control volume in the axial direction gives [26]:

$$\mathrm{q''}\tilde{\mathrm{p}}\mathrm{d}\tilde{\mathrm{x}}=\rho\tilde{\mathrm{A}}\mathrm{u_b c_p dT_m} \Rightarrow \frac{\mathrm{d}\tilde{\mathrm{T}}_{\mathrm{m}}}{\mathrm{d}\tilde{\mathrm{x}}}=\frac{\mathrm{q''}\tilde{\mathrm{p}}}{\rho\tilde{\mathrm{A}}\mathrm{u_b c_p}}=\mathrm{cte} \tag{5}$$

The following condition exists for fully developed convective heat transfer in a closed channel:

$$\frac{\partial}{\partial\tilde{x}}\left(\frac{\tilde{T}-\tilde{T}_w}{\tilde{T}_m-\tilde{T}_w}\right)=0 \tag{6}$$

Using Eqs. (5), and (6), and considering the constant heat flux at walls ($q''=h(\tilde{T}_w-\tilde{T}_m)$), we have [28]:

$$\frac{d\tilde{T}_m}{d\tilde{x}}=\frac{\partial\tilde{T}}{\partial\tilde{x}}=\frac{d\tilde{T}_w}{d\tilde{x}}=\frac{q''\tilde{p}}{\rho\tilde{A}u_b c_p}=Cons\tan t \tag{7}$$

Inserting Eqs. (2) and (7) into Eq. (1c) and considering the fully developed rectilinear flow (setting the transverse velocity components to zero), the following dimensionless form of the heat transfer equation is obtained:

$$\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=4u(y,z) \tag{8}$$

The main flow velocity and flow rate of rectilinear flow in a duct with an elliptical cross-section are as follows [29]:

$$\tilde{u}(\tilde{y},\tilde{z})=\frac{1}{2\mu}\left(-\frac{d\tilde{p}}{d\tilde{x}}\right)\frac{\tilde{a}^2\tilde{b}^2}{\tilde{a}^2+\tilde{b}^2}\left(1-\frac{\tilde{y}^2}{\tilde{a}^2}-\frac{\tilde{z}^2}{\tilde{b}^2}\right) \tag{9a}$$

$$\tilde{Q}=\frac{\pi}{4\mu}\left(-\frac{d\tilde{p}}{d\tilde{x}}\right)\frac{\tilde{a}^3\tilde{b}^3}{\tilde{a}^2+\tilde{b}^2} \tag{9b}$$

The bulk velocity ($\tilde{u}_b$) can be obtained as:

$$\tilde{u}_b=\frac{\tilde{Q}}{\tilde{A}}=\frac{1}{4\mu}\left(-\frac{d\tilde{p}}{d\tilde{x}}\right)\frac{\tilde{a}^2\tilde{b}^2}{\tilde{a}^2+\tilde{b}^2} \tag{10}$$

Based on the Eqs. (9) and (10), the dimensionless main flow velocity can be expressed as follows:

$$u=2\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right) \tag{11}$$

Inserting the Eq. (11) into the Eq. (8), the heat transfer equation of fully developed flow in a straight elliptical pipe is obtained:

$$\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=8\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right) \tag{12}$$

Based on the definition of dimensionless temperature, the boundary condition of Eq. (11) can be expressed as (at wall: $\tilde{T}=\tilde{T}_w$):

$$at \quad \frac{y^2}{a^2}+\frac{z^2}{b^2}=1: \qquad T=0 \tag{13}$$

3- Exact Solution

The governing equation of non-dimensional temperature is expressed in Eq. (12). The final solution of this partial differential equation is the summation of the general and particular solution. According to a lemma in calculus, if we found a particular solution which satisfies the boundary conditions of a differential equation, then the general solution will be zero. In this condition, the final solution is completely equal to the particular solution. In this study, the particular solution is polynomial because Eq. (12) is linear and the non-homogenous term of this equation is a polynomial. Here, we found a particular solution to Eq. (12) which satisfies the boundary condition (Eq. (13)). According to the mentioned lemma, the general solution is zero and the particular solution is the final solution. Due to the symmetry properties of temperature around y and z axis, only even powers of these variables should be considered. Thus a fourth-degree polynomial regarding y and z has been assumed for the dimensionless temperature distribution (T) as follows:

$$T(y,z)=C_1y^4+C_2z^4+C_3y^2+C_4z^2 +C_5y^2z^2+C_6 \tag{14}$$

It should be noted that according to Eq. (12) a higher degree of the polynomial (e.g. 6, 8 …) leads to zero the coefficients of these terms.

By substituting the Eq. (14) into Eq. (12), we have:

$$T_{yy}+T_{zz}=\left(12C_1+2C_5\right)y^2+\left(12C_2+2C_5\right)z^2$$
$$+2\left(C_3+C_4\right)=8\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right) \tag{15}$$

By comparing the terms of polynomial coefficients on both sides of the equation, we have:

$$6C_1+C_5=-\frac{4}{a^2} \tag{16a}$$

$$6C_2+C_5=-\frac{4}{b^2} \tag{16b}$$

$$C_3+C_4=4 \tag{16c}$$

By applying the Eq. (15) into boundary condition (Eq. (13)) and comparing the terms of polynomial coefficients, three other equations obtained:

$$\frac{a^4}{b^4}C_1+C_2-\frac{a^2}{b^2}C_5=0 \tag{16d}$$

$$-2\frac{a^4}{b^2}C_1-\frac{a^2}{b^2}C_3+C_4+a^2C_5=0 \tag{16e}$$

$$a^4C_1+a^2C_3+C_6=0 \tag{16f}$$

By solving the Eqs. (16), six unknown coefficients are achieved as:

$$C_1=\frac{-2\left(b^4+5a^2b^2\right)}{3a^2\left(a^4+b^4+6a^2b^2\right)} \tag{17a}$$

$$C_2=\frac{-2\left(a^4+5a^2b^2\right)}{3b^2\left(a^4+b^4+6a^2b^2\right)} \tag{17b}$$

$$C_3=\frac{4\left(16a^2b^4+5a^4b^2+3b^6\right)}{3\left(a^2+b^2\right)\left(a^4+b^4+6a^2b^2\right)} \tag{17c}$$

$$C_4=\frac{4\left(16a^4b^2+5a^2b^4+3a^6\right)}{3\left(a^2+b^2\right)\left(a^4+b^4+6a^2b^2\right)} \tag{17d}$$

$$C_5=\frac{-4\left(a^2+b^2\right)}{a^4+b^4+6a^2b^2} \tag{17e}$$

$$C_6=\frac{-2\left(26a^4b^4+5a^6b^2+5a^2b^6\right)}{3\left(a^2+b^2\right)\left(a^4+b^4+6a^2b^2\right)} \tag{17f}$$

Based on the Eqs. (2) and (3a), we have:

$$a^2=\frac{\tilde{a}^2}{d_h^{\,2}}=\left(\frac{E(e)}{\pi}\right)^2\frac{1}{\eta^2} \tag{18a}$$

$$b^2=\frac{\tilde{b}^2}{d_h^{\,2}}=\left(\frac{E(e)}{\pi}\right)^2 \tag{18b}$$

Eq. (17) can be presented in terms of aspect ratio using Eqs. (2) and (18):

$$C_1=-\left(\frac{\pi}{E(e)}\right)^2\frac{2\eta^4\left(\eta^2+5\right)}{3\left(\eta^4+6\eta^2+1\right)} \tag{19a}$$

$$C_2=-\left(\frac{\pi}{E(e)}\right)^2\frac{2\left(5\eta^2+1\right)}{3\left(\eta^4+6\eta^2+1\right)} \tag{19b}$$

$$C_3=\frac{4\eta^2\left(3\eta^4+16\eta^2+5\right)}{3\left(\eta^6+7\eta^4+7\eta^2+1\right)} \tag{19c}$$

$$C_4=\frac{4\left(5\eta^4+16\eta^2+3\right)}{3\left(\eta^6+7\eta^4+7\eta^2+1\right)} \tag{19d}$$

$$C_5=-\left(\frac{\pi}{E(e)}\right)^2\frac{4\eta^2\left(\eta^2+1\right)}{\eta^4+6\eta^2+1} \tag{19e}$$

$$C_6=-\left(\frac{E(e)}{\pi}\right)^2\frac{2\left(5\eta^4+26\eta^2+5\right)}{3\left(\eta^6+7\eta^4+7\eta^2+1\right)} \tag{19f}$$

Finally, by substituting Eq. (19) into Eq. (14), the non-dimensional temperature profile is obtained as follows:

$$\begin{aligned}T(y,z)=&\frac{-2\left(\frac{\pi}{E(e)}\right)^2}{3\left(\eta^6+7\eta^4+7\eta^2+1\right)}\Bigg\{\\&\eta^4\left(\eta^4+6\eta^2+5\right)y^4+\left(5\eta^4+6\eta^2+1\right)z^4+\\&-2\left(\frac{E(e)}{\pi}\right)^2\left[\eta^2\left(3\eta^4+16\eta^2+5\right)y^2\right.\\&\left.+\left(5\eta^4+16\eta^2+3\right)z^2\right]+6\eta^2\left(\eta^4+2\eta^2+1\right)y^2z^2\\&+\left(\frac{E(e)}{\pi}\right)^4 5\eta^4+26\eta^2+5\Bigg\}\end{aligned} \tag{20}$$

By substituting y=0 and z=0 in Eq. (20), the value of dimensionless temperature at the center of cross section is obtained as:

$$T_C=-\left(\frac{E(e)}{\pi}\right)^2\frac{2\left(5\eta^4+26\eta^2+5\right)}{3\left(\eta^6+7\eta^4+7\eta^2+1\right)} \tag{21}$$

The value of $E(e)$ can be calculated approximately as [27]:

$$E(e)\approx\frac{\pi}{4}(\eta+1)\left[1+\frac{3s}{10+\sqrt{4-3s}}\right] \tag{22}$$

Where:

$$s=\frac{(1-\eta)^2}{(1+\eta)^2} \tag{23}$$

Thus the approximate value of T_c is obtained as:

$$T_C\approx-\frac{\left(5\eta^4+26\eta^2+5\right)(\eta+1)^2}{24\left(\eta^6+7\eta^4+7\eta^2+1\right)}\times\left[1+\frac{3s}{10+\sqrt{4-3s}}\right]^2 \tag{24}$$

The convection coefficient can be found using the following relation:

$$h = \frac{q''}{\tilde{T}_w - \tilde{T}_m} \tag{25}$$

Therefore, using Eq. (25) and non-dimensional form of T_m from Eq. (2), the Nusselt number is calculated using the following equation:

$$Nu = -\frac{1}{T_m} = -\frac{1}{\frac{1}{A}\int_A uTdA} \tag{26}$$

Using the obtained temperature distribution, the Nusselt number can be calculated as:

$$Nu = \frac{9\left(a^2+b^2\right)\left(a^4+b^4+6a^2b^2\right)}{a^2b^2\left(98a^2b^2+17a^4+17b^4\right)} \tag{27}$$

Using Eqs. (18), Eq. (27) is simplified as:

$$Nu = 9\left(\frac{\pi}{E(e)}\right)^2 \frac{\eta^6+7\eta^4+7\eta^2+1}{17\eta^4+98\eta^2+17} \tag{28}$$

Eq. (28) presents the Nusselt number of elliptical pipes in terms of the aspect ratio.

Using Eq. (22) and (23) the approximate value of *Nu* can be obtained as:

$$Nu \approx 144\left(\frac{10+\sqrt{4-3s}}{10+3s+\sqrt{4-3s}}\right)^2 \times \frac{\eta^6+7\eta^4+7\eta^2+1}{(\eta+1)^2\left(17\eta^4+98\eta^2+17\right)} \tag{29}$$

4- Results and Discussion

Fig. 2 shows the comparison of the Nu number versus aspect ratio (η) for forced convection heat transfer in elliptical pipes between the current analytical solution (Eq. (29)) and the study of Shah and London [2]. As this figure indicates, both results are in a suitable agreement.

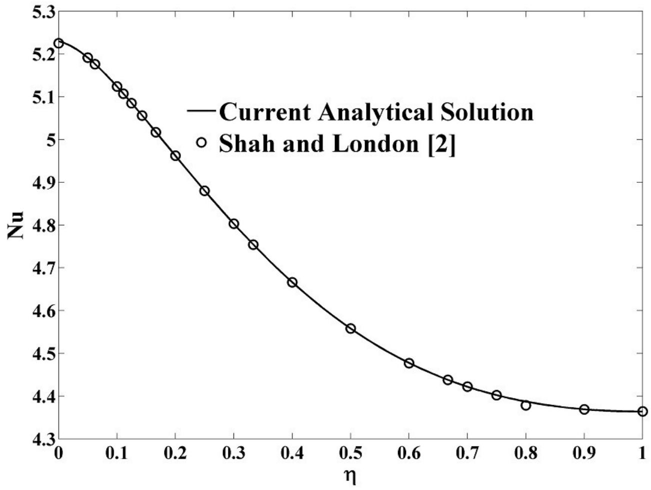


Fig. 2. The compression of Nusselt number versus aspect ratio (η) for forced convection heat transfer in elliptical pipes between the current analytical solution (Eq. (28)) and study of Shah and London [2]

As the second validation test, the results of the present analytical solution for convective heat transfer in straight ducts with the circular cross section under the constant heat flux at the wall are presented. For circular ducts ($\tilde{a}=\tilde{b}$), we have:

$$\tilde{a} = \tilde{b} \text{ or } \eta = 1 \tag{30}$$

And also we have [27]:

$$e = 0 \text{ and } E(0) = \frac{\pi}{2} \tag{31}$$

By substituting Eqs. (30) and (31) in Eq. (28) or Eq. (29), the Nusselt number is calculated as:

$$Nu = \frac{48}{11} \tag{32}$$

The above value is the Nusselt number of fully developed flow and heat transfer in straight circular pipes under the constant heat flux at the wall which is reported in the literature. This finding indicates that the result of the present study is valid for this test case.

Eq. (29) calculates the same value for the Nusselt number at $\eta \to 0$ and $\eta \to \infty$ as:

$$Nu \approx \frac{4356}{833} \tag{33}$$

It is important to remember that these two limit cases are similar for ellipses (refer to the definition of aspect ratio in ellipses) and the current exact solution presents an identical result for them.

Fig. 3 shows contours of dimensionless temperature distribution for some different aspect ratios. This figure indicated that the isothermal lines are in forms of concentric ellipses. To further evaluate, Fig. 4(a) depicts the absolute value of dimensionless temperature distribution at the center of the cross section as a function of aspect ratio. As seen in this figure, the centerline dimensionless temperature reaches a peak value of 9/24 at η=1 (circular cylinder) and reduces by increasing (or decreasing) the value of aspect ratio and tends to 245/726 by substituting $\eta \to \infty$ (or η=0) into Eq. (24). Although as Fig. 4(a) indicates, by increasing the value of aspect ratio from 24, the dimensionless temperature at the center of duct almost remains constant. To examine the effect of aspect ratio on heat transfer processes, the values of the average Nusselt number as a function of aspect ratios are shown in Fig. 4(b). As evident, the minimum value of the Nusselt number is observed for the circular cross section (48/11). This finding indicated that the minimum amount of heat transfer coefficient (*h*) is related to the circular shape of cross section.

Due to the constant value of heat flux at the walls, the temperature difference between the wall temperature and mean temperature should be maximized in a circular pipe to obtain the minimum value for Nusselt number. The dimensionless temperature is defined based on the temperature difference between the wall and flow (see Eq. (2)). Therefore, the dimensionless temperature at the center of the cross section is maximum in the circular pipe which is shown in Fig. 4(a). Fig. 4(b) also indicates that the value of Nusselt number increases by changing the cross section from circle to ellipse. It means that changing the cross section from circular to elliptical cylinder leads to an enhancement in heat transfer rate.

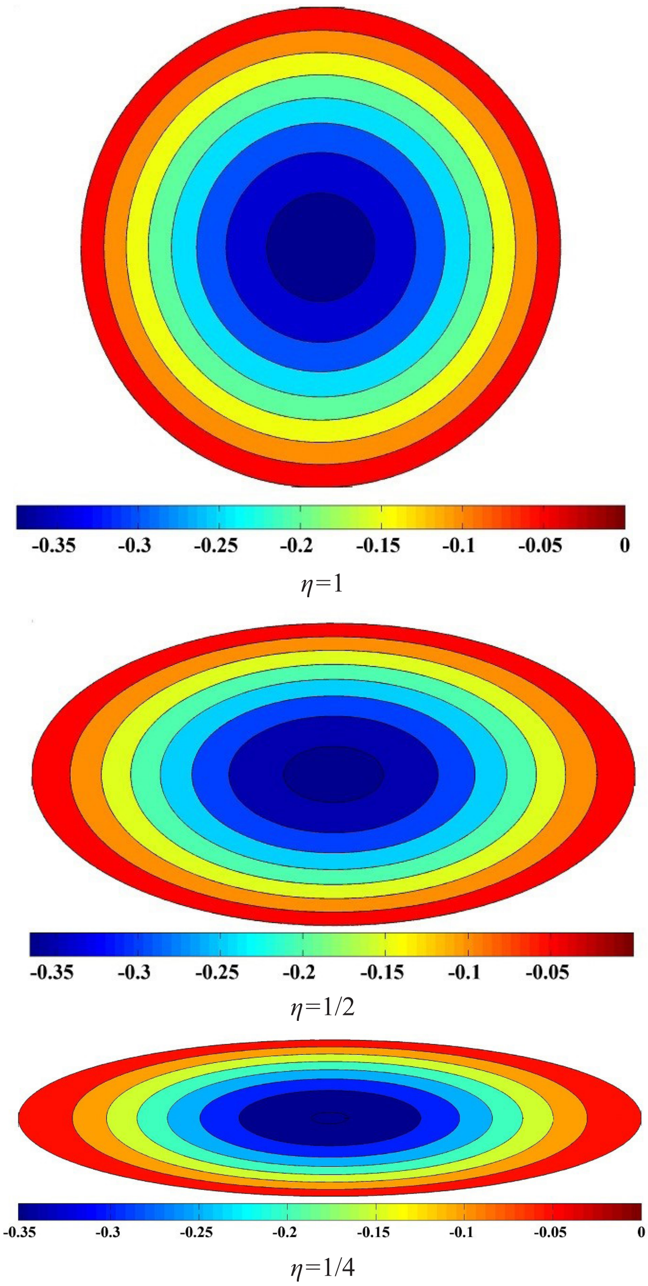
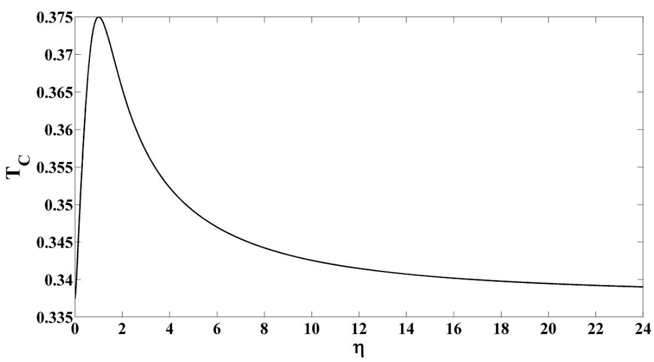
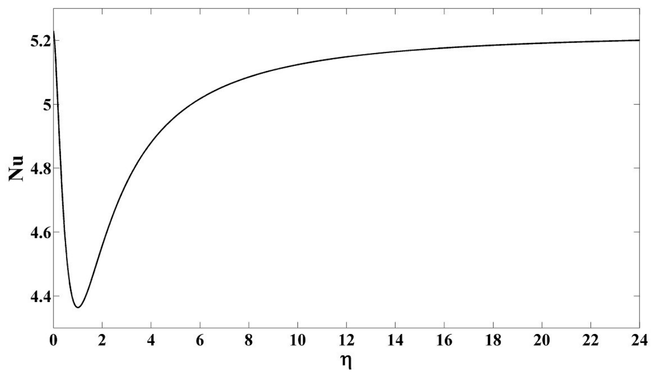


Fig. 3. Dimensionless contours of temperature at different aspect ratios



(a)



(b)

Fig. 4. Diagrams of the (a) absolute value of dimensionless temperature at the center of cross section and (b) Nusselt number versus the aspect ratio

According to Eq. (29), the Nusselt number tends to the value of 4356/833 at large enough aspect ratios. However, as Fig. 4(b) illustrates, by increasing the value of aspect ratio from 24 Nusselt number reaches almost the fixed value. It means that when $\eta \geq 24$ aspect ratio does not affect heat transfer rate from the elliptical cylinder. Table 1 shows an increase in Nu number of the elliptical cylinder to the circular cylinder with respect to a maximum increase in Nu to circular cylinder versus the aspect ratio. In this table Nu_C and Nu_∞ are the Nusselt number for circular and elliptical cylinders, respectively when $\eta \rightarrow \infty$.

Table 1. Increase in Nu number of elliptical cylinder to circular cylinder with respect to maximum increase versus the aspect ratio

$\frac{(Nu - Nu_C)}{(Nu_\infty - Nu_C)}$	0.5	0.6	0.7	0.8	0.9	0.95
η	3.2	4.03	5.13	7	11.46	18.36

As this table indicates, 95% of the increase in Nu number to the circular cylinder is related to η=18.36. Hence, by choosing this aspect ratio for the elliptical cylinder, 95% of maximum heat transfer rate could be obtained.

Fig. 5 shows the distribution of Nusselt number in terms of dimensional major and minor axes of the elliptical cross section. As shown in the figure again, the minimum amount of Nusselt number is related to the circular pipes and changing the cross section from circular to elliptical duct leads to an increase in heat transfer rate.

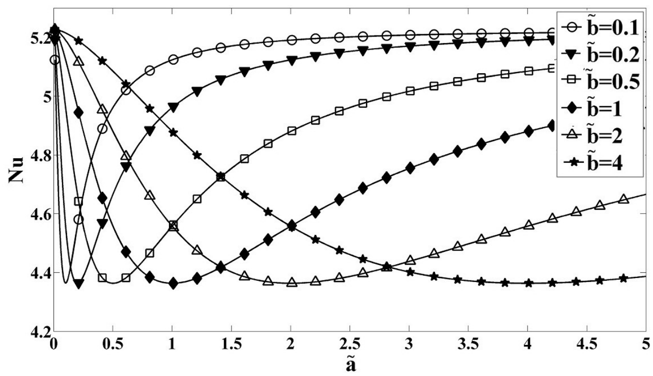


Fig. 5. Diagrams of Nusselt number in terms of the major and minor axes of the elliptical cross-section

5- Conclusion

In this paper, an analytical solution for convective heat transfer in straight elliptical pipes is presented. The solution is obtained using finite series expansion method for fully developed heat transfer under the constant heat flux at walls. The correlation of temperature distribution is obtained in the Cartesian coordinate system and also the variations of Nusselt number as well as the dimensionless temperature at the center of cross section are expressed in terms of aspect ratio as the new equations. These correlations have been validated with the previous investigations. The solution indicated that the Nusselt number is increased by changing the geometry of cross section from circular to elliptical shape from 48/11 to 4356/833 for large enough aspect ratios. Because of constant value of heat flux at walls, the dimensionless temperature at the center of the cylinder which is the difference between the wall and mean temperatures should be maximized in a circular pipe to obtain the minimum value for Nusselt number. This temperature decreases from 9/24 for the circular cylinder to 245/726 for large enough aspect ratio of an elliptical cylinder. Our correlations show that the results are identical for $\eta \to 0$ and $\eta \to \infty$ because these two limit cases are related to a similar geometry. Our results also show that 95% of the increase in the Nusselt number of the elliptical cylinder to circular cylinder happens in $\eta=18.36$. The authors believe that the present method of solution could be used to obtain the exact solution of convective heat transfer in elliptical pipes for other thermal boundary conditions and fluid rheological behaviors.

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