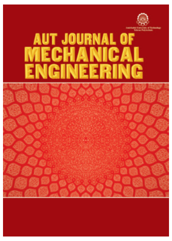

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Robustness of Controlled Lagrangian Method to the Structured Uncertainties

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ABSTRACT: Controlled Lagrangian method uses the inherent geometric structure of the energy of the mechanical systems to provide a stabilizing algorithm for underactuated mechanical systems. The presented method belongs to a larger family of nonlinear control algorithms, namely energy shaping methods in which the controller is designed by providing necessary modifications in the mechanical energy of the system. This paper presents a sensitivity analysis of Controlled Lagrangian method. It is shown that the method presents a suitable performance under the effect of structured (or parametric) uncertainties such as masses values, their positions and their influence on the inertia tensor. Then, the sequel investigates the robustness level of the designed controller in the presence of structured uncertainties. A detailed robustness proof of the scheme is established in this paper. Simulations are provided for a linear inverted pendulum cart system to validate analytical results of robustness to parametric uncertainties. Simulation results confirm that the designed controller for the inverted pendulum, which is unstable and underactuated, is well robust against parametric uncertainties as the analytical studies predicted. The method was also compared with the sliding mode approach, which showed a superior robustness against parametric uncertainties and a more practical control input value.

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1- Introduction

Controlled Lagrangian method is an energy shaping method designed to control underactuated mechanical systems. It uses kinetic energy shaping as well as potential energy shaping to control underactuated systems. This method was initially introduced in [1-4]. It is shown that in order to stabilize an underactuated system in its unstable equilibrium point, shaping the potential energy alone is not sufficient and a modification in kinetic energy is also essential ([3, 4]). Bloch et al. [3, 4] introduced a method to solve nonlinear PDEs of kinetic and potential energy shaping. Auckly et al. [5- 7] transformed the nonlinear PDEs of kinetic shaping to a system of linear PDEs, called λ method where the solutions were more straightforward. Change [8] used gyroscopic forces in the Controlled Lagrangian method to attain more freedom in controller parameters; consequently, he successfully stabilized the Fruta pendulum. Woolsey et al. [9] studied the effect of physical dissipation on the performance of Controlled Lagrangian method and showed that the physical dissipation may deteriorate the controller performance. In order to avoid difficulties of solving the PDEs of shaping the total energy, some assumptions were proposed in [17]; however, the assumptions were such restrictive that only a narrow class of mechanical systems was able to satisfy them. These works were done in Lagrangian framework. Similar endeavors were also presented in Hamiltonian framework, entitled as Interconnection and damping assignment passivity-based control or IDA-PBC [10- 12]. Controlled Lagrangian method and IDA-PBC are shown to be equivalent [8]. Using Controlled Lagrangian method, a simple control law for stabilizing inverted cart pendulum system was presented in [13]. The scheme has found other applications such as speed regulation of biped walking robots [18] and trajectory generation for a cluster of mobile robots [19].

Slotin and Li [14] divided model imprecision into two categories: structured uncertainties and unmodeled dynamics. Dissipation and Colomb friction are included in unmodelled dynamics and their effects on Controlled Lagrangian method was studied in [9]. It was shown in [20] that IDA-PBC is able to reject external disturbances for a class of underactuated mechanical systems, including an Inertia Wheel. Donaire et al. [21] robustified IDA-PBC method for a specific class of disturbances by adding a nonlinear PID to the outer loop of the controller. Using an adaptive algorithm, the performance of IDA-PBC were improved for an Inertia Wheel [22]. All of the aforementioned works were done in Hamiltonian framework. In the present paper, effects of structured uncertainties on Controlled Lagrangian method are studied in Lagrangian framework. It is shown analytically that the method is well robust to structured uncertainties. In order to validate this assertion, a controller is proposed for the inverted pendulum cart system using Controlled Lagrangian method. Then, the effects of structured uncertainties are studied exhaustively by simulations. Moreover, the method was compared with the robust sliding mode approach which showed a superior performance and a more practical control input value.

The organization of this paper is as follows. In section 2, the general method of controlled Lagrangian is presented. Section 3 presents structured uncertainties and their effect on controlled Lagrangian method; then, the stability of the method is established in the presence of structured uncertainties. In

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section 4, the robustness of Controlled Lagrangian method to the structured uncertainties is illustrated by a series of simulations. These simulations are performed for an inverted pendulum on a cart system. Finally, conclusion and further remarks are presented in section 5.

2- General Method of Controlled Lagrangian

As mentioned in [5, 8], Controlled Lagrangian method is a control strategy for underactuated mechanical systems with a regular Lagrangian. A regular Lagrangian system, by its definition, is a Lagrangian with the property of $[(\partial^2 L)/(\partial \dot{q}^i \partial \dot{q}^j)] \neq 0$. This expression is equivalent to the existence of a non-degenerative inertia tensor in a mechanical system. The method shapes both potential energy and kinetic energy of the system to obtain asymptotical stability in a naturally unstable point. The main idea of controlled Lagrangian method is based on a simple fact, viz. different Lagrangians can produce the same equation of motion (see [15] for more details). Different developers of this method use different notations and in this paper notations and symbols of [8] are used.

For a simple Lagrangian system the triple (L,F,W) is defined as the controlled Lagrangian system where $L=T-U$ is the Lagrangian of the system, F is an external force acting on the system and W is the control bundle of the system. Underactuation of a system is depicted by the inequality: rank $W <$ dim Q (Q is the configuration space of the system). Equations of motion of this system is then given by

$$\mathcal{EL}(L)=\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}-\frac{\partial L}{\partial q}=F+[W]_{(n\times m)}u \tag{1}$$

where n = dim Q and m = rank W.

It is assumed that two different Lagrangians, i.e. (L,F,W) and $(\hat{L},\hat{F},\hat{W})$ generate same equations of motion. Hence, expression $\ddot{q}=\hat{\ddot{q}}$ is valid for all $q\in Q$ and a direct calculation shows the following expression is also valid for all $q\in Q$.

$$\begin{aligned}[W]u &= \mathcal{EL}(L)-F-\left[\frac{\partial^2 L}{\partial \dot{q}\partial \dot{q}}\right]\left[\frac{\partial^2 \hat{L}}{\partial \dot{q}\partial \dot{q}}\right]^{-1}\left(\mathcal{EL}(\hat{L})-\hat{F}\right)\\ &+\left[\frac{\partial^2 L}{\partial \dot{q}\partial \dot{q}}\right]\left[\frac{\partial^2 \hat{L}}{\partial \dot{q}\partial \dot{q}}\right]^{-1}[\hat{W}]\hat{u}.\end{aligned} \tag{2}$$

For a simple mechanical system, Equation (2) simplifies to

$$\begin{aligned}[W]u &= \left([\mathbf{C}]-[\mathbf{M}][\hat{\mathbf{M}}]^{-1}[\hat{\mathbf{C}}]\right)\{\dot{\mathbf{q}}\}+\{g\}-[\mathbf{M}][\hat{\mathbf{M}}]^{-1}\{\hat{g}\}\\ &-F+[\mathbf{M}][\hat{\mathbf{M}}]^{-1}\hat{F}+[\mathbf{M}][\hat{\mathbf{M}}]^{-1}[\hat{W}]\hat{u},\end{aligned} \tag{3}$$

where $[M]$ is inertia tensor, $[C]$ is Coriolis and centripetal matrix and $\{g\}$ is $\partial U/\partial q$.

Because of the underactuation, $[W]$ is not a full rank matrix; therefore, it has a nonzero left annihilator, more specifically, a matrix $[W^{\perp}]$ can be defined with two properties: First, $T^*Q = W \oplus W^{\perp}$ and second, $[W^{\perp}][W]=0$. These properties are satisfied by a matrix whose rows are defined by

$$\{v \in T^*Q \mid \forall \alpha \in W, \langle v,\alpha\rangle = 0\} \tag{4}$$

Left multiplication of Equation (3) by $[W^{\perp}]$ results in

$$\begin{aligned}0 &= [W^{\perp}]\left([\mathbf{C}]-[\mathbf{M}][\hat{\mathbf{M}}]^{-1}[\hat{\mathbf{C}}]\right)\{\dot{\mathbf{q}}\}\\ &+[W^{\perp}]\left(\{g\}-[\mathbf{M}][\hat{\mathbf{M}}]^{-1}\{\hat{g}\}\right)\\ &+[W^{\perp}]\left(-F+[\mathbf{M}][\hat{\mathbf{M}}]^{-1}\hat{F}+[\mathbf{M}][\hat{\mathbf{M}}]^{-1}[\hat{W}]\hat{u}\right).\end{aligned} \tag{5}$$

It can be shown that in order to satisfy Equation (2), both control bundle $\hat{W}$ and the external force should be in the form of

$$[\hat{W}]=[\hat{M}][M]^{-1}[W], \tag{6}$$

$$\hat{\mathbf{F}}=[\hat{M}][M]^{-1}\mathbf{F} \tag{7}$$

By inserting Equation (6) and (7) into Equation (5), it reduces to two equations given as

$$0=[W^{\perp}]\left([\mathbf{C}]-[\mathbf{M}][\hat{\mathbf{M}}]^{-1}[\hat{\mathbf{C}}]\right), \tag{8}$$

$$0=[W^{\perp}]\left(\{g\}-[\mathbf{M}][\hat{\mathbf{M}}]^{-1}\{\hat{g}\}\right) \tag{9}$$

Equations (8) and (9) are known as matching equations. First, Equation (8) should be solved in order to find all elements of $[\hat{M}]$ (that shapes kinetic energy) and then, by substituting $[\hat{M}]$ for Equation (9), the potential energy is to be shaped to determine the potential function $\hat{U}$.

In practice, Equation (8) consists of a set of first-order nonlinear PDEs whose solution can be quite challenging. However, using a new variable, called λ, nonlinear PDEs (8) and (9) can be transformed to a set of triangular first order linear PDEs. This variable is defined by [5]:

$$\lambda=[\hat{\mathbf{M}}]^{-1}[\mathbf{M}]. \tag{10}$$

The λ equations in local coordinates are given by [5] as,

$$\frac{\partial(m_{\alpha i}\lambda_{\beta}^{i})}{\partial q^{k}}-[\alpha k,i]\lambda_{\beta}^{i}-[\beta k,i]\lambda_{\alpha}^{i}=0, \tag{11}$$

$$\lambda_{\alpha}^{k}\frac{\partial \hat{m}_{ij}}{\partial q^{k}}+\frac{\partial \lambda_{\alpha}^{k}}{\partial q^{i}}\hat{m}_{kj}+\frac{\partial \lambda_{\alpha}^{k}}{\partial q^{j}}\hat{m}_{ki}=\frac{\partial m_{ij}}{\partial q^{\alpha}}, \tag{12}$$

$$\lambda_{\alpha}^{k}\frac{\partial \hat{U}}{\partial q^{k}}=\frac{\partial U}{\partial q^{\alpha}}, \tag{13}$$

$$\hat{F}_{i}=\hat{m}_{ij}m^{jk}F_{k}. \tag{14}$$

In Equations (11) to (14), Latin indices vary from 1 to a total number of freedom and Greek indices vary from 1 to the number of underactuation. These equations are triangular, i.e. Equation (11) is solved to determine λ_{α}^{i} then, using λ, Equation (12) is solved for $\hat{m}_{ij}$, and the shaped potential energy ($\hat{U}$), is obtained from Equation (13). Finally, external force $\hat{F}$ is obtained from Equation (14).

Equations (11) to (14) indicate that the two Lagrangian (L,F,W) and $(\hat{L},\hat{F},\hat{W})$ generate the same equations of motion. In solving Equations (11) to (14), some arbitrary functions and constants remain unknown. These arbitrary unknowns are defined as control gains and are used to achieve stability of the system. In the rest of this section, it is assumed that no external force is applied on the system. Later, in section 3, an external force is employed to introduce structural uncertainties.

In controlled Lagrangian method, the Lyapunov candidate for the system is its mechanical energy ($E = T + U$). Here, the energy-momentum method is used to establish the stability of the system. The general theory of energy-momentum method with its details is given in [16]. In this method, a simple mechanical system is stable at a specific point if the second variation of the mechanical energy function of the system is positive-definite at that point. In other words, if all eigenvalues of the Hessian matrix of mechanical energy function have a positive sign, the system is stable. The second variation or Hessian matrix is:

$$\delta^2 E_{2n\times 2n} = \begin{bmatrix} \left[\dfrac{\partial^2 E}{\partial q^2}\right]_{n\times n} & \left[\dfrac{\partial^2 E}{\partial q \partial \dot{q}}\right]_{n\times n} \\ \left[\dfrac{\partial^2 E}{\partial \dot{q} \partial q}\right]_{n\times n} & \left[\dfrac{\partial^2 E}{\partial \dot{q}^2}\right]_{n\times n} \end{bmatrix}. \tag{15}$$

For a simple mechanical system, the second variation matrix at the origin ($q = 0$, $\dot{q} = 0$) reduces to

$$\delta^2 E = \begin{bmatrix} \left[\dfrac{\partial^2 U}{\partial q^2}\right] & 0 \\ 0 & M \end{bmatrix}, \tag{16}$$

where U is the potential function and M is the inertia tensor. All of the arbitrary functions and constants should be chosen in a proper manner to guarantee the positive-definiteness of the Hessian matrix. It should be noted that the positive-definiteness of the Hessian matrix indicates only stability of the system. For asymptotic stability, a proper control force $\hat{u}$ should be implemented to the system. Control bundle $[\hat{W}]$ is obtained from solving Equation 8. It is assumed here that m in Equation (1) is 1, so only one degree of freedom is actuated and $[\hat{W}]$ is reduced to a column matrix whose elements are w_i , $i = 1 \ldots n$. In this case, if $\hat{u}$ is chosen as

$$\hat{u} = -c_0\left(w_1\dot{q}^1 + \cdots w_n\dot{q}^n\right) \quad c_0 > 0, \tag{17}$$

It can be shown that the system $(\hat{L},0,\hat{W})$ becomes asymptotically stable. Finally, the equivalent control force u such that the system $(L,0,W)$ becomes asymptotically stable is obtained by employing Equation (2) or Equation (3).

3- Robustness of Controlled Lagrangian to Structured Uncertainties

Structured uncertainties are some kinds of inaccuracies on the terms included in the model, but their values are different in the model and in the real plant e.g. masses, the center of masses, etc. [14]. In this section, these uncertainties are introduced in the equations of motion. Then, the effect of structured uncertainties on the stability of the system is illustrated by modifying the Hessian matrix.

For a simple mechanical system with no external force, Equations of motion (1) in local coordinate are

$$M_{ij}\ddot{q}^j + [kj,i]\dot{q}^k\dot{q}^j + \frac{\partial U}{\partial q^i} = W_{ij}u^j, \tag{18}$$

where $[kj,i]$ are Christoffel symbols of first kind and their relation to Coriolis and Centripetal matrix $\mathbf{C}$ is

$$C_{ij} = [kj,i]\dot{q}^k .$$

It should be mentioned that Christoffel symbols are not independent quantities and strictly depend on the inertia tensor $\mathbf{M}$ by

$$[ij,k] = \frac{1}{2}\left(\frac{\partial M_{ik}}{\partial q^j} + \frac{\partial M_{jk}}{\partial q^i} - \frac{\partial M_{ij}}{\partial q^k}\right).$$

Structured uncertainties are included in the model, but their values are not precise. In this paper, these imprecisions are shown by tilde symbol ~. Considering these uncertainties, Equation (18) becomes

$$\left(M_{ij} + \tilde{M}_{ij}\right)\ddot{q}^j + \left([kj,i] + [\widetilde{kj,i}]\right)\dot{q}^k\dot{q}^j + \frac{\partial U}{\partial q^i} + \frac{\partial \tilde{U}}{\partial q^i} = W_{ij}u^j, \tag{19}$$

Where

$$[\widetilde{kj,i}] = \frac{1}{2}\left(\frac{\partial \tilde{M}_{ik}}{\partial q^j} + \frac{\partial \tilde{M}_{jk}}{\partial q^i} - \frac{\partial \tilde{M}_{ij}}{\partial q^k}\right).$$

These uncertainties can be modeled as external forces in the right hand of Equation (19),

$$M_{ij}\ddot{q}^j + [kj,i]\dot{q}^k\dot{q}^j + \frac{\partial U}{\partial q^i} = F_i + W_{ij}u^j, \tag{20}$$

where

$$F_i = -\tilde{M}_{ij}\ddot{q}^j - [\widetilde{kj,i}]\dot{q}^k\dot{q}^j - \frac{\partial \tilde{U}}{\partial q^i} \tag{21}$$

Now, suppose a controller u is designed by the Controlled Lagrangian method and asymptotic stability of the original system $(L,0,W)$ is attained. For this controller, shaped kinetic energy $\hat{T}=(1/2)\hat{m}_{ij}\dot{q}^i\dot{q}^j$, shaped potential energy $\hat{U}$ and dissipation force $\hat{u}$ are found such that they satisfy equations (11) to (13) and (17). Also, the associated Hessian matrix $\delta^2\hat{E}$ with the designed controller is positive-definite at the origin. For this system equations of motion are

$$\hat{M}_{ij}\ddot{q}^j + [\widehat{kj,i}]\dot{q}^k\dot{q}^j + \frac{\partial \hat{U}}{\partial q^i} = \hat{F}_i + \hat{W}_{ij}\hat{u}^j, \tag{22}$$

where $\hat{F}_i$=0 because F_i=0. Hessian matrix of this system is

$$\delta^2\hat{E} = \begin{bmatrix} \left[\dfrac{\partial^2\hat{U}}{\partial q^2}\right] & 0 \\ 0 & \hat{M} \end{bmatrix} > 0 \tag{23}$$

Referring to equation (14), if F_i exists in system (L,F,W), its equivalent in system $(\hat{L},\hat{F},\hat{W})$ is $\hat{F}_i=\hat{M}_{ij}M^{jk}F_k$. For the structured uncertainties F_i in Equation (21), the equivalent force in system $(\hat{L},\hat{F},\hat{W})$ is

$$\hat{F}_i = -\hat{M}_{ij}M^{jk}\tilde{M}_{ks}\ddot{q}^s - \hat{M}_{ij}M^{jk}[\widetilde{sr,k}]\dot{q}^s\dot{q}^r - \hat{M}_{ij}M^{jk}\frac{\partial\tilde{U}}{\partial q^k}, \tag{24}$$

and hence, the equations of motion (22) become

$$\hat{M}_{ij}\ddot{q}^j + [\widehat{kj,i}]\dot{q}^k\dot{q}^j + \frac{\partial \hat{U}}{\partial q^i} = \hat{W}_{ij}\hat{u}^j - \hat{M}_{ij}M^{jk}\tilde{M}_{ks}\ddot{q}^s - \hat{M}_{ij}M^{jk}[\widetilde{sr,k}]\dot{q}^s\dot{q}^r - \hat{M}_{ij}M^{jk}\frac{\partial\tilde{U}}{\partial q^k}, \tag{25}$$

$$\left(\hat{M}_{is} + \hat{M}_{ij}M^{jk}\tilde{M}_{ks}\right)\ddot{q}^s + \left([\widehat{sr,k}] + \hat{M}_{ij}M^{jk}[\widetilde{sr,k}]\right)\dot{q}^s\dot{q}^r + \left(\frac{\partial\hat{U}}{\partial q^i} + \hat{M}_{ij}M^{jk}\frac{\partial\tilde{U}}{\partial q^k}\right) = \hat{W}_{ij}\hat{u}^j. \tag{26}$$

Equation (26) presents equations of motion of a Lagrangian system with inertia tensor

$$\hat{\mathbf{M}} + \hat{\mathbf{M}}\mathbf{M}^{-1}\tilde{\mathbf{M}},$$

and potential function $\hat{U} + \hat{\tilde{U}}$, where $\hat{\tilde{U}}$ is defined by

$$\frac{\partial\hat{\tilde{U}}}{\partial\mathbf{q}} = \hat{\mathbf{M}}\mathbf{M}^{-1}\frac{\partial\tilde{U}}{\partial\mathbf{q}}$$

According to the energy-momentum stability criterion, Lagrangian system with Equations of motion (26) is stable if the associated Hessian matrix is positive-definite. Hessian matrix of (26) is

$$\delta^2\hat{E} = \begin{bmatrix} \left[\dfrac{\partial^2\hat{U}}{\partial q^2} + \dfrac{\partial}{\partial\mathbf{q}}\left(\hat{\mathbf{M}}\mathbf{M}^{-1}\dfrac{\partial\tilde{U}}{\partial\mathbf{q}}\right)\right] & 0 \\ 0 & \hat{\mathbf{M}} + \hat{\mathbf{M}}\mathbf{M}^{-1}\tilde{\mathbf{M}} \end{bmatrix}. \tag{27}$$

For the exact model, $\tilde{M},\hat{U}$=0; hence, Hessian matrix (27) reduces to (23) and is positive-definite. In the uncertain model (19), two terms $\hat{M}M^{-1}\tilde{M}$ and $(\hat{M}M^{-1}(\partial\tilde{U}/\partial q))$ affect Hessian matrix and change its eigenvalues. The system remains stable as long as the eigenvalues are positive. Starting from zero, the uncertainty terms $\tilde{M}$, $\check{U}$ change the eigenvalues of the Hessian matrix gradually. Trajectories of the perturbed system remain bounded as long as the positive-definiteness of Hessian matrix is preserved.

To establish the asymptotic stability of the perturbed system, we use mechanical energy of the Lagrangian $(L',0,\hat{W})$ as the Lyapunov function and show that it is a negative semidefinite. Here we assume L' is the Lagrangian of the perturbed system with the inertia tensor $M'=\hat{M}+\hat{M}M^{-1}\tilde{M}$ and the potential function $U'=\hat{U}+\hat{\tilde{U}}$. It is also assumed that the perturbed system is stable (the Hessian matrix (27) is positive-definite). For this Lagrangian, it is easy to show that the energy term is $E'=(\partial L'/\partial\dot{q}^i)$-$L'$. Hence, the time derivative of E' produces the equations of motions as

$$\begin{aligned}
\frac{d}{dt}E' &= \frac{\partial L'}{\partial\dot{q}^i}\ddot{q}^i + \dot{q}^i\frac{d}{dt}\left(\frac{\partial L'}{\partial\dot{q}^i}\right) - \frac{dL'}{dt} \\
&= \frac{\partial L'}{\partial\dot{q}^i}\ddot{q}^i + \frac{\partial^2 L'}{\partial q^i\partial\dot{q}^j}\dot{q}^i\dot{q}^j + \frac{\partial^2 L'}{\partial\dot{q}^i\partial\dot{q}^j}\dot{q}^i\ddot{q}^j \\
&\quad - \frac{\partial L'}{\partial q^i}\dot{q}^i - \frac{\partial L'}{\partial\dot{q}^i}\ddot{q}^i = \dot{q}^i\left(\frac{\partial^2 L'}{\partial\dot{q}^i\partial\dot{q}^j}\ddot{q}^j + \frac{\partial^2 L'}{\partial q^i\partial\dot{q}^j}\dot{q}^j - \frac{\partial L'}{\partial q^i}\right) \\
&= \dot{q}^i\left(M'_{ij}\ddot{q}^j + C'_{ij}\dot{q}^j + \frac{\partial U'}{\partial q^i}\right) = \dot{q}^i\left(\hat{W}_{ij}\hat{u}_j\right),
\end{aligned}$$

where in the last equation, we used (18). Then, substituting $\hat{u}_j$ as (17) reduces the time derivative of E' as

$$\frac{d}{dt}E' = -c_0\left(\hat{w}_1\dot{q}^1 + \hat{w}_2\dot{q}^2 + \cdots + \hat{w}_n\dot{q}^n\right)^2 \le 0.$$

Hence, the asymptotic stability is established by the virtue of LaSalle's theorem.

4- Controller Design and Simulations

In this section, Controlled Lagrangian method is employed to stabilize an inverted pendulum cart system, then the structured (parametric) uncertainties are introduced to the system. The robustness of the controller in the presence of parametric uncertainties is demonstrated by analytical means as well as a number of simulations. In the analytical study, the materials of section 3 are used frequently. In order to evaluate the performance of the proposed method, the results are compared with those of a second-order sliding mode approach [23] under similar circumstances.

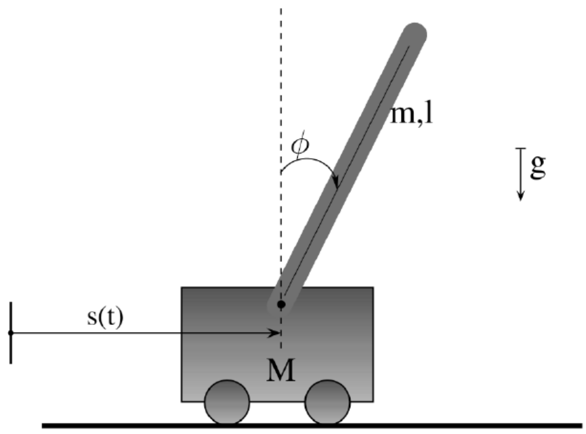


Fig. 1. Inverted pendulum cart system

4- 1- Controller design

The Lagrangian of the inverted pendulum cart system (Figure 1) is

$$L = \frac{1}{2}\begin{bmatrix} \dot{\phi} & \dot{s} \end{bmatrix}\begin{bmatrix} A & B\cos(\phi) \\ B\cos(\phi) & D \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{s} \end{bmatrix} - K\cos(\phi), \qquad (28)$$

where $A = ml^2$, $B = ml$, $D = m + M$, $K = mgl$ and M, m, l are cart mass, pendulum mass and pendulum center of mass of pendulum, respectively. Control bundle of the system is

$$[\mathbf{W}] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

During the designing process, we assume that no external force exists, $\boldsymbol{F} = 0$.

The shaped inertia tensor $[\hat{\boldsymbol{M}}]$, the shaped potential function $\hat{U}$ and the control bundle $[\hat{\boldsymbol{W}}]$ of the equivalent Lagrangian $(\hat{L}, \hat{F}, \hat{W})$ are obtained by solving Equations (11) to (13)[1]. as

$$[\hat{\mathbf{M}}] = \begin{bmatrix} AC_1 - C_1^2 C_2 \left(\cos\left(\phi(t)\right)\right)^2 (B - C_2 C_3) & C_1 \cos\left(\phi(t)\right)(B - C_2 C_3) \\ C_1 \cos\left(\phi(t)\right)(B - C_2 C_3) & C_3 \end{bmatrix}, \qquad (29)$$

$$\tilde{U} = KC_1\cos(\phi) + \frac{1}{2}\epsilon\left(s - C_1 C_2 \sin(\phi)\right)^2 \qquad (30)$$

$$[\hat{\mathbf{W}}] = [\hat{\mathbf{M}}][\mathbf{M}]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{AC_3 - BC_1(B - C_2C_3)\cos^2(\phi)}{AD - B^2\cos^2(\phi)}\begin{bmatrix} -C_1C_2\cos(\phi) \\ 1 \end{bmatrix}. \qquad (31)$$

Second variation matrix of the energy function $\hat{E} = \hat{T} + \hat{U}$, or its Hessian matrix is obtained by applying Equation (15) to the energy function $\hat{E}$ at the origin $(s, \dot{s}, \phi, \dot{\phi} = 0)$

1 Equations 11 to 13 are sets of PDEs and have general solutions. In order to obtain the control law, a particular solution would be sufficient. Other solutions lead to different control laws and different controller gains.

$$\delta^2\hat{E} = \begin{bmatrix} -KC_1 + \epsilon C_1^2 C_2^2 & -\epsilon C_1 C_2 & 0 & 0 \\ -\epsilon C_1 C_2 & \epsilon & 0 & 0 \\ 0 & 0 & AC_1 - C_1^2 C_2 B + C_1^2 C_2^2 C_3 & C_1 B - C_1 C_2 C_3 \\ 0 & 0 & C_1 B - C_1 C_2 C_3 & C_3 \end{bmatrix}. \qquad (32)$$

In Equations (29) to (32) constants $A,B,K > 0$ are systems' constants which were defined beneath Equation [28], and C_1, C_2, C_3, ϵ are controller gains. These controller gains should be chosen in a proper manner to assure the positive-definiteness of Hessian matrix (32).

In order to achieve asymptotic stability, feedback dissipation control force $\hat{u}$ must be implemented. According to Equation (17) and control bundle (31), the dissipation control force is

$$\hat{u} = -c_0 \frac{AC_3 - BC_1(B - C_2C_3)\cos^2(\phi)}{AD - B^2\cos^2(\phi)}\left(-C_1C_2\cos(\phi)\dot{\phi} + \dot{s}\right), \qquad c_0 > 0, \qquad (33)$$

where c_0 is another controller gain that indicates the dissipation rate of the energy. The equivalent controller force for the inverted pendulum cart system is obtained by implementing Equation (3).

4- 2- Structured uncertainties

In the inverted pendulum cart system, values of cart mass M, pendulum mass m and pendulum center of mass l could be different in the controller model and in the real model. In this subsection, for simplicity, it is assumed that the structured uncertainty is in cart mass. Following a similar approach to what was done in Equations (20) and (21), the external force due to the cart mass uncertainty is

$$\mathbf{F} = -\begin{bmatrix} 0 & 0 \\ 0 & \tilde{M} \end{bmatrix}\begin{bmatrix} \ddot{s} \\ \ddot{\phi} \end{bmatrix}, \qquad (34)$$

hence, the equivalent external force $\hat{F}$ of the Force (34) is obtained by employing Equation (14).

$$\hat{\mathbf{F}} = -\hat{\mathbf{M}}\mathbf{M}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{M} \end{bmatrix}\begin{bmatrix} \ddot{s} \\ \ddot{\phi} \end{bmatrix}. \qquad (35)$$

Force (35) changes the inertia tensor $\hat{\boldsymbol{M}}$ to

$$\hat{\mathbf{M}}\left(\mathbf{I} + \frac{1}{AD - B^2\cos^2(\phi)}\begin{bmatrix} 0 & -B\tilde{M}\cos(\phi) \\ 0 & D\tilde{M} \end{bmatrix}\right), \qquad (36)$$

while the potential energy remains intact. Continuous change in $\tilde{M}$, starting from 0, changes the eigenvalues of inertia tensor $\hat{\boldsymbol{M}}$ gradually. The system remains stable until the value of $\tilde{M}$ becomes large enough to change the sign of one of the eigenvalues of $\hat{\boldsymbol{M}}$. Other structured uncertainties act in a similar manner, except that they cause a change in both potential energy and kinetic energy.

4- 3- Simulations

In order to verify the robustness of Controlled Lagrangian method to structured uncertainties, a series of simulations has been done. In addition to these simulations, analytical results of the previous subsection are used to determine

Table 1. Physical specification of inverted pendulum cart system and controller gains

M	m	l	C_1	C_2	C_3	ϵ	c_0
1.79 kg	0.127 kg	17.78 cm	-1	-3.68	0.012	0.01	5

the allowable range of cart mass uncertainty in which the controller remains stable. Simulations are performed by physical specifications and the controller are gains given in Table 1. Using these values, Equation (36) indicates that the system (28) with controller (3) remains stable if $\tilde{M}$ varies within the interval (-0.8M , +0.1M). Figures 2 to 4 show the controller performance and its input history in the presence of the structured uncertainty in cart mass. The simulation results, shown in Figures 2 to 4, confirm that if $\tilde{M}$ varies within the interval (-0.8M , +0.1M), the system remains stable. Similar argument to what has been done in Equation (36) can be used to find limits of $\tilde{m}$ and $\tilde{l}$. Figures 5 to 10 illustrate the effect of $\tilde{m}$ and $\tilde{l}$ on the controller performance. Additionally, the limits of $\tilde{m}$ and $\tilde{l}$ for which the system remains stable are displayed in these figures.

4- 4- Second-order sliding mode control of Inverted pendulum cart system

Second-order sliding mode control scheme is a branch of sliding mode method that is tailored to deal with a class of underactuated systems [23]. The key element in this method is to find a diffeomorphism in which the zero dynamics of the transformed system becomes locally asymptotically stable. For system (38), the coordinate transformation

$$\eta = s - \ln\left(1+\frac{\sin\phi}{\cos\phi}\right) \tag{37-a}$$

$$\xi = \tan\phi - \lambda_1\eta - \lambda_2\dot{\eta} \tag{37-b}$$

transforms the system into a proper system in which the internal dynamics $\ddot{\eta}=g(\eta,\dot{\eta},\xi,\dot{\xi})$ is locally asymptotically stable in (0,0,0,0). Hence, the dynamics $\ddot{\xi}=f(\eta,\dot{\eta},\xi,\dot{\xi})+u$ is easily controllable with the control law [23]

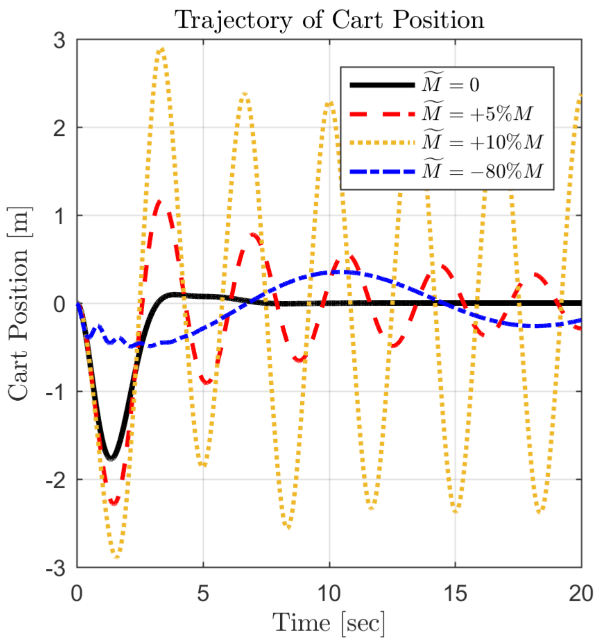
$$u\left(\eta,\dot{\eta},\xi,\dot{\xi}\right) = -f\left(\eta,\dot{\eta},\xi,\dot{\xi}\right) - \alpha\,\text{sign}\left(\xi\right) - \beta\,\text{sign}\left(\dot{\xi}\right) - h\xi - p\dot{\xi}, \tag{38}$$

where sign(.) is the sign function and λ_1, λ_2, α, β, h, p denote controller's gains. For the physical specifications of Table 1, we used the control gains λ_1=32 , λ_2=1, α=0.2 , β=0.01 , h=5, p=0.5.

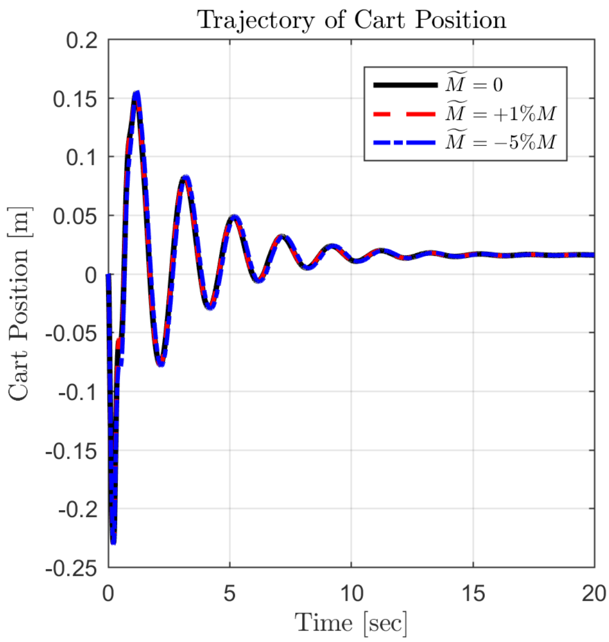
The results of the second-order sliding mode control are shown in Figures 2 to 10 for different parametric uncertainties. In Figures 2 to 4, the effect of uncertainty in the cart mass value on the controller's performance and the input history is depicted. Figures 5 to 7 are devoted to uncertainty in the pendulum mass, and Figures 8 to 10 are dedicated to uncertainty in the length of the link.

4- 5- Discussion

The lower boundaries of $\tilde{M}$, $\tilde{m}$ and $\tilde{l}$ should always be greater than -M, -m and -l, respectively. For instance, $\tilde{m}$=-m means that the pendulum has no mass which is a physically impossible circumstance. From the analytical point of view, $\tilde{m}$=-m causes the mass inertia tensor **M** to become degenerated, which is an indication of instability. Simulation results of Figures 5 to 7 confirm this assessment. Surprisingly, in these Figures, as the uncertainties approach the lower limits, the controller

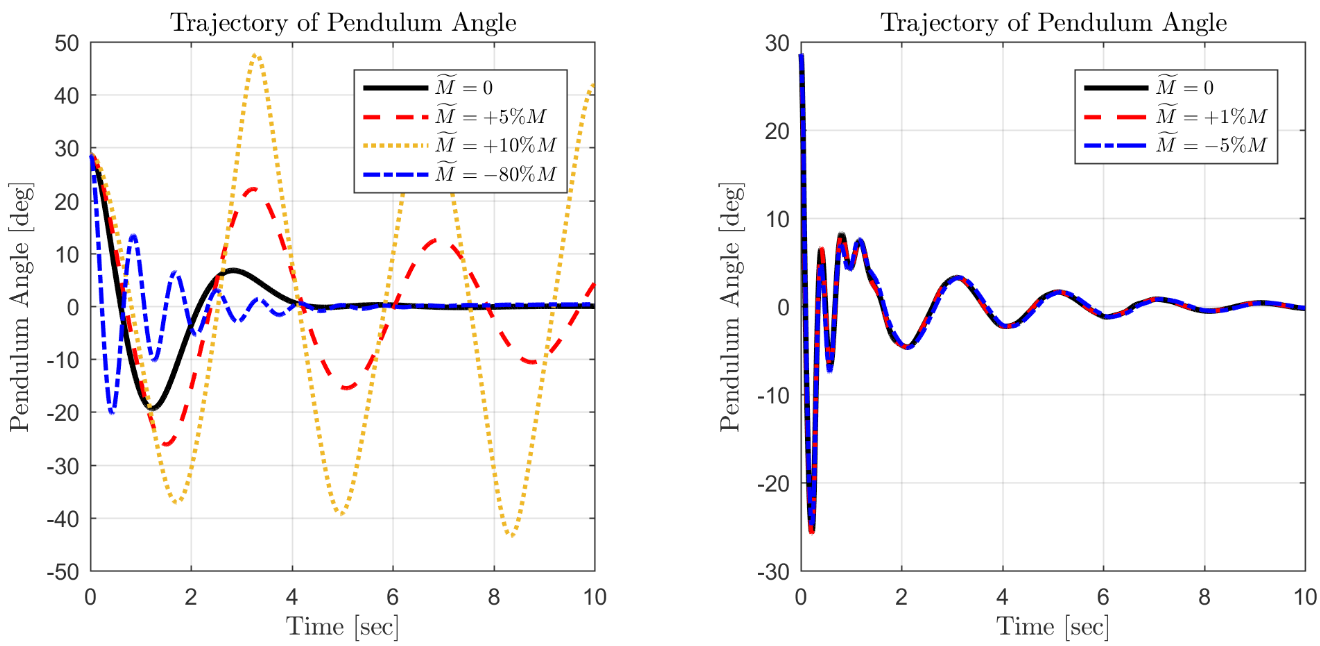


(a) Controlled Lagrangian



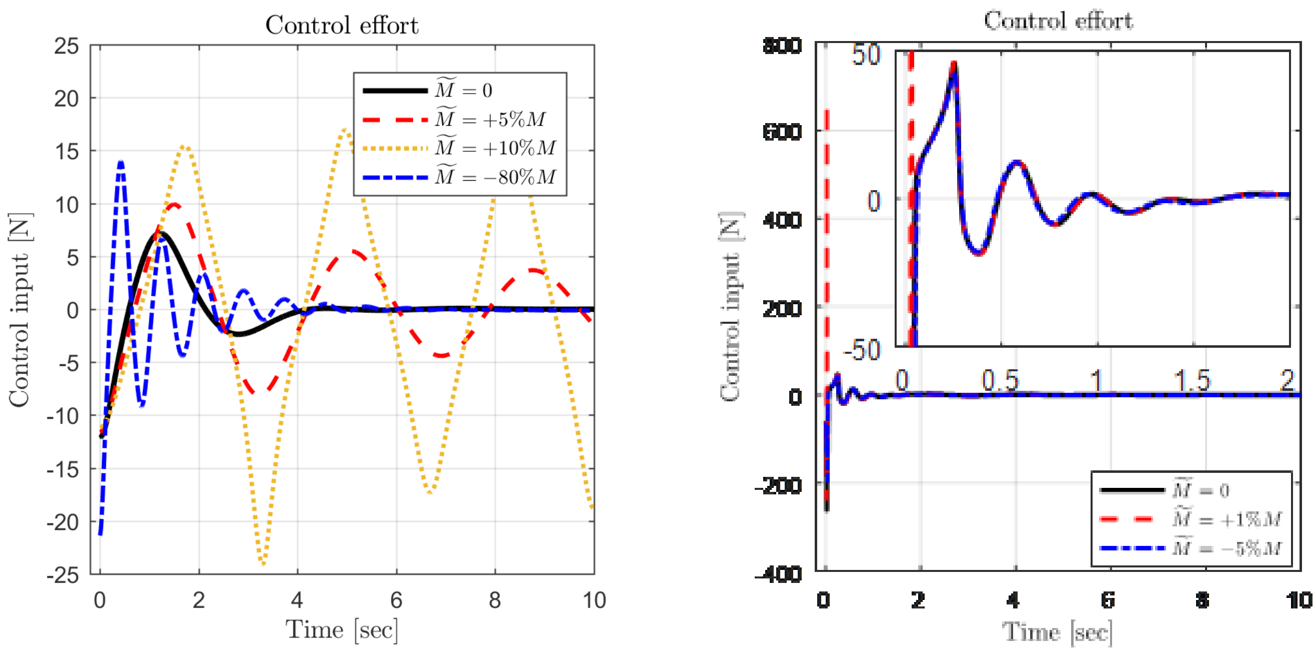
(b) Sliding mode

Fig. 2. An example of a figure Comparison of controllers' performance to control the cart position in the presence of cart mass uncertainty $\tilde{M}$



(a) Controlled Lagrangian (b) Sliding mode

Fig. 3. Comparison of controllers' performance to control the pendulum angle in the presence of cart mass uncertainty $\tilde{M}$

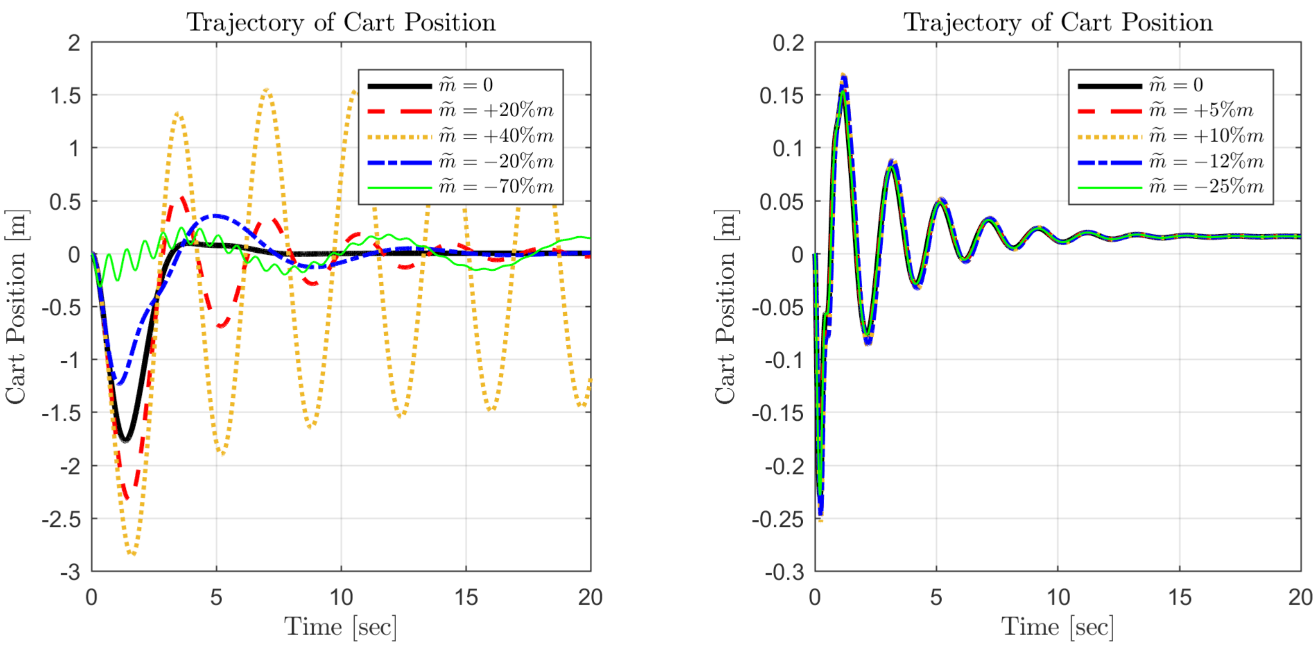


(a) Controlled Lagrangian (b) Sliding mode

Fig. 4. Comparison of control input in the presence of cart mass uncertainty $\tilde{M}$

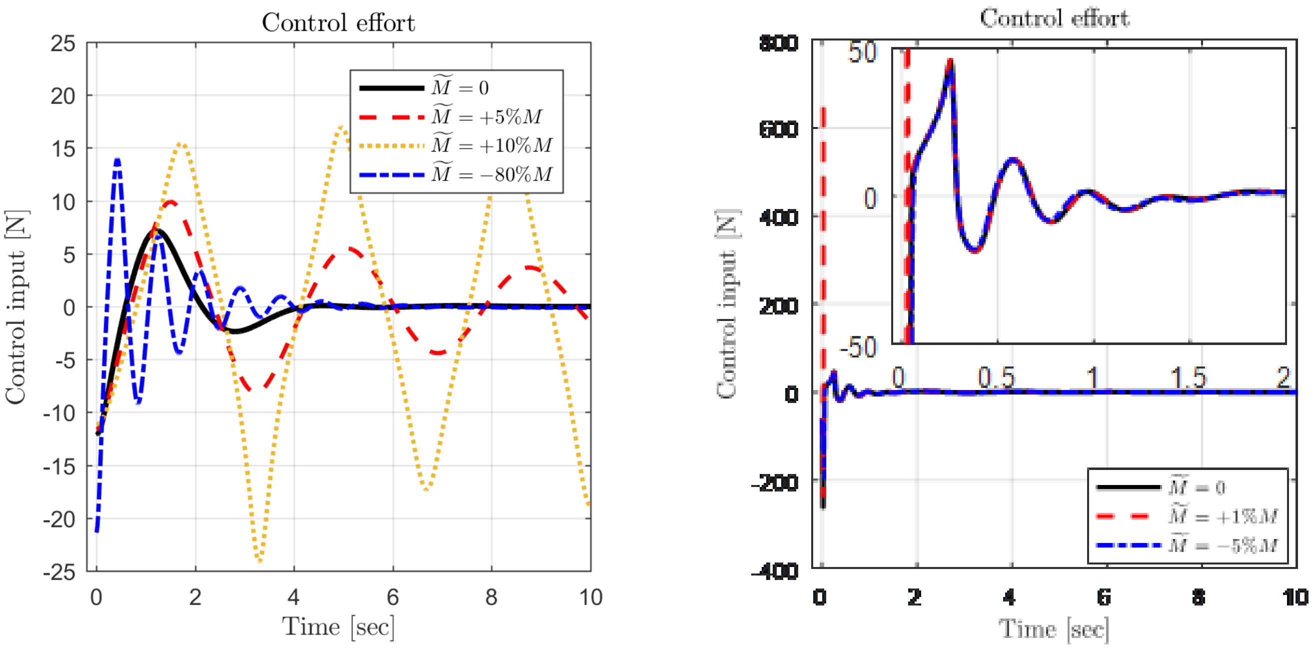
shows a better performance. This behaviour can be explained by the energy term of the system. In the designing process of the controller, it is assumed that the physical parameters are known exactly and these parameters are used to calculate dissipative control force (see Equation (17)). Negative structured uncertainties mean an overestimation of physical parameters in the controller; therefore, a relatively larger dissipative control force, to what is needed by the plant, is produced by the controller. Hence, the controller leads to a better performance. On the other hand, while uncertainties approach the upper limits, although the controller may not meet asymptotic stability, stability is still achieved. This result is shown in Figures 2 to 10 for $\tilde{M}$= 0.1M, $\tilde{m}$= 0.4m and $\tilde{l}$ = 0.4l.

The upper boundaries of uncertainties also depend on the controller gains. It should be mentioned that only C_1, C_2 and C_3 affect upper boudaries. Although ϵ enhances asymptotic stability, it is used to regulate cart position and has no effect on the stability of pendulum angle. c_0 is used to dissipate energy from the system and has no effect on stability (It is



(a) Controlled Lagrangian (b) Sliding mode

Fig. 5. Comparison of controllers' performance to control the cart position in the presence of pendulum mass uncertainty m̃

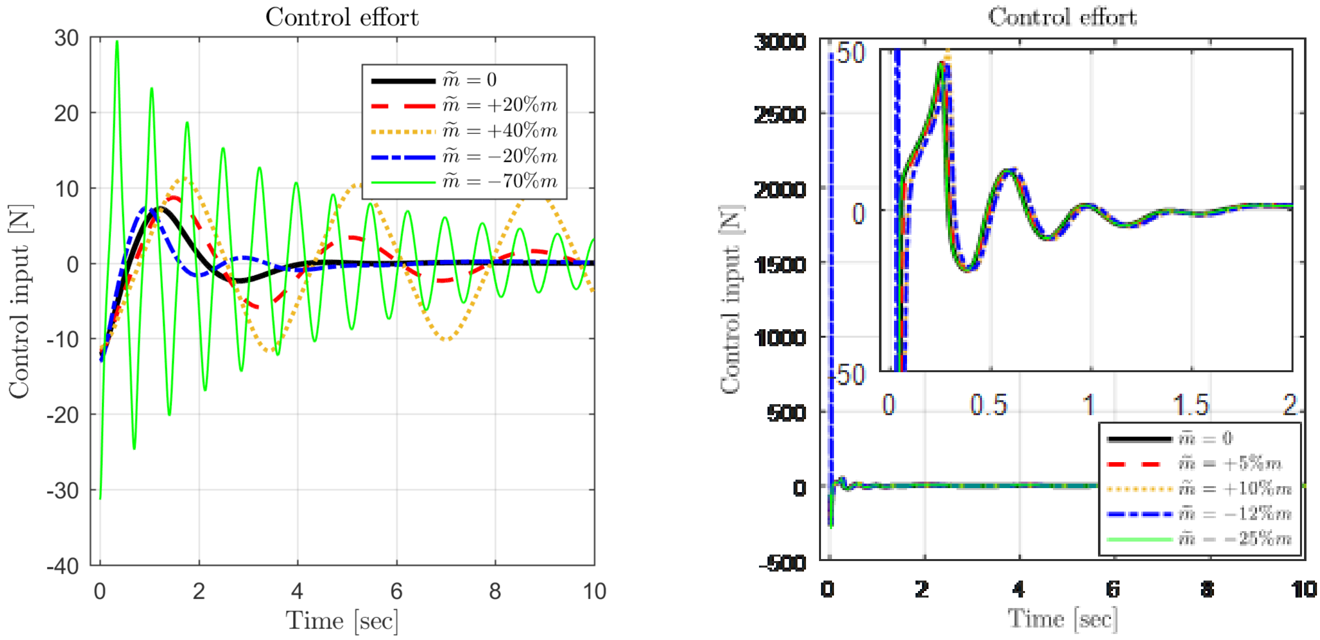


(a) Controlled Lagrangian (b) Sliding mode

Fig. 6. Comparison of controllers' performance to control the pendulum angle in the presence of pendulum mass uncertainty m̃

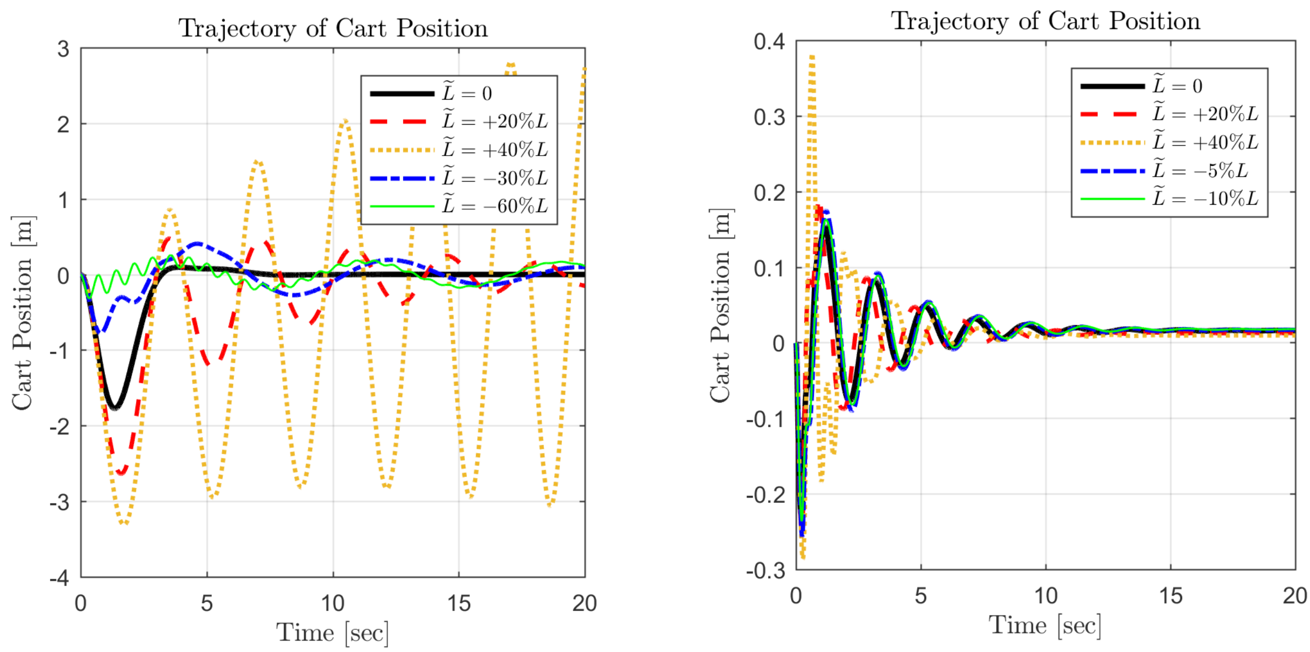
shown in section 3 that Hessian matrix is independent of c_0). Overall, the upper boundaries can be extended by changing the controller gains C_1, C_2 and C_3 in the Table 1. If the numerical value of C_1, for instance, changes to -5, the upper boundary of $\tilde{M}$ extends from $0.1M$ to $2.5M$. This implies that choosing control gains properly is an important issue in the robustness of the controller because they define the limits of uncertainties that can be tolerated by the controller.

For further evaluation, the comparison between our proposed method and second-order sliding mode control is conducted in Figures 2 to 10. The figures show that the controlled Lagrangian method is more robust than the sliding mode scheme. For instance, according to Figures 2 and 3, the limits of $\tilde{M}$ in Controlled Lagrangian approach is $\tilde{M}\in[-0.8M,+0.05M]$, while it is only $\tilde{M}\in[-0.05M,+0.01M]$ for the sliding mode approach. Similar situations are observable in Figures 5 and 6 for limits of $\tilde{m}$, and in Figures 8 and 9 for limits of $\tilde{l}$. On the other hand, the performance of sliding mode approach in its robust region of attraction (the values of uncertainties for which the controller is robust against)



(a) Controlled Lagrangian (b) Sliding mode

Fig. 7. Comparison of control input in the presence of pendulum mass uncertainty $\tilde{m}$



(a) Controlled Lagrangian (b) Sliding mode

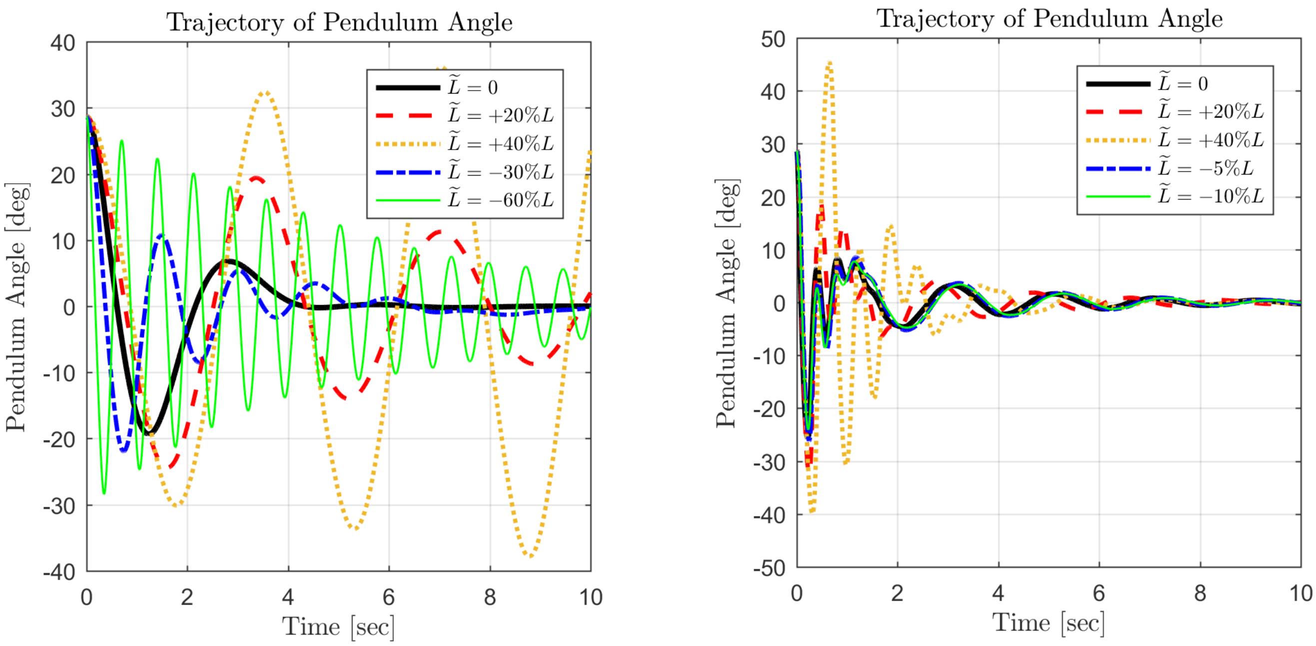
Fig. 8. Comparison of controllers' performance to control the cart position in the presence of link length uncertainty $\tilde{l}$

is superior than the performance of Controlled Lagrangian approach. Worded other way, the sliding mode approach produces better results than controlled Lagrangian method when it is stabilizeable, but the regain of the attraction of Controlled Lagrangian method is larger than the sliding mode's.

Comparison of the control input history between the two methods in Figures 4, 7 and 10 suggests that the sliding mode approach produces much larger values of control forces. These high values of control forces can easily saturate the actuators of the plant which results in instability of the scheme. However, Controlled Lagrangian approach produced more practical values for various uncertainties. Although the input values have increased in the presence of uncertainties, the level of the increase is tolerable in the practical applications where saturation is an important issue.

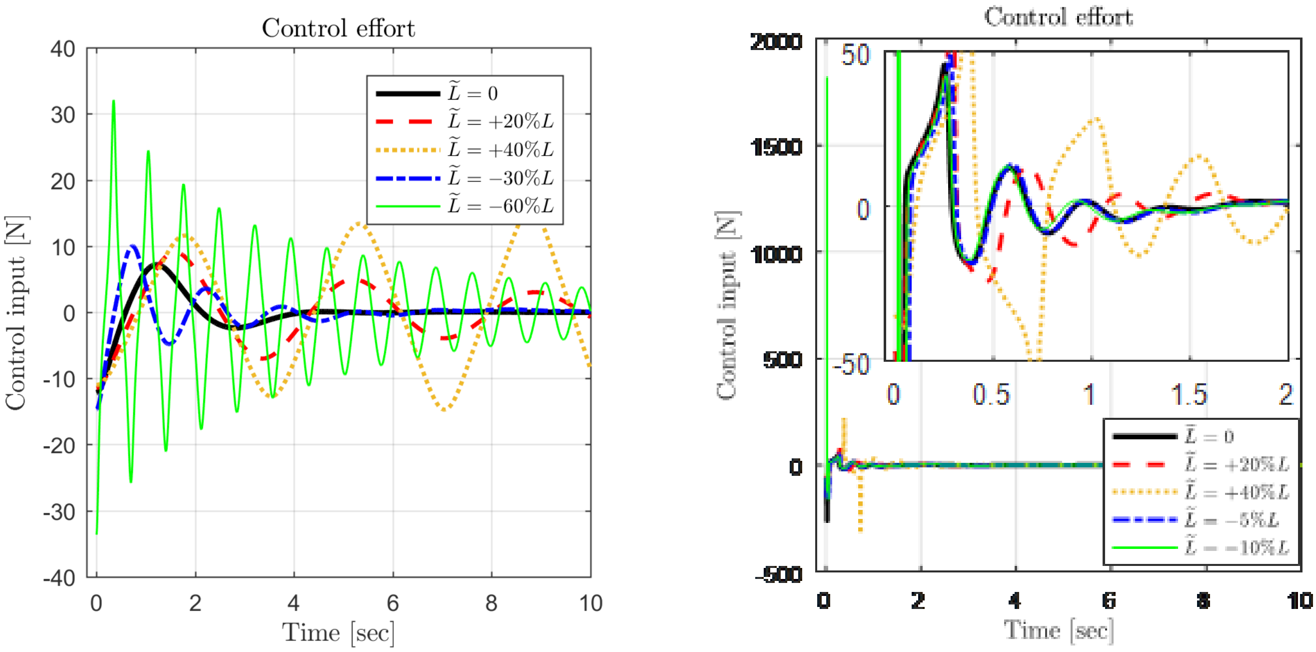
5- Conclusion and Further Remarks

Effects of structured uncertainties on the performance and the control input of Controlled Lagrangian method were studied



(a) Controlled Lagrangian (b) Sliding mode

Fig. 9. Comparison of controllers' performance to control the pendulum angle in the presence of link length uncertainty $\tilde{l}$



(a) Controlled Lagrangian (b) Sliding mode

Fig. 10. Comparison of control input in the presence of link length uncertainty $\tilde{l}$

in this article. Since Controlled Lagrangian method is an energy-based method, the structured uncertainties effect has been studied via energy term of the system, more specifically through its Hessian matrix. Consequently, the robustness of Controlled Lagrangian method to the structured uncertainties was investigated by examining positive-definiteness of Hessian matrix. It is concluded that the control gains that shape the inertia tensor provide limits of uncertainties for which the controller remains stable. In order to verify analytical results, extensive simulations were performed on the well-known inverted pendulum cart system. Comparison of the proposed scheme with the robust sliding mode approach showed a better robustness of the scheme. Additionally, it was observed that the proposed method provided a smaller control input that is more practical for the situations in which the actuator saturation is essential.

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