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Exact Closed-Form Solution for Vibration Analysis of Beams Carrying Lumped Masses with Rotary Inertias

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ABSTRACT: In this paper, an exact closed-form solution is presented for free vibration analysis of Bernoulli–Euler beams carrying attached masses with rotary inertias. The proposed technique explicitly provides frequency equation and corresponding mode as functions with two integration constants which should be determined by external boundary conditions implementation and leads to the solution to a two by two eigenvalue problem. The concentrated masses and their rotary inertia are modeled using Dirac's delta generalized functions without implementation of continuity conditions. The non-dimensional inhomogeneous differential equation of motion is solved by applying integration procedure. Using the fundamental solutions which are made of the appropriate linear composition of trigonometric and hyperbolic functions leads to making the implementation of boundary conditions much easier. The proposed technique is employed to study the effects of quantity, position and translational and rotational inertia of the concentrated masses on the dynamic behavior of the beam for all standard boundary conditions. Unlike many of the previous exact approaches, the presented solution has no limitation in a number of concentrated masses.

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1- Introduction

Studying dynamic characteristics of systems with flexible links or components is an essential research that can provide a successful design of mechanisms, robots, machines, and structures. Thus, vibration analysis of the beams carrying concentrated elements is a classical problem in the structural dynamics.

There is a weak possibility to find an exact closed form solution for nonlinear vibration analysis of beams and plates carrying concentrated masses and most of the relevant papers used numerical and approximate approaches. However, hitherto many studies have investigated the linear vibration characteristics of beams carrying various concentrated elements such as linear and rotational springs, point masses, rotary inertias, spring-mass systems, multi-span beams, etc. Chen [1] analytically studied the dynamic behavior of a simply supported beam carrying a concentrated mass at its center, considering the mass by the Dirac's delta function. A frequency analysis of a Bernoulli-Euler beam, carrying a concentrated mass at an arbitrary position was presented by Low [2]. He used the modified Dunkerley formula to obtain frequencies of vibration of beams, carrying concentrated masses. Laura et al. [3] obtained an analytical solution for the determination of natural frequencies and mode shapes of a clamped-free beam which was carrying a mass at the free end. In a comprehensive paper, Dowell [4] focused on the effects of mass and stiffness added to a dynamical system. Laura et al. [5] presented a note on the transverse vibration of continuous beams subjecting an axial force and carrying concentrated masses by applying the Rayleigh–Ritz method. Gürgöze [6] studied the approximate determination of the fundamental frequency and first mode shape of a beam with local springs and point masses. Also, in another paper, he investigated the vibration of restrained beams with heavy masses [7]. Liu et al. [8] employed the Laplace transformation technique to formulate the frequency equation for beams with elastically restrained ends, carrying concentrated masses. Using differential quadrature element method (DQEM), Torabi et al. [9] presented a numerical solution for vibration analysis of cantilever Timoshenko beams with non-uniform thickness carrying multiple concentrated masses. Torabi et al. [10] modeled concentrated masses by the Dirac's delta function and presented an exact closed-form solution for vibration analysis of truncated conical and tapered beams carrying multiple concentrated masses.

In most of the above literature, the effect of the rotary inertia of the attached masses has not been considered. Regarding the optimized Rayleigh methodology, Laura et al. [11] investigated the fundamental frequency of vibration of beams and plates elastically restrained against the rotation at the supports and carried the finite masses and rotary inertias. The free and forced vibrations of a uniform beam elastically restrained against rotation at one end, against translation at the other end, and carrying a lumped mass having rotary inertia and external loading at an arbitrary intermediate point was analyzed by Hamdan and Jubran [12]. Chang [13] considered a simply supported Rayleigh beam which was carrying a rigidly attached centered mass. He specified the natural frequencies and normal modes of the system while the position of the mass was supposed to be fixed. Zhang et al. [14] presented the transverse vibration analysis for Bernoulli-Euler beams, carrying concentrated masses and took into account their rotary inertia at both ends. An exact solution for the transverse vibration of Bernoulli–Euler beams, carrying

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point masses and taking into account their rotary inertia was investigated in closed-form fashion by Maiz and his co-workers [15]. They modeled general boundary conditions by means of translational and rotational springs at both ends and described the determination of the natural frequencies of vibration for a beam with general boundary conditions. Like most of the presented papers, their proposed method was limited to a finite number of masses existing on the beam, because of the increasing number of masses that leads to a lot of computational effort and complexity. For instance, when the discussed model was a beam with two concentrated masses, three piecewise functions had to be considered and twelve boundary conditions had to be applied to the governing equations. While in the present investigation, the formulation of governing equations in the presented technique is derived as an infinite series of terms, including the effect of concentrated masses and their rotary inertias. Therefore, using this technique, a beam carrying an unlimited number of masses can be solved with the less calculation.

Recently, transfer matrix method (TMM) has been used by some authors to study the vibration analysis of beams with concentered elements; e.g. Wu and Chang [16] studied free vibration of axial-loaded multi-step Timoshenko beam carrying arbitrary concentrated elements. Based on both Bernoulli-Euler and Timoshenko beam theories, Torabi et al. [17] studied free transverse vibration analysis of multi-step beams carrying concentrated masses having rotary inertia. In another work, they investigated the whirling analysis of axial-loaded multi-step Timoshenko rotor carrying concentrated masses [18]. Depending on the type of boundary conditions, natural frequencies were obtained through the solution for a determinant of order two or four for any number of lumped elements. Unfortunately, in TMM an increase in the number of point elements leads to a rise in the number of matrices which should be multiplied consecutively and therefore leads to a great increase in the size of components of the matrix in the final determinant. This weakness increases computation effort and limits this method in the number of concentrated elements. In order to overcome this weakness, in this paper using the concept of Dirac's delta function, an exact closed-form solution is presented for vibration analysis of beams carrying attached masses with rotary inertias. Effects of quantity, position and translational and rotational inertia of the concentrated masses on the dynamic behavior of the beam are investigated for various boundary conditions.

2- Mathematical procedure

According to Figure 1, a uniform beam with concentrated masses located at spatial coordinates x_i, is considered. As the figure shows, M_i and J_i are translational and rotational inertia of the i-th attached mass, respectively. The transverse displacement and transverse force per unit length are respectively denoted by $y(x,t)$ and $q(x,t)$. The beam parameters are a cross-sectional area, cross-sectional moment of inertia about the neutral axis, mass density, and elastic modulus of material which are represented by A, I, ρ and E, respectively. The translational inertia of the any attached mass can be assumed as a function of the spatial coordinates x as follows:

$$m_i(x) = \bar{m}_i\left[u(x - x_i) - u(x - x_i - dx)\right], \tag{1}$$

where $u(x - x_i)$ is the well-known unit step (Heaviside)

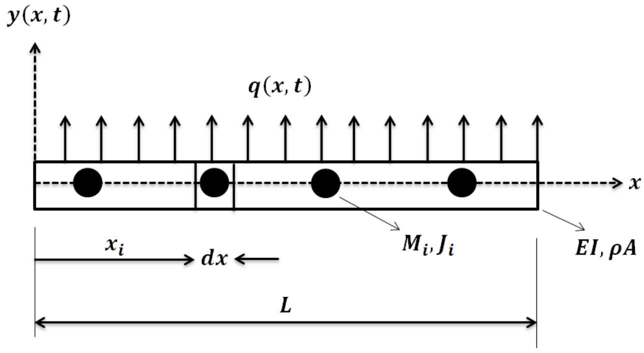


Fig. 1. The Bernoulli-Euler beam with multiple concentrated masses and rotary inertia.

function and

$$\bar{m}_i = \frac{M_i}{dx}. \tag{2}$$

By considering the attached masses as point elements, differential length dx should be led to zero, thus

$$\begin{aligned}&\lim_{dx\to 0} m_i(x)\\ &= \lim_{dx\to 0}\frac{M_i}{dx}\left[u(x - x_i) - u(x - x_i - dx)\right]\\ &= M_i\delta(x - x_i),\end{aligned} \tag{3}$$

and in a similar manner, the rotational inertia of the any attached mass can be expressed as

$$\begin{aligned}&\lim_{dx\to 0}\frac{J_i}{dx}\left[u(x - x_i) - u(x - x_i - dx)\right]\\ &= J_i\delta(x - x_i).\end{aligned} \tag{4}$$

Figure 2 displays the free body diagram for a beam element regarding the Bernoulli-Euler beam theory [19], where $V(x,t)$ and $M(x,t)$ represent the shearing force and bending moment, respectively. The force and moment equations of motion for the free vibration analysis of the beam can be written as [20]

$$\begin{aligned}&V - \left(V + \frac{\partial V}{\partial x}dx\right)\\ &= \left[\rho A\, dx + M_i\delta(x - x_i)dx\right]\frac{\partial^2 y}{\partial t^2}\end{aligned} \tag{5}$$

$$\begin{aligned}&M + \frac{\partial M}{\partial x}dx - M - \left(V + \frac{\partial V}{\partial x}dx\right)\frac{dx}{2}\\ &- V\frac{dx}{2} = J_i\delta(x - x_i)dx\,\frac{\partial^3 y}{\partial t^2\partial x}.\end{aligned} \tag{6}$$

Neglecting the terms involving second powers in dx, Eqs. (5) and (6) can be simplified as

$$\frac{\partial V}{\partial x} + \left[\rho A + M_i\delta(x - x_i)\right]\frac{\partial^2 y}{\partial t^2} = 0 \tag{7}$$

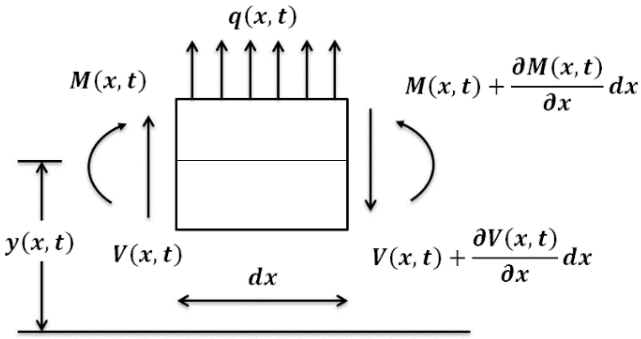


Fig. 2. The element of Bernoulli-Euler beam.

$$V=\frac{\partial M}{\partial x}-J_i\delta(x-x_i)\frac{\partial^3 y}{\partial t^2\partial x}. \tag{8}$$

Inserting Eq. (8) into Eq. (7), leads to

$$\begin{aligned}&\frac{\partial^2 M}{\partial x^2}-J_i\left[\delta(x-x_i)\frac{\partial^4 y}{\partial t^2\partial x^2}+\delta'(x-x_i)\frac{\partial^3 y}{\partial t^2\partial x}\right]\\&+\left[\rho A+M_i\delta(x-x_i)\right]\frac{\partial^2 y}{\partial t^2}=0.\end{aligned} \tag{9}$$

The above equation must be satisfied over $0<x<L$ domain. Also, with respect to the spatial coordinates, the derivative is denoted by the prime. The relationship between the bending moment and deformation in the Bernoulli-Euler beam theory is given as [20]

$$M(x,t)=EI\frac{\partial^2 y(x,t)}{\partial x^2}. \tag{10}$$

Inserting Eq. (10) into Eq. (9), the differential equation of motion can be obtained as

$$\begin{aligned}&EI\frac{\partial^4 y}{\partial x^4}+\left[\rho A+M_i\delta(x-x_i)\right]\frac{\partial^2 y}{\partial t^2}\\&-J_i\left[\delta(x-x_i)\frac{\partial^4 y}{\partial t^2\partial x^2}+\delta'(x-x_i)\frac{\partial^3 y}{\partial t^2\partial x}\right]=0.\end{aligned} \tag{11}$$

Non-dimensional spatial coordinate and transverse displacement can be introduced as

$$\zeta=\frac{x}{L}\quad w=\frac{y}{L} \tag{12}$$

and the transverse displacement functions can be considered as

$$w(\zeta,t)=\phi(\zeta)e^{i\omega t}, \tag{13}$$

where ω is the natural circular frequency. Hence, by introducing non-dimensional following terms:

$$\alpha_i=\frac{M_i}{\rho AL}\quad \beta_i=\frac{J_i}{\rho AL^3}=\alpha_i c_i^2 \tag{14}$$

$$c_i=\frac{r_i^g}{L}\qquad \lambda^4=\frac{\rho AL^4\omega^2}{EI},$$

the dynamic equation for transverse vibration can be written as in the following:

$$\begin{aligned}&\frac{\partial^4\phi(\zeta)}{\partial\zeta^4}-\lambda^4\phi(\zeta)=\\&\lambda^4\left\{\begin{aligned}&\alpha_i\delta(\zeta-\zeta_i)\phi(\zeta)\\&-\beta_i\left[\delta(\zeta-\zeta_i)\frac{\partial^2\phi(\zeta)}{\partial\zeta^2}+\delta'(\zeta-\zeta_i)\frac{\partial\phi(\zeta)}{\partial\zeta}\right]\end{aligned}\right\}.\end{aligned} \tag{15}$$

It should be noted that in deriving the last equation, the following property of Dirac's delta function has been utilized [21, 22]

$$\delta\left[L(\zeta-\zeta_i)\right]=\frac{1}{L}\delta(\zeta-\zeta_i). \tag{16}$$

and in Eq. (14), $r_i^g=\sqrt{J_i/M_i}$ is the radius of gyration of i-th point mass.
Introducing the function $A(\zeta)$ as the collection of all the terms with Dirac's deltas and their derivatives as

$$\begin{aligned}&A(\zeta)=\\&\lambda^4\sum_{i=1}^{N}\left\{\begin{aligned}&\alpha_i\delta(\zeta-\zeta_i)\phi(\zeta)\\&-\beta_i\left[\delta(\zeta-\zeta_i)\frac{\partial^2\phi(\zeta)}{\partial\zeta^2}+\delta'(\zeta-\zeta_i)\frac{\partial\phi(\zeta)}{\partial\zeta}\right]\end{aligned}\right\},\end{aligned} \tag{17}$$

the non-dimensional differential equation takes the following form:

$$\frac{\partial^4\phi(\zeta)}{\partial\zeta^4}-\lambda^4\phi(\zeta)=A(\zeta). \tag{18}$$

The governing differential equation given by Eq. (18) for specified boundary conditions, leads to the evaluation of the mode shapes and the corresponding frequencies. In order to solve Eq. (18), it can be observed that the solution of $\Phi(\zeta)$ must be in the same form with the eigen-mode of the bare beam. Therefore, a solution for the overall beam is assumed as a combination of the standard trigonometric and hyperbolic functions in which the coefficients of the combination are the generalized functions according to the following general form:

$$\begin{aligned}&\phi(\zeta)=d_1(\zeta)\sin(\lambda\zeta)+d_2(\zeta)\cos(\lambda\zeta)\\&+d_3(\zeta)\sinh(\lambda\zeta)+d_4(\zeta)\cosh(\lambda\zeta).\end{aligned} \tag{19}$$

The functions $d_1(\zeta)$, $d_2(\zeta)$, $d_3(\zeta)$ and $d_4(\zeta)$ appearing in Eq. (19), are unknown generalized functions determined according to the procedure outlined in Appendix A. The expressions of $d_1(\zeta)$-$d_4(\zeta)$ depend on four integration constants c_1, c_2, c_3 and c_4 and are defined as

$$
\begin{aligned}
&d_1(\zeta)= \\
&-\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\cos(\lambda\zeta_i)\phi(\zeta_i)\\ -\beta_i\lambda\sin(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+c_1 \\
&d_2(\zeta)= \\
&-\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\sin(\lambda\zeta_i)\phi(\zeta_i)\\ +\beta_i\lambda\cos(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+c_2 \\
&d_3(\zeta)= \\
&-\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\cosh(\lambda\zeta_i)\phi(\zeta_i)\\ +\beta_i\lambda\sinh(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+c_3 \\
&d_4(\zeta)= \\
&-\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\sinh(\lambda\zeta_i)\phi(\zeta_i)\\ +\beta_i\lambda\cosh(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+c_4
\end{aligned}
, \qquad (20)
$$

where c_1, c_2, c_3, c_4 are the integration constants. Meanwhile, inserting Eq. (20) into Eq.(19), $\Phi(\zeta)$ can be expressed as

$$
\begin{aligned}
&\phi(\zeta)= \\
&\left\{-\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\cos(\lambda\zeta_i)\phi(\zeta_i)\\ -\beta_i\lambda\sin(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+c_1\right\}\times \\
&\sin(\lambda\zeta)+ \\
&\left\{-\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\sin(\lambda\zeta_i)\phi(\zeta_i)\\ +\beta_i\lambda\cos(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+c_2\right\}\times \\
&\cos(\lambda\zeta)+ \\
&\left\{-\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\cosh(\lambda\zeta_i)\phi(\zeta_i)\\ +\beta_i\lambda\sinh(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+c_3\right\}\times \\
&\sinh(\lambda\zeta)+ \\
&\left\{-\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\sinh(\lambda\zeta_i)\phi(\zeta_i)\\ +\beta_i\lambda\cosh(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+c_4\right\}\times \\
&\cosh(\lambda\zeta),
\end{aligned}
\qquad (21)
$$

and Eq. (21) can be simplified as

$$
\begin{aligned}
&\phi(\zeta)= \\
&\lambda\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i T(\lambda\{\zeta-\zeta_i\})\phi(\zeta_i)\\ -\beta_i\lambda S(\lambda\{\zeta-\zeta_i\})\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\}+C(\lambda\zeta),
\end{aligned}
\qquad (22)
$$

where

$$
\begin{aligned}
&T(\zeta)=0.5\left[\sinh(\zeta)-\sin(\zeta)\right] \\
&S(\zeta)=0.5\left[\cosh(\zeta)-\cos(\zeta)\right]. \\
&C(\zeta)=c_1\sin(\zeta)+c_2\cos(\zeta) \\
&+c_3\sinh(\zeta)+c_4\cosh(\zeta)
\end{aligned}
\qquad (23)
$$

The function $\Phi(\zeta_j)$ can be selected by applying the product with Dirac's delta as the next equation [22, 23].

$$
\begin{aligned}
&\phi(\zeta_j)=\int_{-\infty}^{+\infty}\phi(\zeta)\delta(\zeta-\zeta_j)d\zeta= \\
&\lambda\sum_{i=1}^{j-1}\left\{\begin{matrix}\alpha_i T(\lambda[\zeta_j-\zeta_i])\phi(\zeta_i)\\ -\beta_i\lambda S(\lambda[\zeta_j-\zeta_i])\phi'(\zeta_i)\end{matrix}\right\}+C(\lambda\zeta_j).
\end{aligned}
\qquad (24)
$$

By derivation of Eq. (22) with respect to the spatial variable ζ, it can be written that

$$
\begin{aligned}
&\phi'(\zeta)= \\
&\lambda^2\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i T'(\lambda\{\zeta-\zeta_i\})\phi(\zeta_i)\\ -\beta_i\lambda S'(\lambda\{\zeta-\zeta_i\})\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\} \\
&+\lambda C'(\lambda\zeta).
\end{aligned}
\qquad (25)
$$

Also, the function $\Phi'(\zeta_j)$ can be selected by applying the product with Dirac's delta as follows:

$$
\begin{aligned}
&\phi'(\zeta_j)=\int_{-\infty}^{+\infty}\phi'(\zeta)\delta(\zeta-\zeta_j)d\zeta \\
&=\lambda^2\sum_{i=1}^{j-1}\left\{\begin{matrix}\alpha_i T'(\lambda[\zeta_j-\zeta_i])\phi(\zeta_i)\\ -\beta_i\lambda S'(\lambda[\zeta_j-\zeta_i])\phi'(\zeta_i)\end{matrix}\right\} \\
&+\lambda C'(\lambda\zeta_j).
\end{aligned}
\qquad (26)
$$

The recurrence expressions of Eqs. (24) and (26) can be given by the following explicit form:

$$
\begin{aligned}
&\phi(\zeta_j)=c_1\bar{\mu}_j+c_2\bar{\eta}_j+c_3\bar{\gamma}_j+c_4\bar{\kappa}_j \\
&\phi'(\zeta_j)=c_1\bar{\upsilon}_j+c_2\bar{\theta}_j+c_3\bar{\sigma}_j+c_4\bar{\tau}_j
\end{aligned}
, \qquad (27)
$$

where

$$
\begin{aligned}
&\bar{\mu}_j=\lambda\sum_{i=1}^{j-1}\left\{\begin{matrix}\alpha_i T(\lambda[\zeta_j-\zeta_i])\bar{\mu}_i\\ -\beta_i\lambda S(\lambda[\zeta_j-\zeta_i])\bar{\upsilon}_i\end{matrix}\right\}+\sin(\lambda\zeta_j) \\
&\bar{\eta}_j=\lambda\sum_{i=1}^{j-1}\left\{\begin{matrix}\alpha_i T(\lambda[\zeta_j-\zeta_i])\bar{\eta}_i\\ -\beta_i\lambda S(\lambda[\zeta_j-\zeta_i])\bar{\theta}_i\end{matrix}\right\}+\cos(\lambda\zeta_j) \\
&\bar{\gamma}_j=\lambda\sum_{i=1}^{j-1}\left\{\begin{matrix}\alpha_i T(\lambda[\zeta_j-\zeta_i])\bar{\gamma}_i\\ -\beta_i\lambda S(\lambda[\zeta_j-\zeta_i])\bar{\sigma}_i\end{matrix}\right\}+\sinh(\lambda\zeta_j) \\
&\bar{\kappa}_j=\lambda\sum_{i=1}^{j-1}\left\{\begin{matrix}\alpha_i T(\lambda[\zeta_j-\zeta_i])\bar{\theta}_i\\ -\beta_i\lambda S(\lambda[\zeta_j-\zeta_i])\bar{\tau}_i\end{matrix}\right\}+\cosh(\lambda\zeta_j),
\end{aligned}
\qquad (28)
$$

additionally

$$\bar{\upsilon}_j = \lambda^2 \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i T'\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\bar{\mu}_i \\ -\beta_i \lambda S'\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\bar{\upsilon}_i \end{Bmatrix} + \lambda\cos\left(\lambda\zeta_j\right)$$

$$\bar{\theta}_j = \lambda^2 \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i T'\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\bar{\eta}_i \\ -\beta_i \lambda S'\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\bar{\theta}_i \end{Bmatrix} - \lambda\sin\left(\lambda\zeta_j\right)$$

$$\bar{\sigma}_j = \lambda^2 \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i T'\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\bar{\gamma}_i \\ -\beta_i \lambda S'\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\bar{\sigma}_i \end{Bmatrix} + \lambda\cosh\left(\lambda\zeta_j\right) \quad (29)$$

$$\bar{\tau}_j = \lambda^2 \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i T'\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\bar{\theta}_i \\ -\beta_i \lambda S'\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\bar{\tau}_i \end{Bmatrix} + \lambda\sinh\left(\lambda\zeta_j\right).$$

The exact solution of the eigen-mode governing Eq. (15), is given by Eq. (22), and through Eq. (27) can be stated in the following explicit form:

$$\begin{aligned}\phi(\zeta) = {} & c_1 \begin{bmatrix} \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i T\left(\lambda[\zeta - \zeta_i]\right)\bar{\mu}_i \\ -\beta_i \lambda S\left(\lambda[\zeta - \zeta_i]\right)\bar{\upsilon}_i \end{Bmatrix} u\left(\zeta - \zeta_i\right) \\ + \sin\left(\lambda\zeta\right) \end{bmatrix} + \\ & c_2 \begin{bmatrix} \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i T\left(\lambda[\zeta - \zeta_i]\right)\bar{\eta}_i \\ -\beta_i \lambda S\left(\lambda[\zeta - \zeta_i]\right)\bar{\theta}_i \end{Bmatrix} u\left(\zeta - \zeta_i\right) \\ + \cos\left(\lambda\zeta\right) \end{bmatrix} + \\ & c_3 \begin{bmatrix} \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i T\left(\lambda[\zeta - \zeta_i]\right)\bar{\gamma}_i \\ -\beta_i \lambda S\left(\lambda[\zeta - \zeta_i]\right)\bar{\sigma}_i \end{Bmatrix} u\left(\zeta - \zeta_i\right) \\ + \sinh\left(\lambda\zeta\right) \end{bmatrix} + \\ & c_4 \begin{bmatrix} \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i T\left(\lambda[\zeta - \zeta_i]\right)\bar{\kappa}_i \\ -\beta_i \lambda S\left(\lambda[\zeta - \zeta_i]\right)\bar{\tau}_i \end{Bmatrix} u\left(\zeta - \zeta_i\right) \\ + \cosh\left(\lambda\zeta\right) \end{bmatrix}. \end{aligned} \quad (30)$$

Instead of a combination of the standard trigonometric and hyperbolic functions, the expressions for displacement and its derivation may be expressed in a more convenient form in terms of four fundamental solutions as follows:

$$\begin{aligned} g_1(x) &= \frac{1}{2}\left[\cosh(x) + \cos(x)\right] \\ g_2(x) &= \frac{1}{2}\left[\sinh(x) + \sin(x)\right] \\ g_3(x) &= \frac{1}{2}\left[\cosh(x) - \cos(x)\right] = S(x) \\ g_4(x) &= \frac{1}{2}\left[\sinh(x) - \sin(x)\right] = T(x) \end{aligned} \quad (31)$$

Then $g_i(x)$, $i = 1,...,4$, are a better choice of merit functions than standard trigonometric and hyperbolic functions since these functions have several properties which help to implement the boundary conditions easily. There is the following relation between derivatives of these functions:

$$\begin{aligned} \frac{d}{dx} g_p(x) &= g_{p-1}(x), \quad p = 1,...,4, \\ g_0(x) &= g_4(x). \end{aligned} \quad (32)$$

Moreover, the values of them at zero point are similar to Kronicker's delta function as

$$\frac{d^j}{dx^j} g_p(x)\Big|_{x=0} = \delta_{p(j+1)} \quad \begin{matrix} p = 1,...,4 \\ j = 0,...,3 \end{matrix} \quad (33)$$

Regarding Eqs. (22), (23) with the aforementioned fundamental solutions in Eq. (31), it can be expressed that

$$\begin{aligned} C(x) &= e_1 g_1(x) + e_2 g_2(x) \\ &+ e_3 g_3(x) + e_4 g_4(x) = \sum_{k=1}^{4} e_k g_p(x), \end{aligned} \quad (34)$$

and also

$$\begin{aligned} \phi(\zeta_i) &= e_1\mu_i + e_2\eta_i + e_3\gamma_i + e_4\kappa_i \\ \phi'(\zeta_i) &= e_1\upsilon_i + e_2\theta_i + e_3\sigma_i + e_4\tau_i \end{aligned}, \quad (35)$$

the new definition of the coefficients is obtained from the following relations:

$$\begin{aligned} \mu_j &= \lambda \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i g_4\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\mu_i \\ -\beta_i \lambda g_3\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\upsilon_i \end{Bmatrix} + g_1\left(\lambda\zeta_j\right) \\ \eta_j &= \lambda \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i g_4\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\eta_i \\ -\beta_i \lambda g_3\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\theta_i \end{Bmatrix} + g_2\left(\lambda\zeta_j\right) \\ \gamma_j &= \lambda \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i g_4\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\gamma_i \\ -\beta_i \lambda g_3\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\sigma_i \end{Bmatrix} + g_3\left(\lambda\zeta_j\right) \\ \kappa_j &= \lambda \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i g_4\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\kappa_i \\ -\beta_i \lambda g_3\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\tau_i \end{Bmatrix} + g_4\left(\lambda\zeta_j\right) \end{aligned} \quad (36)$$

$$\begin{aligned} \upsilon_j &= \lambda^2 \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i g_3\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\mu_i \\ -\beta_i \lambda g_2\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\upsilon_i \end{Bmatrix} + \lambda g_4\left(\lambda\zeta_j\right) \\ \theta_j &= \lambda^2 \sum_{i=1}^{j-1} \begin{Bmatrix} \alpha_i g_3\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\eta_i \\ -\beta_i \lambda g_2\left(\lambda\left[\zeta_j - \zeta_i\right]\right)\theta_i \end{Bmatrix} + \lambda g_1\left(\lambda\zeta_j\right) \end{aligned} \quad (37)$$

$$\sigma_j=\lambda^2\sum_{i=1}^{j-1}\left\{\begin{array}{l}\alpha_i g_3\left(\lambda\left[\zeta_j-\zeta_i\right]\right)\gamma_i\\-\beta_i\lambda g_2\left(\lambda\left[\zeta_j-\zeta_i\right]\right)\sigma_i\end{array}\right\}+\lambda g_2\left(\lambda\zeta_j\right)$$
$$\tau_j=\lambda^2\sum_{i=1}^{j-1}\left\{\begin{array}{l}\alpha_i g_3\left(\lambda\left[\zeta_j-\zeta_i\right]\right)\kappa_i\\-\beta_i\lambda g_2\left(\lambda\left[\zeta_j-\zeta_i\right]\right)\tau_i\end{array}\right\}+\lambda g_3\left(\lambda\zeta_j\right).$$

Finally, the exact solution of the Eigen-mode in explicit form with the use of fundamental solutions can be derived as

$$\begin{aligned}\phi(\zeta)=&\\ e_1&\left[\begin{array}{l}\lambda\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_4\left(\lambda[\zeta-\zeta_i]\right)\mu_i\\-\beta_i\lambda g_3\left(\lambda[\zeta-\zeta_i]\right)\upsilon_i\end{array}\right\}u\left(\zeta-\zeta_i\right)\\+g_1\left(\lambda\zeta\right)\end{array}\right]+\\ e_2&\left[\begin{array}{l}\lambda\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_4\left(\lambda[\zeta-\zeta_i]\right)\eta_i\\-\beta_i\lambda g_3\left(\lambda[\zeta-\zeta_i]\right)\theta_i\end{array}\right\}u\left(\zeta-\zeta_i\right)\\+g_2\left(\lambda\zeta\right)\end{array}\right]+\\ e_3&\left[\begin{array}{l}\lambda\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_4\left(\lambda[\zeta-\zeta_i]\right)\gamma_i\\-\beta_i\lambda g_3\left(\lambda[\zeta-\zeta_i]\right)\sigma_i\end{array}\right\}u\left(\zeta-\zeta_i\right)\\+g_3\left(\lambda\zeta\right)\end{array}\right]+\\ e_4&\left[\begin{array}{l}\lambda\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_4\left(\lambda[\zeta-\zeta_i]\right)\kappa_i\\-\beta_i\lambda g_3\left(\lambda[\zeta-\zeta_i]\right)\tau_i\end{array}\right\}u\left(\zeta-\zeta_i\right)\\+g_4\left(\lambda\zeta\right)\end{array}\right].\end{aligned} \quad (38)$$

3- Frequency Equation

In this section, frequency equation will be derived by enforcing the standard boundary conditions, including pinned–pinned (PP), clamped-clamped (CC), cantilever (CF), and clamped–pinned (CP). The frequency equations will be derived from the determinant of a matrix 2x2 for any type of boundary conditions and will be numerically solved in order to obtain the frequency parameters (λ) and corresponding vibration modes ($\phi(\zeta)$).

3- 1- Pinned-Pinned

The boundary conditions of the pinned-pinned beam can be expressed as follows:

$$\phi(0)=0,\quad \phi''(0)=0,\quad \phi(1)=0,\quad \phi''(1)=0. \quad (39)$$

Accounting for Eqs. (38), (39), the following conditions for the integration constants e_1, e_2, e_3, e_4, can be indicated as

$$e_1=e_3=0 \quad (40)$$

$$\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{Bmatrix}e_2\\ e_4\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}, \quad (41)$$

where

$$\begin{aligned}A_{11}&=\lambda\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_4\left(\lambda\varepsilon_i\right)\eta_i\\-\beta_i\lambda g_3\left(\lambda\varepsilon_i\right)\theta_i\end{array}\right\}+g_2(\lambda)\\ A_{12}&=\lambda\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_4\left(\lambda\varepsilon_i\right)\kappa_i\\-\beta_i\lambda g_3\left(\lambda\varepsilon_i\right)\tau_i\end{array}\right\}+g_4(\lambda)\\ A_{21}&=\lambda\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_2\left(\lambda\varepsilon_i\right)\eta_i\\-\beta_i\lambda g_1\left(\lambda\varepsilon_i\right)\theta_i\end{array}\right\}+g_4(\lambda)\\ A_{22}&=\lambda\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_2\left(\lambda\varepsilon_i\right)\kappa_i\\-\beta_i\lambda g_1\left(\lambda\varepsilon_i\right)\tau_i\end{array}\right\}+g_2(\lambda),\end{aligned} \quad (42)$$

and ε_i=1- ζ_i .
The frequency equation of the pinned-pinned beam carrying multiple concentrated masses with a rotary inertia can be obtained by evaluating the second-order determinant of the system of Eq. (41) as

$$A_{11}A_{22}-A_{12}A_{21}=0 \quad (43)$$

The zeros of the Eq. (43) indicate the values of the frequency parameters. By inserting the obtained frequency parameters in the boundary conditions system of Eq. (41), the value of the integration constants that provides the vibration mode can be obtained as the follows:

$$e_4=1,\ e_2=-\frac{A_{12}}{A_{11}}. \quad (44)$$

Inserting the last relations into Eq. (38), the values of the integration constants given by Eqs. (40), and (44), the closed-form expressions of the vibration modes can be obtained as

$$\begin{aligned}\phi_k(\zeta)=&\\ &-\frac{\lambda_k\sum_{i=1}^{N}\left\{\alpha_i g_4\left(\lambda_k\varepsilon_i\right)\kappa_i-\beta_i\lambda_k g_3\left(\lambda_k\varepsilon_i\right)\tau_i\right\}+g_4\left(\lambda_k\right)}{\lambda_k\sum_{i=1}^{N}\left\{\alpha_i g_4\left(\lambda_k\varepsilon_i\right)\eta_i-\beta_i\lambda_k g_3\left(\lambda_k\varepsilon_i\right)\theta_i\right\}+g_2\left(\lambda_k\right)}\\ &\times\left[\lambda_k\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_4\left(\lambda_k[\zeta-\zeta_i]\right)\eta_i\\-\beta_i\lambda_k g_3\left(\lambda_k[\zeta-\zeta_i]\right)\theta_i\end{array}\right\}u\left(\zeta-\zeta_i\right)+g_2\left(\lambda_k\zeta\right)\right]\\ &+\lambda_k\sum_{i=1}^{N}\left\{\begin{array}{l}\alpha_i g_4\left(\lambda_k[\zeta-\zeta_i]\right)\kappa_i\\-\beta_i\lambda_k g_3\left(\lambda_k[\zeta-\zeta_i]\right)\tau_i\end{array}\right\}u\left(\zeta-\zeta_i\right)+g_4\left(\lambda_k\zeta\right).\end{aligned} \quad (45)$$

3- 2- Clamped-Clamped

The boundary conditions of the clamped-clamped beam can be expressed as follows:

$$\phi(0)=0,\quad \phi'(0)=0,\quad \phi(1)=0,\quad \phi'(1)=0. \quad (46)$$

Accounting for Eqs. (38), (46), the following conditions for the integration constants e_1, e_2, e_3, e_4, can be indicated as

$$e_1 = e_2 = 0 \tag{47}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{Bmatrix} e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{48}$$

where

$$\begin{aligned} A_{11} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_4(\lambda\varepsilon_i)\gamma_i \\ -\beta_i \lambda g_3(\lambda\varepsilon_i)\sigma_i \end{Bmatrix} + g_3(\lambda) \\ A_{12} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_4(\lambda\varepsilon_i)\kappa_i \\ -\beta_i \lambda g_3(\lambda\varepsilon_i)\tau_i \end{Bmatrix} + g_4(\lambda) \\ A_{21} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_3(\lambda\varepsilon_i)\gamma_i \\ -\beta_i \lambda g_2(\lambda\varepsilon_i)\sigma_i \end{Bmatrix} + g_2(\lambda) \\ A_{22} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_3(\lambda\varepsilon_i)\kappa_i \\ -\beta_i \lambda g_2(\lambda\varepsilon_i)\tau_i \end{Bmatrix} + g_3(\lambda). \end{aligned} \tag{49}$$

The frequency equation of the clamped-clamped beam carrying multiple concentrated masses with a rotary inertia can be obtained by evaluating the second-order determinant of the system of Eq. (48) and in a similar manner, the closed-form expressions of the vibration modes can be achieved as in the following:

$$\begin{aligned} &\phi_k(\zeta) = \\ &-\frac{\lambda_k \sum_{i=1}^{N} \{\alpha_i g_4(\lambda_k \varepsilon_i)\kappa_i - \beta_i \lambda_k g_3(\lambda_k \varepsilon_i)\tau_i\} + g_4(\lambda_k)}{\lambda_k \sum_{i=1}^{N} \{\alpha_i g_4(\lambda_k \varepsilon_i)\gamma_i - \beta_i \lambda_k g_3(\lambda_k \varepsilon_i)\sigma_i\} + g_3(\lambda_k)} \\ &\times\left[\lambda_k \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_4(\lambda_k[\zeta-\zeta_i])\gamma_i \\ -\beta_i \lambda_k g_3(\lambda_k[\zeta-\zeta_i])\sigma_i \end{Bmatrix} u(\zeta-\zeta_i) + g_3(\lambda_k\zeta) \right] \\ &+\lambda_k \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_4(\lambda_k[\zeta-\zeta_i])\kappa_i \\ -\beta_i \lambda_k g_3(\lambda_k[\zeta-\zeta_i])\tau_i \end{Bmatrix} u(\zeta-\zeta_i) + g_4(\lambda_k\zeta). \end{aligned} \tag{50}$$

3- 3- Clamped-Free

The boundary conditions of the clamped-free beam can be expressed as

$$\phi(0)=0, \quad \phi'(0)=0, \quad \phi''(1)=0, \quad \phi'''(1)=0. \tag{51}$$

Accounting for Eqs. (38), (51), the following conditions for the integration constants e_1, e_2, e_3, e_4, are written as

$$e_1 = e_2 = 0 \tag{52}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{Bmatrix} e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{53}$$

in which

$$\begin{aligned} A_{11} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_2(\lambda\varepsilon_i)\gamma_i \\ -\beta_i \lambda g_1(\lambda\varepsilon_i)\sigma_i \end{Bmatrix} + g_1(\lambda) \\ A_{12} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_2(\lambda\varepsilon_i)\kappa_i \\ -\beta_i \lambda g_1(\lambda\varepsilon_i)\tau_i \end{Bmatrix} + g_2(\lambda) \\ A_{21} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_1(\lambda\varepsilon_i)\gamma_i \\ -\beta_i \lambda g_4(\lambda\varepsilon_i)\sigma_i \end{Bmatrix} + g_4(\lambda) \\ A_{22} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_1(\lambda\varepsilon_i)\kappa_i \\ -\beta_i \lambda g_4(\lambda\varepsilon_i)\tau_i \end{Bmatrix} + g_1(\lambda) \end{aligned} . \tag{54}$$

The frequency equation of the clamped-free beam carrying multiple concentrated masses with a rotary inertia can be represented by evaluating the second-order determinant of the system of Eq. (53) and similarly, the closed-form expressions of the vibration modes is given by

$$\begin{aligned} &\phi_k(\zeta) = \\ &-\frac{\lambda_k \sum_{i=1}^{N} \{\alpha_i g_2(\lambda_k \varepsilon_i)\kappa_i - \beta_i \lambda_k g_1(\lambda_k \varepsilon_i)\tau_i\} + g_2(\lambda_k)}{\lambda_k \sum_{i=1}^{N} \{\alpha_i g_2(\lambda_k \varepsilon_i)\gamma_i - \beta_i \lambda_k g_1(\lambda_k \varepsilon_i)\sigma_i\} + g_1(\lambda_k)} \\ &\times\left[\lambda_k \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_4(\lambda_k[\zeta-\zeta_i])\gamma_i \\ -\beta_i \lambda_k g_3(\lambda_k[\zeta-\zeta_i])\sigma_i \end{Bmatrix} u(\zeta-\zeta_i) + g_3(\lambda_k\zeta) \right] \\ &+\lambda_k \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_4(\lambda_k[\zeta-\zeta_i])\kappa_i \\ -\beta_i \lambda_k g_3(\lambda_k[\zeta-\zeta_i])\tau_i \end{Bmatrix} u(\zeta-\zeta_i) + g_4(\lambda_k\zeta). \end{aligned} \tag{55}$$

3- 4- Clamped-Pinned

The boundary conditions of the clamped-pinned beam can be considered as

$$\phi(0)=0, \quad \phi'(0)=0, \quad \phi(1)=0, \quad \phi''(1)=0. \tag{56}$$

Accounting for Eqs. (38) and (56), the following conditions for the integration constants e_1, e_2, e_3, e_4, are expressed as

$$e_1 = e_2 = 0 \tag{57}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{Bmatrix} e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{58}$$

in which

$$\begin{aligned} A_{11} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_4(\lambda\varepsilon_i)\gamma_i \\ -\beta_i \lambda g_3(\lambda\varepsilon_i)\sigma_i \end{Bmatrix} + g_3(\lambda) \\ A_{12} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_4(\lambda\varepsilon_i)\kappa_i \\ -\beta_i \lambda g_3(\lambda\varepsilon_i)\tau_i \end{Bmatrix} + g_4(\lambda) \\ A_{21} &= \lambda \sum_{i=1}^{N} \begin{Bmatrix} \alpha_i g_2(\lambda\varepsilon_i)\gamma_i \\ -\beta_i \lambda g_1(\lambda\varepsilon_i)\sigma_i \end{Bmatrix} + g_1(\lambda) \end{aligned} \tag{59}$$

$$A_{22} = \lambda \sum_{i=1}^{N} \left\{ \begin{array}{l} \alpha_i g_2\left(\lambda \varepsilon_i\right) \kappa_i \\ -\beta_i \lambda g_1\left(\lambda \varepsilon_i\right) \tau_i \end{array} \right\} + g_2\left(\lambda\right)$$

The frequency equation of the clamped-pinned beam carrying multiple concentrated mass with a rotary inertia is obtained by evaluating the second-order determinant of the system of Eq. (58) and in a similar manner, the closed-form expressions of the vibration modes is represented as

$$\begin{aligned} &\phi_k\left(\zeta\right) = \\ &-\frac{\lambda_k \sum_{i=1}^{N} \left\{ \alpha_i g_4\left(\lambda_k \varepsilon_i\right) \kappa_i - \beta_i \lambda_k g_3\left(\lambda_k \varepsilon_i\right) \tau_i \right\} + g_4\left(\lambda_k\right)}{\lambda_k \sum_{i=1}^{N} \left\{ \alpha_i g_4\left(\lambda_k \varepsilon_i\right) \gamma_i - \beta_i \lambda_k g_3\left(\lambda_k \varepsilon_i\right) \sigma_i \right\} + g_3\left(\lambda_k\right)} \\ &\times \left[\lambda_k \sum_{i=1}^{N} \left\{ \begin{array}{l} \alpha_i g_4\left(\lambda_k \left[\zeta - \zeta_i\right]\right) \gamma_i \\ -\beta_i \lambda_k g_3\left(\lambda_k \left[\zeta - \zeta_i\right]\right) \sigma_i \end{array} \right\} u\left(\zeta - \zeta_i\right) + g_3\left(\lambda_k \zeta\right) \right] \\ &+\lambda_k \sum_{i=1}^{N} \left\{ \begin{array}{l} \alpha_i g_4\left(\lambda_k \left[\zeta - \zeta_i\right]\right) \kappa_i \\ -\beta_i \lambda_k g_3\left(\lambda_k \left[\zeta - \zeta_i\right]\right) \tau_i \end{array} \right\} u\left(\zeta - \zeta_i\right) + g_4\left(\lambda_k \zeta\right). \end{aligned} \tag{60}$$

4- Numerical Results and Discussion

In order to validate the results of the presented technique, indicated in Tables 1 to 7, initially, the first five frequency parameters of a beam with two or four attached masses are calculated and listed for various cases in position and value of the mass and inertia parameters (α & c). It can be observed that the proposed technique is in a very good agreement with other exact solutions, are given by [15].

The maximum error presented at the bottom of Tables 1 to 7 is less than 1 % which confirms a high accuracy of the proposed solution. It is worth mentioning that this small difference may be created through the diversity of employed numerical methods and divergence benchmark in the solution of the final algebraic equation, presented in Ref. [15].

In addition, it can be concluded that as the value of the mass and inertia parameters increase, the value of all frequency parameters decreases. Of course, it is well worth mentioning that the reduction of the frequency parameters due to the rotary inertia parameter is lower than the reduction concerning with the mass parameter; rather as will be shown in the following it depends on the position of the mass.

Table 1. First five frequency parameters for a clamped–clamped beam with two symmetric masses.

$\zeta_1 = 0.25$; $\zeta_2 = 0.75$		$c_1 = c_2 = 0$		$c_1 = c_2 = 0.01$		$c_1 = c_2 = 0.05$		$c_1 = c_2 = 0.1$	
		Ref [15]	Present	Ref [15]	Present	Ref [15]	Present	Ref [15]	Present
$\alpha_1=\alpha_2=0.01$	λ_1	4.7126	4.7126	4.7125	4.7125	4.7112	4.7112	4.7071	4.7071
	λ_2	7.7732	7.7732	7.7731	7.7731	7.7723	7.7723	7.7696	7.7696
	λ_3	10.8958	10.8958	10.8956	10.8956	10.8899	10.8898	10.8714	10.8714
	λ_4	14.1150	14.1148	14.1125	14.1124	14.0520	14.0518	13.8602	13.8601
	λ_5	17.2557	17.2543	17.2513	17.2515	17.1426	17.1413	16.7908	16.7890
$\alpha_1=\alpha_2=0.1$	λ_1	4.5668	4.5668	4.5663	4.5663	4.5554	4.5554	4.5217	4.5217
	λ_2	7.1911	7.1910	7.1908	7.1908	7.1855	7.1855	7.1671	7.1671
	λ_3	10.2346	10.2346	10.2325	10.2325	10.1796	10.1796	9.9795	9.9795
	λ_4	13.9713	13.9712	13.9472	13.9471	13.3525	13.3525	11.7542	11.7543
	λ_5	17.1148	17.1172	17.0715	17.0724	15.9720	15.9720	13.5895	13.5895
$\alpha_1=\alpha_2=0.5$	λ_1	4.0973	4.0973	4.0961	4.0961	4.0663	4.0663	3.9755	3.9755
	λ_2	5.8984	5.8984	5.8980	5.8980	5.8893	5.8893	5.8555	5.8555
	λ_3	9.1453	9.1453	9.1356	9.1356	8.8716	8.8716	7.9804	7.9804
	λ_4	13.7527	13.7528	13.6401	13.6400	11.2437	11.2437	8.5500	8.5500
	λ_5	16.9258	16.8841	16.7178	16.6903	12.9941	12.9940	10.8372	10.8372
$\alpha_1=\alpha_2=1$	λ_1	3.7335	3.7335	3.7320	3.7320	3.6959	3.6959	3.5868	3.5868
	λ_2	5.1746	5.1746	5.1743	5.1743	5.1656	5.1656	5.1306	5.1306
	λ_3	8.7418	8.7418	8.7220	8.7220	8.1800	8.1800	6.9010	6.9010
	λ_4	13.6791	13.6784	13.4578	13.4564	9.8682	9.8682	7.2687	7.2687
	λ_5	16.8681	16.9985	16.4514	16.4461	11.6278	11.6279	10.2256	10.2256
$\alpha_1=\alpha_2=2$	λ_1	3.3053	3.3053	3.3037	3.3037	3.2659	3.2659	3.1514	3.1514
	λ_2	4.4574	4.4574	4.4571	4.4571	4.4491	4.4491	4.4160	4.4160
	λ_3	8.4667	8.4667	8.4261	8.4261	7.3841	7.3841	5.8827	5.8827
	λ_4	13.6312	13.6297	13.1901	13.1898	8.4819	8.4819	6.1460	6.1460
	λ_5	16.8320	16.7198	15.9869	15.9472	10.6332	10.6332	9.8684	9.8684

Maximum error=0.7671 %

A beam with one, three or five similar attached masses is assumed. The first five frequency parameters of the beam for one single attached mass, for three masses, and for five masses are respectively listed in Tables 8 to10. It can be observed from these tables that, as expected, whatever quantity of masses increases, the value of the frequency parameters decreases for all boundary conditions.

Table 2. First five frequency parameters for a clamped–clamped beam with two asymmetric masses.

$\zeta_1 = 0.25$; $\zeta_2 = 0.5$		$c_1 = c_2 = 0$		$c_1 = c_2 = 0.01$		$c_1 = c_2 = 0.05$		$c_1 = c_2 = 0.1$	
		Ref [15]	Present	Ref [15]	Present	Ref [15]	Present	Ref [15]	Present
α_1=α_2=0.01	λ_1	4.6921	4.6921	4.6921	4.6921	4.6915	4.6915	4.6895	4.6895
	λ_2	7.8128	7.8128	7.8125	7.8125	7.8061	7.8061	7.7861	7.7861
	λ_3	10.8932	10.8932	10.8931	10.8931	10.8903	10.8903	10.8810	10.8810
	λ_4	14.1262	14.1261	14.1236	14.1236	14.0593	14.0591	13.8577	13.8575
	λ_5	17.1859	17.1856	17.1836	17.1836	17.1287	17.1292	16.9486	16.9476
α_1=α_2=0.1	λ_1	4.4053	4.4053	4.4051	4.4051	4.4003	4.4003	4.3856	4.3856
	λ_2	7.4860	7.4860	7.4841	7.4841	7.4361	7.4361	7.2818	7.2818
	λ_3	10.2227	10.2227	10.2217	10.2217	10.1940	10.1940	10.0654	10.0654
	λ_4	14.0604	14.0602	14.0336	14.0335	13.3904	13.3904	11.9149	11.9149
	λ_5	16.6703	16.6707	16.6471	16.6455	16.0376	16.0359	13.7828	13.7828
α_1=α_2=0.5	λ_1	3.7027	3.7027	3.7022	3.7022	3.6922	3.6922	3.6606	3.6606
	λ_2	6.4814	6.4814	6.4778	6.4778	6.3855	6.3855	6.0575	6.0575
	λ_3	9.2683	9.2683	9.2606	9.2606	9.0218	9.0218	8.0269	8.0269
	λ_4	13.9693	13.9694	13.8313	13.8315	11.3901	11.3901	9.4410	9.4410
	λ_5	16.0876	16.0812	15.9755	15.9827	12.9703	12.9703	10.2982	10.2982
α_1=α_2=1	λ_1	3.2772	3.2772	3.2768	3.2768	3.2658	3.2658	3.2314	3.2314
	λ_2	5.7693	5.7693	5.7658	5.7658	5.6755	5.6755	5.3312	5.3312
	λ_3	9.0003	9.0003	8.9827	8.9827	8.3750	8.3750	6.9111	6.9111
	λ_4	13.9388	13.9387	13.6573	13.6559	10.2652	10.2652	8.0475	8.0475
	λ_5	15.9243	15.9157	15.7069	15.7013	11.3195	11.3195	9.8784	9.8784
α_1=α_2=2	λ_1	2.8399	2.8399	2.8394	2.8394	2.8287	2.8287	2.7949	2.7949
	λ_2	5.0077	5.0077	5.0046	5.0046	4.9240	4.9240	4.5992	4.5992
	λ_3	8.8463	8.8463	8.8084	8.8084	7.5086	7.5086	5.8790	5.8790
	λ_4	13.9185	13.9198	13.3490	13.3480	9.0757	9.0757	6.8012	6.8012
	λ_5	15.8239	15.8957	15.3957	15.3982	10.2276	10.2276	9.6737	9.6737

Maximum error=0.4517 %

Table 3. First five frequency parameters for a pinned-pinned beam with two symmetric masses.

$\zeta_1 = 0.25$; $\zeta_2 = 0.75$		$c_1 = c_2 = 0$		$c_1 = c_2 = 0.01$		$c_1 = c_2 = 0.05$		$c_1 = c_2 = 0.1$	
		Ref [15]	Present	Ref [15]	Present	Ref [15]	Present	Ref [15]	Present
α_1=α_2=0.01	λ_1	3.1261	3.1261	3.1261	3.1261	3.1257	3.1257	3.1246	3.1246
	λ_2	6.2218	6.2218	6.2218	6.2218	6.2218	6.2218	6.2218	6.2218
	λ_3	9.3790	9.3790	9.3786	9.3786	9.3687	9.3687	9.3376	9.3376
	λ_4	12.5664	12.5664	12.5644	12.5644	12.5167	12.5167	12.3679	12.3679
	λ_5	15.6328	15.6329	15.6309	15.6309	15.5845	15.5847	15.4321	15.4320
α_1=α_2=0.1	λ_1	3.0013	3.0013	3.0012	3.0012	2.9983	2.9983	2.9892	2.9892
	λ_2	5.7745	5.7745	5.7745	5.7745	5.7745	5.7745	5.7745	5.7745
	λ_3	9.0595	9.0595	9.0559	9.0559	8.9674	8.9674	8.6820	8.6820
	λ_4	12.5664	12.5664	12.5465	12.5466	12.0741	12.0741	10.8225	10.8225
	λ_5	15.1713	15.1714	15.1541	15.1537	14.6979	14.6978	13.3007	13.3007

	λ_1	2.6393	2.6393	2.6390	2.6390	2.6315	2.6315	2.6085	2.6085
	λ_2	4.7664	4.7664	4.7664	4.7664	4.7664	4.7664	4.7664	4.7664
$\alpha_1=\alpha_2=0.5$	λ_3	8.4744	8.4744	8.4594	8.4594	8.0892	8.0892	7.1123	7.1123
	λ_4	12.5664	12.5664	12.4671	12.4670	10.4963	10.4963	8.0784	8.0784
	λ_5	14.5617	14.5598	14.4846	14.4825	12.5720	12.5720	10.8300	10.8300
	λ_1	2.3832	2.3832	2.3828	2.3828	2.3740	2.3740	2.3469	2.3469
	λ_2	4.1920	4.1920	4.1920	4.1920	4.1920	4.1920	4.1920	4.1920
$\alpha_1=\alpha_2=1$	λ_3	8.2394	8.2394	8.2114	8.2114	7.5328	7.5328	6.2114	6.2114
	λ_4	12.5664	12.5668	12.3679	12.3679	9.3276	9.3276	6.8955	6.8955
	λ_5	14.3802	14.3801	14.2279	14.2273	11.4423	11.4423	10.2253	10.2253
	λ_1	2.0960	2.0960	2.0956	2.0956	2.0864	2.0864	2.0583	2.0583
	λ_2	3.6171	3.6171	3.6171	3.6171	3.6171	3.6171	3.6171	3.6171
$\alpha_1=\alpha_2=2$	λ_3	8.0730	8.0730	8.0190	8.0190	6.8399	6.8399	5.3282	5.3282
	λ_4	12.5664	12.5663	12.1712	12.1713	8.0784	8.0784	5.8419	5.8419
	λ_5	14.2680	14.2668	13.9592	13.9499	10.5691	10.5691	9.8684	9.8684

Maximum error=0.0667 %

Table 4. First five frequency parameters for a pinned-pinned beam with two asymmetric masses.

$\zeta_1 = 0.25$; $\zeta_2 = 0.50$		$c_1 = c_2 = 0$		$c_1 = c_2 = 0.01$		$c_1 = c_2 = 0.05$		$c_1 = c_2 = 0.1$	
		Ref [15]	Present	Ref [15]	Present	Ref [15]	Present	Ref [15]	Present
	λ_1	3.1185	3.1185	3.1184	3.1184	3.1183	3.1183	3.1177	3.1177
	λ_2	6.2524	6.2524	6.2523	6.2523	6.2494	6.2494	6.2403	6.2403
$\alpha_1=\alpha_2=0.01$	λ_3	9.3558	9.3558	9.3556	9.3556	9.3509	9.3509	9.3356	9.3356
	λ_4	12.5664	12.5664	12.5644	12.5644	12.5168	12.5168	12.3684	12.3684
	λ_5	15.5961	15.5960	15.5951	15.5960	15.5714	15.5705	15.4950	15.4953
	λ_1	2.9415	2.9415	2.9414	2.9414	2.9401	2.9401	2.9359	2.9359
	λ_2	6.0161	6.0161	6.0151	6.0151	5.9914	5.9914	5.9175	5.9175
$\alpha_1=\alpha_2=0.1$	λ_3	8.8650	8.8650	8.8637	8.8637	8.8302	8.8302	8.6981	8.6981
	λ_4	12.5664	12.5663	12.5465	12.5465	12.0735	12.0735	10.8986	10.8986
	λ_5	14.9527	14.9521	14.9422	14.9421	14.6718	14.6717	13.4418	13.4418
	λ_1	2.4946	2.4946	2.4945	2.4945	2.4916	2.4916	2.4824	2.4824
	λ_2	5.3428	5.3428	5.3403	5.3403	5.2788	5.2788	5.0881	5.0881
$\alpha_1=\alpha_2=0.5$	λ_3	7.9643	7.9643	7.9604	7.9604	7.8441	7.8441	7.2183	7.2183
	λ_4	12.5664	12.5663	12.4664	12.4663	10.5152	10.5152	8.8187	8.8187
	λ_5	14.2171	14.2176	14.1610	14.1605	12.4431	12.4432	9.6262	9.6262
	λ_1	2.2162	2.2162	2.2161	2.2161	2.2128	2.2128	2.2027	2.2027
	λ_2	4.8384	4.8384	4.8355	4.8355	4.7649	4.7649	4.5445	4.5445
$\alpha_1=\alpha_2=1$	λ_3	7.6317	7.6317	7.6240	7.6240	7.3718	7.3718	6.3101	6.3101
	λ_4	12.5664	12.5663	12.3649	12.3650	9.4562	9.4562	7.9257	7.9257
	λ_5	14.0212	14.0208	13.9073	13.9072	10.8964	10.8964	8.5582	8.5582
	λ_1	1.9256	1.9256	1.9254	1.9254	1.9222	1.9222	1.9121	1.9121
	λ_2	4.2553	4.2553	4.2525	4.2525	4.1825	4.1825	3.9609	3.9609
$\alpha_1=\alpha_2=2$	λ_3	7.4180	7.4180	7.4019	7.4019	6.8212	6.8212	5.4070	5.4070
	λ_4	12.5664	12.5655	12.1594	12.1594	8.4533	8.4533	6.7602	6.7602
	λ_5	13.9050	13.9096	13.6744	13.6782	9.4571	9.4571	8.1863	8.1863

Maximum error=0.0331 %

Table 5. First five frequency parameters for a cantilever beam with two symmetric masses.

$\zeta_1 = 0.25$; $\zeta_2 = 0.75$		$c_1 = c_2 = 0$		$c_1 = c_2 = 0.01$		$c_1 = c_2 = 0.05$		$c_1 = c_2 = 0.1$	
		Ref [15]	Present	Ref [15]	Present	Ref [15]	Present	Ref [15]	Present
α_1=α_2=0.01	λ_1	1.8669	1.8669	1.8669	1.8669	1.8668	1.8668	1.8665	1.8665
	λ_2	4.6851	4.6851	4.6850	4.6850	4.6827	4.6827	4.6757	4.6757
	λ_3	7.7887	7.7887	7.7887	7.7887	7.7869	7.7869	7.7813	7.7813
	λ_4	10.9048	10.9047	10.9046	10.9046	10.8999	10.8999	10.8850	10.8850
	λ_5	14.1171	14.1170	14.1147	14.1145	14.0569	14.0566	13.8735	13.8733
α_1=α_2=0.1	λ_1	1.8003	1.8003	1.8002	1.8002	1.7994	1.7994	1.7967	1.7967
	λ_2	4.6083	4.6083	4.6074	4.6074	4.5867	4.5867	4.5240	4.5240
	λ_3	7.3191	7.3191	7.3184	7.3184	7.3026	7.3026	7.2516	7.2516
	λ_4	10.3067	10.3067	10.3050	10.3050	10.2639	10.2639	10.1052	10.1052
	λ_5	13.9865	13.9863	13.9634	13.9634	13.3953	13.3952	11.8800	11.8800
α_1=α_2=0.5	λ_1	1.6000	1.6000	1.5999	1.5999	1.5976	1.5976	1.5903	1.5903
	λ_2	4.3191	4.3191	4.3162	4.3162	4.2466	4.2466	4.0495	4.0495
	λ_3	6.3836	6.3836	6.3800	6.3800	6.2961	6.2961	6.0715	6.0715
	λ_4	9.3381	9.3381	9.3312	9.3312	9.1379	9.1379	8.2312	8.2312
	λ_5	13.7841	13.7837	13.6761	13.6755	11.4015	11.4015	9.2243	9.2243
α_1=α_2=1	λ_1	1.4529	1.4529	1.4528	1.4528	1.4499	1.4499	1.4411	1.4411
	λ_2	4.0343	4.0343	4.0305	4.0305	3.9408	3.9408	3.6874	3.6874
	λ_3	5.9799	5.9799	5.9712	5.9712	5.7797	5.7797	5.3853	5.3853
	λ_4	8.9843	8.9843	8.9709	8.9709	8.5646	8.5646	7.0960	7.0960
	λ_5	13.7146	13.7137	13.5026	13.5016	10.1527	10.1527	8.5116	8.5116
α_1=α_2=2	λ_1	1.2838	1.2838	1.2837	1.2837	1.2806	1.2806	1.2712	1.2712
	λ_2	3.6358	3.6358	3.6319	3.6319	3.5381	3.5381	3.2631	3.2631
	λ_3	5.7009	5.7009	5.6803	5.6803	5.2724	5.2724	4.6783	4.6783
	λ_4	8.7435	8.7435	8.7169	8.7168	7.8393	7.8393	6.0303	6.0303
	λ_5	13.6691	13.6659	13.2466	13.2465	9.0712	9.0712	8.0572	8.0572

Maximum error=0.0234 %

Table 6. First five frequency parameters for a cantilever beam with two asymmetric masses.

$\zeta_1 = 0.25$; $\zeta_2 = 0.50$		$c_1 = c_2 = 0$		$c_1 = c_2 = 0.01$		$c_1 = c_2 = 0.05$		$c_1 = c_2 = 0.1$	
		Ref [15]	Present	Ref [15]	Present	Ref [15]	Present	Ref [15]	Present
α_1=α_2=0.01	λ_1	1.8728	1.8728	1.8728	1.8728	1.8727	1.8727	1.8724	1.8724
	λ_2	4.6627	4.6627	4.6626	4.6626	4.6620	4.6620	4.6602	4.6602
	λ_3	7.8141	7.8141	7.8138	7.8138	7.8078	7.8078	7.7888	7.7888
	λ_4	10.8925	10.8925	10.8924	10.8924	10.8896	10.8896	10.8803	10.8803
	λ_5	14.1262	14.1261	14.1235	14.1236	14.0592	14.0591	13.8573	13.8574
α_1=α_2=0.1	λ_1	1.8523	1.8523	1.8522	1.8522	1.8514	1.8514	1.8490	1.8490
	λ_2	4.4279	4.4279	4.4277	4.4277	4.4232	4.4232	4.4090	4.4090
	λ_3	7.4885	7.4885	7.4866	7.4866	7.4417	7.4417	7.2971	7.2971
	λ_4	10.2160	10.2159	10.2149	10.2149	10.1879	10.1879	10.0621	10.0621
	λ_5	14.0603	14.0602	14.0335	14.0336	13.3891	13.3890	11.9090	11.9090

	λ_1	1.7711	1.7711	1.7709	1.7709	1.7677	1.7677	1.7579	1.7579
	λ_2	3.8880	3.8880	3.8875	3.8875	3.8759	3.8759	3.8384	3.8384
$\alpha_1=\alpha_2=0.5$	λ_3	6.5059	6.5059	6.5026	6.5026	6.4207	6.4207	6.1329	6.1329
	λ_4	9.2404	9.2404	9.2331	9.2331	9.0069	9.0069	8.0333	8.0333
	λ_5	13.9690	13.9684	13.8307	13.8299	11.3806	11.3806	9.4421	9.4421
	λ_1	1.6881	1.6881	1.6879	1.6879	1.6828	1.6828	1.6676	1.6676
	λ_2	3.5984	3.5984	3.5977	3.5977	3.5809	3.5809	3.5261	3.5261
$\alpha_1=\alpha_2=1$	λ_3	5.8179	5.8179	5.8151	5.8151	5.7418	5.7418	5.4687	5.4687
	λ_4	8.9619	8.9619	8.9453	8.9453	8.3707	8.3707	6.9231	6.9231
	λ_5	13.9383	13.9371	13.6562	13.6545	10.2484	10.2484	8.0525	8.0525
	λ_1	1.5636	1.5636	1.5633	1.5633	1.5565	1.5565	1.5363	1.5363
	λ_2	3.3385	3.3385	3.3374	3.3374	3.3101	3.3101	3.2221	3.2221
$\alpha_1=\alpha_2=2$	λ_3	5.0967	5.0967	5.0946	5.0946	5.0421	5.0421	4.8403	4.8403
	λ_4	8.8007	8.8007	8.7651	8.7651	7.5268	7.5268	5.9004	5.9004
	λ_5	13.9179	13.9117	13.3470	13.3476	9.0692	9.0692	6.8091	6.8091

Maximum error=0.0446 %

Table 7. First five frequency parameters for a clamped–clamped beam with four symmetric masses.

$\zeta_1=0.125$; $\zeta_2=0.375$ $\zeta_3=0.625$; $\zeta_4=0.875$		$c_1=c_2=0$		$c_1=c_2=0.01$		$c_1=c_2=0.05$		$c_1=c_2=0.1$	
		Ref [15]	**Present**	**Ref [15]**	**Present**	**Ref [15]**	**Present**	**Ref [15]**	**Present**
	λ_1	4.6840	4.6840	4.6840	4.6840	4.6826	4.6826	4.6782	4.6782
	λ_2	7.7796	7.7796	7.7792	7.7792	7.7697	7.7697	7.7399	7.7399
$\alpha_1=\alpha_2=0.01$	λ_3	10.9328	10.9328	10.9310	10.9310	10.8880	10.8879	10.7556	10.7556
	λ_4	13.8857	13.8858	13.8851	13.8849	13.8706	13.8705	13.8239	13.8238
	λ_5	17.0445	17.0431	17.0409	17.0398	16.9556	16.9560	16.6877	16.6898
	λ_1	4.3491	4.3491	4.3487	4.3487	4.3392	4.3392	4.3099	4.3099
	λ_2	7.2352	7.2352	7.2325	7.2325	7.1689	7.1689	6.9764	6.9764
$\alpha_1=\alpha_2=0.1$	λ_3	10.3944	10.3944	10.3819	10.3818	10.0848	10.0847	9.2616	9.2616
	λ_4	12.3091	12.3089	12.3056	12.3056	12.2171	12.2171	11.8915	11.8914
	λ_5	15.7406	15.7385	15.7063	15.7050	14.9448	14.9462	13.3483	13.3483
	λ_1	3.5945	3.5945	3.5937	3.5937	3.5757	3.5757	3.5210	3.5210
	λ_2	5.9801	5.9801	5.9753	5.9753	5.8617	5.8617	5.5285	5.5285
$\alpha_1=\alpha_2=0.5$	λ_3	8.7765	8.7765	8.7569	8.7569	8.2499	8.2499	6.9649	6.9649
	λ_4	9.6195	9.6195	9.6142	9.6142	9.4698	9.4698	8.9339	8.9339
	λ_5	14.3751	14.3785	14.1862	14.1859	11.5622	11.5622	9.5382	9.5382
	λ_1	3.1633	3.1633	3.1625	3.1625	3.1435	3.1435	3.0865	3.0865
	λ_2	5.2591	5.2591	5.2542	5.2542	5.1380	5.1380	4.7995	4.7995
$\alpha_1=\alpha_2=1$	λ_3	7.7375	7.7375	7.7194	7.7194	7.2232	7.2232	5.9649	5.9649
	λ_4	8.3269	8.3269	8.3217	8.3217	8.1776	8.1776	7.6522	7.6522
	λ_5	14.0575	14.0228	13.6736	13.6799	9.9683	9.9683	8.0941	8.0941
	λ_1	2.7309	2.7309	2.7301	2.7301	2.7119	2.7119	2.6577	2.6577
	λ_2	4.5370	4.5370	4.5325	4.5325	4.4235	4.4235	4.1072	4.1072
$\alpha_1=\alpha_2=2$	λ_3	6.6774	6.6774	6.6616	6.6616	6.2123	6.2123	5.0638	5.0638
	λ_4	7.1145	7.1145	7.1097	7.1097	6.9766	6.9766	6.4975	6.4975
	λ_5	13.8827	13.8724	13.1317	13.0827	8.4936	8.4936	6.8383	6.8383

Maximum error=0.3745 %

Table 8. First five frequency parameters of a beam with a single mass for different values of mass and rotary inertia and various boundary conditions.

$\zeta = 0.5$		$c_1 = 0.05$			$\alpha = 0.1$		
		$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 1$	$c = 0$	$c = 0.01$	$c = 0.1$
Pinned-Pinned (PP)	λ_1	3.1261	3.0013	2.3832	3.0013	3.0013	3.0013
	λ_2	6.2801	6.2522	5.9773	6.2832	6.2819	6.1592
	λ_3	9.3790	9.0595	8.2394	9.0595	9.0595	9.0595
	λ_4	12.5414	12.3043	10.2964	12.5664	12.5564	11.4751
	λ_5	15.6329	15.1708	26.7079	15.1708	15.1710	15.1714
Clamped-Clamped (CC)	λ_1	4.7007	4.4698	3.4378	4.4698	4.4698	4.4698
	λ_2	7.8468	7.7888	7.2123	7.8532	7.8506	7.5927
	λ_3	10.9430	10.5888	9.7855	10.5888	10.5888	10.5888
	λ_4	14.1016	13.7546	11.2575	14.1371	14.1230	12.5819
	λ_5	17.1960	16.7049	25.309	16.7049	16.7048	25.1510
Clamped-Free (CF)	λ_1	1.8729	1.8534	1.6966	1.8540	1.8540	1.8516
	λ_2	4.6705	4.4886	3.7652	4.4889	4.4888	4.4876
	λ_3	7.8487	7.7936	7.2490	7.8545	7.8521	7.6085
	λ_4	10.9423	10.5830	9.7611	10.5830	10.5830	10.5830
	λ_5	14.1014	13.7536	11.2449	14.1369	14.1229	12.5770
Clamped- Pinned (CP)	λ_1	3.9063	3.7433	2.9481	3.7437	3.7437	3.7419
	λ_2	7.0590	6.9817	6.5143	7.0203	7.0188	6.8638
	λ_3	10.1663	9.8616	8.9826	9.8806	9.8799	9.7927
	λ_4	13.3167	13.0119	10.9130	13.2796	13.2697	12.1355
	λ_5	16.4169	15.9233	14.5701	16.0175	16.0158	15.4584

Table 9. First five frequency parameters of a beam with three similar masses for different values of mass and rotary inertia and various boundary conditions.

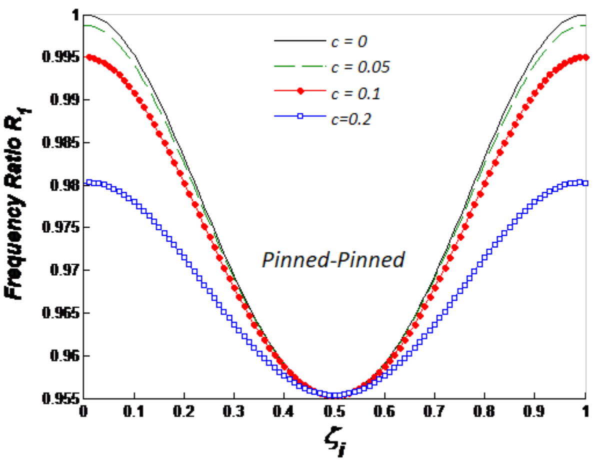
$\zeta = [0.3\ 0.5\ 0.7]$		$c_1 = 0.05$			$\alpha = 0.1$		
		$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 1$	$c = 0$	$c = 0.01$	$c = 0.1$
Pinned-Pinned (PP)	λ_1	3.1061	2.8554	2.0371	2.8570	2.8570	2.8503
	λ_2	6.2240	5.7877	4.2114	5.8127	5.8117	5.7142
	λ_3	9.3511	8.7860	6.4884	8.9184	8.9131	8.4087
	λ_4	12.4689	11.7865	8.9510	12.2595	12.2415	10.4809
	λ_5	15.4841	14.1476	10.4922	14.1868	14.1856	13.9119
Clamped-Clamped (CC)	λ_1	4.6724	4.2723	3.0185	4.2793	4.2790	4.2514
	λ_2	7.7604	7.1074	5.0243	7.1476	7.1460	6.9858
	λ_3	10.8790	10.0126	7.0599	10.1487	10.1434	9.5924
	λ_4	14.0256	13.1478	9.4023	14.0479	14.0117	11.2212
	λ_5	17.0374	15.5790	10.8397	15.8865	15.8795	14.3061
Clamped-Free (CF)	λ_1	1.8660	1.7923	1.4240	1.7937	1.7937	1.7881
	λ_2	4.6520	4.3642	3.4349	4.3782	4.3776	4.3230
	λ_3	7.7723	7.1997	5.5048	7.2389	7.2374	7.0793
	λ_4	10.8831	10.0519	7.3333	10.1763	10.1714	9.6686
	λ_5	14.0260	13.1575	9.4998	14.0411	14.0056	11.2667

Clamped- Pinned (CP)	λ_1	3.8815	3.5640	2.5368	3.5676	3.5675	3.5532
	λ_2	6.9946	6.4607	4.6390	6.4947	6.4933	6.3597
	λ_3	10.1152	9.4070	6.7914	9.5467	9.5412	8.9921
	λ_4	13.2378	12.4131	9.1712	12.9320	12.9132	10.9238
	λ_5	16.2688	14.8898	10.6770	15.2756	15.2580	14.0919

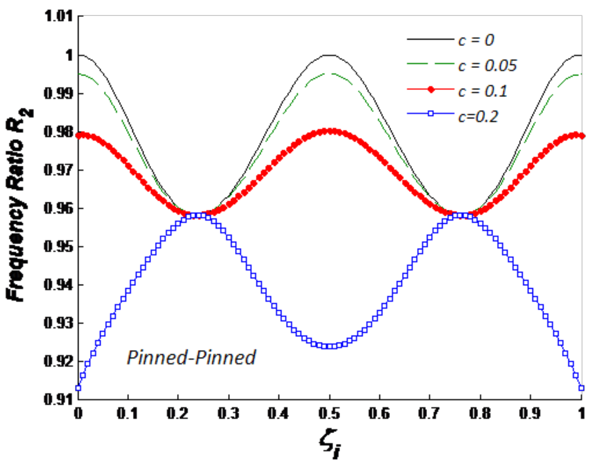
Table 10. First five frequency parameters of a beam with five similar masses for different values of mass and rotary inertia and various boundary conditions.

ζ = [0.1 0.3 0.5 0.7 0.9]		$c_1 = 0.05$			$\alpha = 0.1$		
		$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 1$	$c = 0$	$c = 0.01$	$c = 0.1$
Pinned-Pinned (PP)	λ_1	3.1026	2.8329	1.9970	2.8387	2.8385	2.8158
	λ_2	6.1997	5.6311	3.9337	5.6767	5.6749	5.5026
	λ_3	9.2859	8.3606	5.7543	8.5070	8.5010	7.9544
	λ_4	12.3556	10.9841	7.4042	11.2768	11.2659	10.0416
	λ_5	15.3368	13.1965	8.5992	13.1966	13.1966	13.1965
Clamped-Clamped (CC)	λ_1	4.6710	4.2631	3.0027	4.2742	4.2738	4.2306
	λ_2	7.7476	7.0295	4.8994	7.0996	7.0968	6.8305
	λ_3	10.8329	9.7440	6.6855	9.9651	9.9562	9.1222
	λ_4	13.9210	12.4538	8.4240	13.2043	13.1740	10.7300
	λ_5	16.8666	14.5053	9.4358	14.7316	14.7259	13.8300
Clamped-Free (CF)	λ_1	1.8525	1.6953	1.1998	1.6969	1.6969	1.6904
	λ_2	4.6362	4.2354	2.9880	4.2644	4.2633	4.1533
	λ_3	7.7516	7.0460	4.9251	7.1628	7.1580	6.7346
	λ_4	10.8377	9.7727	6.7276	10.0812	10.0686	8.9762
	λ_5	13.9295	12.5091	8.5011	13.4698	13.4298	10.6298
Clamped- Pinned (CP)	λ_1	3.8777	3.5396	2.4938	3.5481	3.5477	3.5146
	λ_2	6.9740	6.3307	4.4172	6.3873	6.3850	6.1708
	λ_3	10.0589	9.0502	6.2182	9.2268	9.2197	8.5555
	λ_4	13.1315	11.6794	7.8582	12.0500	12.0370	10.4477
	λ_5	16.1171	13.9225	9.1077	14.2363	14.2233	13.4142

In order to study the position of the mass on the frequency parameters, a beam with a single concentrated mass (α=0.1) and variable values of rotary inertia are employed as c=[0 0.05 0.1 0.2]. The first two frequency ratios vs. the variations of the position of the mass are depicted in Figure 3 for various boundary conditions. The frequency ratio (R_n)



(a)



(b)

is considered as the ratio of the frequency parameter to the corresponding one for a bare beam. As shown in Figure 3, when the value of rotary inertia increases, magnitudes of frequency parameters will decrease.

Figure 3, also shows that in each mode of any boundary conditions, there are some points that when mass is located on them, the reduction of frequency parameter is zero when the rotary inertia in neglected. In other words, when mass

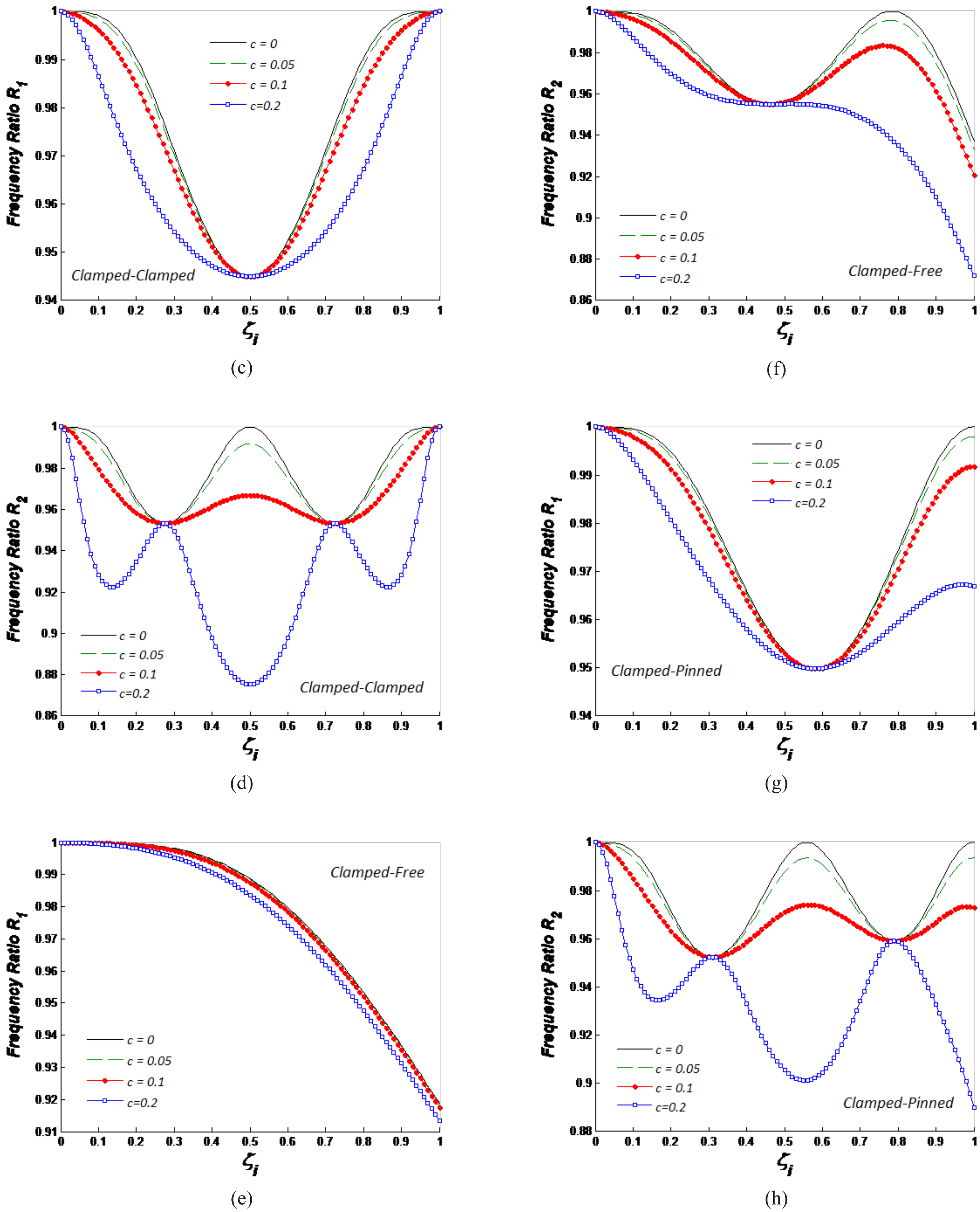


Fig. 3. The first two frequency ratio for a beam with a single attached mass (α=0.1) vs. position of the mass for variable values of rotary inertia and various boundary conditions.

is located at these points, all decreases in corresponding frequency parameter is influenced by the rotary inertia and translational inertia has no effect on the corresponding frequency parameter. These points are nodes in the corresponding mode, i.e. the center point for even modes of symmetric beams. Moreover, there are some points that when the mass is located on them, the reduction of frequency parameters is independent of the rotary inertia. In other words,

when the mass is located at these points, all decreases in corresponding frequency parameter are affected by translational inertia and rotary inertia has no effect on the corresponding frequency parameter. These points are antinodes of the corresponding mode, i.e. the center point for odd modes of symmetric beams. The quantity of nodes and antinodes increases at higher modes.

Figure 4.a, shows the first five mode shapes of the pinned-pinned beam with three similar attached masses (α=0.1 & c=0.05) at positions: ζ_1=0.25, ζ_2=0.5, and ζ_3=0.75. Additionally, Figure 4 (b) represents the first five mode shapes of the clamped-free one with similar attachments.

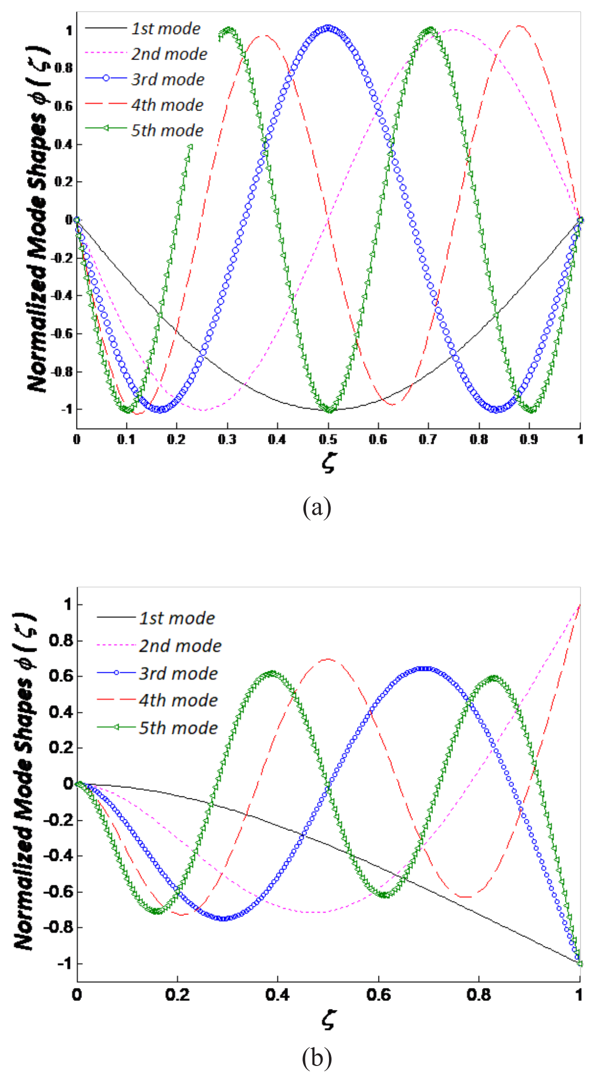


Fig. 4. First five mode shapes of (a) pinned-pinned and (b) cantilever beams with three similar attached masses (α=0.1 & c=0.05) at positions: ζ_1 =0.25, ζ_2=0.5, and ζ_3=0.75.

5- Conclusions

Vibration analysis of uniform Bernoulli-Euler beams carrying multiple concentrated masses and considering their rotary inertia was investigated for all standard boundary conditions. For all boundary conditions, the fourth order partial differential equation was transformed to a quadratic eigenvalue problem. Some typical results calculated by the presented model confirmed an excellent coincidence with the presented results of the other authors. The influence of the mass parameter, the rotary inertia parameter, quantity, and location of mass on the frequency parameters of the beam was studied for various boundary conditions. Based on the results discussed earlier, several conclusions can be addressed as follows:

(1) In general, for a beam with concentrated masses and their rotary inertia, the value of frequency parameters are less than corresponding ones of a bare beam. Therefore, it can be obviously concluded that the increase in the number of concentrated masses always causes more decrease in frequency parameters.

(2) Generally, when the effect of attached masses on vibrating beams is studied, only the translational inertia of the mass is considered. In those cases, it is generally observed that the frequency parameters decrease with respect to the values of the mass, except for the cases in which the masses are located at nodal points of the corresponding normal mode.

(3) When the model takes into account the rotary inertia of the mass too, all frequency parameters decrease.

(4) The translational inertia has its highest influence over a natural frequency when the mass is located at an antinode of the corresponding normal mode. In this situation, the rotary inertia has no effect.

(5) The rotary inertia has the highest influence on a natural frequency when the mass is located at a node of the normal mode. In this case, the translational inertia does not have any effect.

(6) Effect of the mass and rotary inertia on the mode shapes of a beam respectively appear as a reduction in the amplitude and slope in the mass position.

Appendix A

This appendix presents a procedure to determine the functions $d_1(\zeta)$, $d_2(\zeta)$, $d_3(\zeta)$ and $d_4(\zeta)$ which appear in Eq. (19). This equation is given here for convenience:

$$\phi(\zeta)=d_1(\zeta)\sin(\lambda\zeta)+d_2(\zeta)\cos(\lambda\zeta) +d_3(\zeta)\sinh(\lambda\zeta)+d_4(\zeta)\cosh(\lambda\zeta). \tag{A-1}$$

Differentiation of Eq. (A.1) with respect to the ζ leads to

$$\begin{aligned}\phi'(\zeta)&=d_1(\zeta)\lambda\cos(\lambda\zeta)-d_2(\zeta)\lambda\sin(\lambda\zeta)\\ &+d_3(\zeta)\lambda\cosh(\lambda\zeta)+d_4(\zeta)\lambda\sinh(\lambda\zeta)+\\ &d_1'(\zeta)\sin(\lambda\zeta)+d_2'(\zeta)\cos(\lambda\zeta)\\ &+d_3'(\zeta)\sinh(\lambda\zeta)+d_4'(\zeta)\cosh(\lambda\zeta).\end{aligned} \tag{A-2}$$

By imposing the following condition:

$$d_1'(\zeta)\sin(\lambda\zeta)+d_2'(\zeta)\cos(\lambda\zeta)+ d_3'(\zeta)\sinh(\lambda\zeta)+d_4'(\zeta)\cosh(\lambda\zeta)=0, \tag{A-3}$$

the following relation can be obtained:

$$\begin{aligned}\phi''(\zeta) &= -d_1(\zeta)\lambda^2\sin(\lambda\zeta) - d_2(\zeta)\lambda^2\cos(\lambda\zeta) \\ &+ d_3(\zeta)\lambda^2\sinh(\lambda\zeta) + d_4(\zeta)\lambda^2\cosh(\lambda\zeta) \\ &+ d_1'(\zeta)\lambda\cos(\lambda\zeta) - d_2'(\zeta)\lambda\sin(\lambda\zeta) + \\ & d_3'(\zeta)\lambda\cosh(\lambda\zeta) + d_4'(\zeta)\lambda sinh(\lambda\zeta).\end{aligned} \tag{A-4}$$

Furthermore, by imposing the next condition as

$$\begin{aligned}& d_1'(\zeta)\lambda\cos(\lambda\zeta) - d_2'(\zeta)\lambda\sin(\lambda\zeta) \\ &+ d_3'(\zeta)\lambda\cosh(\lambda\zeta) + d_4'(\zeta)\lambda sinh(\lambda\zeta) = 0,\end{aligned} \tag{A-5}$$

next equation can be written as follows:

$$\begin{aligned}\phi'''(\zeta) &= -d_1(\zeta)\lambda^3\cos(\lambda\zeta) + d_2(\zeta)\lambda^3\sin(\lambda\zeta) \\ &+ d_3(\zeta)\lambda^3\cosh(\lambda\zeta) + d_4(\zeta)\lambda^3\sinh(\lambda\zeta) \\ &- d_1'(\zeta)\lambda^2\sin(\lambda\zeta) - d_2'(\zeta)\lambda^2\cos(\lambda\zeta) \\ &+ d_3'(\zeta)\lambda^2\sinh(\lambda\zeta) + d_4'(\zeta)\lambda^2\cosh(\lambda\zeta).\end{aligned} \tag{A-6}$$

and finally by imposing the next condition as

$$\begin{aligned}& -d_1'(\zeta)\lambda^2\sin(\lambda\zeta) - d_2'(\zeta)\lambda^2\cos(\lambda\zeta) \\ &+ d_3'(\zeta)\lambda^2\sinh(\lambda\zeta) + d_4'(\zeta)\lambda^2\cosh(\lambda\zeta) = 0,\end{aligned} \tag{A-7}$$

one can write

$$\begin{aligned}\phi^{iv}(\zeta) &= d_1(\zeta)\lambda^4\sin(\lambda\zeta) + d_2(\zeta)\lambda^4\cos(\lambda\zeta) \\ &+ d_3(\zeta)\lambda^4\sinh(\lambda\zeta) + d_4(\zeta)\lambda^4\cosh(\lambda\zeta) \\ &- d_1'(\zeta)\lambda^3\cos(\lambda\zeta) + d_2'(\zeta)\lambda^3\sin(\lambda\zeta) \\ &+ d_3'(\zeta)\lambda^3\cosh(\lambda\zeta) + d_4'(\zeta)\lambda^3 sinh(\lambda\zeta).\end{aligned} \tag{A-8}$$

Inserting Eq. (A8) in governing equation (18), leads to

$$\begin{aligned}& -d_1'(\zeta)\lambda^3\cos(\lambda\zeta) + d_2'(\zeta)\lambda^3\sin(\lambda\zeta) \\ &+ d_3'(\zeta)\lambda^3\cosh(\lambda\zeta) + d_4'(\zeta)\lambda^3\sinh(\lambda\zeta) = A(\zeta).\end{aligned} \tag{A-9}$$

Incorporating the assumed conditions of Eqs. (A.3), (A.5), (A.7) and (A.9), the generalized functions $d_1'(\zeta)$, $d_2'(\zeta)$, $d_3'(\zeta)$, and $d_4'(\zeta)$ can be achieved by integrating the following system of four differential equations:

$$\begin{pmatrix} \sin(\lambda\zeta) & \cos(\lambda\zeta) & \sinh(\lambda\zeta) & \cosh(\lambda\zeta) \\ \cos(\lambda\zeta) & -\sin(\lambda\zeta) & \cosh(\lambda\zeta) & \sinh(\lambda\zeta) \\ -\sin(\lambda\zeta) & -\cos(\lambda\zeta) & \sinh(\lambda\zeta) & \cosh(\lambda\zeta) \\ -\cos(\lambda\zeta) & \sin(\lambda\zeta) & \cosh(\lambda\zeta) & \sinh(\lambda\zeta) \end{pmatrix} \times \begin{pmatrix} d_1'(\zeta) \\ d_2'(\zeta) \\ d_3'(\zeta) \\ d_4'(\zeta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{A(\zeta)}{\lambda^3} \end{pmatrix}. \tag{A-10}$$

The system of differential equation matrix, Eq. (A.10), can be written under the following uncoupled form:

$$\begin{aligned} d_1'(\zeta) &= -\frac{\cos(\lambda\zeta)}{2\lambda^3}A(\zeta) \\ d_2'(\zeta) &= \frac{\sin(\lambda\zeta)}{2\lambda^3}A(\zeta) \\ d_3'(\zeta) &= \frac{\cosh(\lambda\zeta)}{2\lambda^3}A(\zeta) \\ d_4'(\zeta) &= -\frac{\sinh(\lambda\zeta)}{2\lambda^3}A(\zeta). \end{aligned} \tag{A-11}$$

Inserting $A(\zeta)$ from Eq. (17) into Eqs. (A.11) and integration of the obtained equations lead to

$$\begin{aligned} d_1(\zeta) &= \\ & -\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\cos(\lambda\zeta_i)\phi(\zeta_i) \\ -\beta_i\lambda\sin(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\} + c_1 \\ d_2(\zeta) &= \\ & -\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\sin(\lambda\zeta_i)\phi(\zeta_i) \\ +\beta_i\lambda\cos(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\} + c_2 \\ d_3(\zeta) &= \\ & -\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\cosh(\lambda\zeta_i)\phi(\zeta_i) \\ +\beta_i\lambda\sinh(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\} + c_3 \\ d_4(\zeta) &= \\ & -\frac{\lambda}{2}\sum_{i=1}^{N}\left\{\begin{bmatrix}\alpha_i\sinh(\lambda\zeta_i)\phi(\zeta_i) \\ +\beta_i\lambda\cosh(\lambda\zeta_i)\phi'(\zeta_i)\end{bmatrix}u(\zeta-\zeta_i)\right\} + c_4 \end{aligned} \tag{A-12}$$

where c_1, c_2, c_3, c_4 are the integration constants. Inserting Eqs. (A.12) into Eq. (A.1) provides a suitable form of the Eigen-mode to be used to obtain the explicit closed-form solution of the problem of interest.

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