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Nonlinear Free Transverse Vibration Analysis of Beams Using Variational Iteration Method

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ABSTRACT: In this study, Variational Iteration Method is employed so as to investigate the linear and non-linear transverse vibration of Euler-Bernoulli beams. This method is a very powerful approach with a high convergence speed providing an analytical and semi-analytical solution to the linear equations and is able to be extended to present semi-analytical solution to the non-linear ones. In this method, firstly, Lagrange`s multiplier and Initial Function should be chosen. The suitable choice of these two elements would effectively affect the convergence speed. In this attempt, in addition to presenting a discussion on how to choose these two functions appropriately, the calculated frequencies in the non-linear state are compared with the available results in the literature, and the accuracy and convergence speed are studied, as well.

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1- Introduction

The problem of linear and nonlinear vibration of a beam with a clamped-clamped boundary condition at both ends has been considered in a huge number of studies. In these studies, for linear problems, different numerical and analytical methods have been employed. In nonlinear ones, however, due to the larger amplitudes, researchers tend to apply numerical approaches to obtain the frequencies and mode shapes. The purpose of this study is to present Variational Iteration Method (VIM) which is capable of predicting the free vibration behavior of the beam with a larger amplitude as well as giving semi-analytical response for nonlinear ones. VIM is a powerful method which has both high accuracy in calculations and high convergence speed. It presents analytical and semi-analytical solutions for linear and nonlinear equations, respectively. J. H. He offered Such a method for the first time, in 1999 [1- 5], and afterwards, it was used for various equations, including Fokker-Planck equation [6], quadratic Riccati differential equations [7], nonlinear heat transfer equations [8,9], ordinary nonlinear differential equations [10], fourth-order parabolic equations [11], wave equations [12], and singular fourth-order parabolic partial differential equations [13].

Over the last few years, this method has become a very powerful means to solve complicated equations. Y. Liu and C.S. Gurram [14], investigated the free vibration of Euler-Bernoulli beams and calculated the natural frequencies for different boundary conditions using VIM. They also demonstrated the high convergence speed for this method. He [15] presented variational approach to limit the cycles of nonlinear oscillators. He [16] also suggested Hamiltonian approach to obtain approximate frequency–amplitude relationship of a nonlinear oscillator with acceptable accuracy. The nonlinear free vibration of an oscillator consisting mass and spring was studied by M. Baghani, et al. [17], and the results were compared by those obtained through Homotopy analysis method, which showed a perfect accuracy of the results. In an another attempt, Y. Huang and H. Liu [18], by combining VIM and Homotopy methods, presented a modified VIM which then was used to solve Van Der Pol equations. S. Siddiqi, M. Iftikhar [19], conducted another research and tried to solve seventh order boundary value problems by using He's polynomials and VIM. Moreover, they presented some examples to express the method and to put emphasis on its high convergence speed. Using a nonlinear example, H. Jafari [20] made a comparison between Successive Approximate Method and VIM and concluded that the results have a complete agreement with each other. Later, A. Al-Sawoor and M. Al-Amr [21] carried out a study to solve the reaction-diffusion system through presenting the modified VIM, and reached to the point that the proposed method has a higher convergence speed compared to that of VIM method and would consequently be more suitable for the fast reversible reaction. Wave-like and heat-like equations in large domains were then studied by H. Ghaneai and M. Hosseini [22]. Their approach was to combine VIM with one auxiliary parameter. Y. Chen et al., [23] looked into natural frequencies of marine risers by variational iteration method. M. Daeichi and M.T. Ahmadian [24] converted the nonlinear partial differential equation of motion to a set of coupled ordinary differential equations using the Galerkin technique and analyzed the nonlinear vibration of transversely vibrated beams with large slenderness and immovable ends using the Variational

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Iteration Method. O. Martin [25] used Galerkin's method and a modified form of the variational iteration method (VIM) for the analysis of the isotropic linear viscoelastic beams. S.A. Khuri and A. Sayfy [26] extended the Variational Iteration Method (VIM) for the proper treatment of boundary value problems (BVPs). VIM is presented to solve the second-order singular initial value problems of the Lane–Emden type by A. Ghorbani and M. Bakherad [27].

It should be noted that there is also another method named Adomian which is very similar to VIM. At first, since there were some problems to determine Lagrange`s multiplier, Adomian was more popular. After resolving the available problems, however, VIM, which had a higher convergence speed than Adomian, gained more interests. Those researchers who have worked on both methods have also confessed to this fact [14].

The purpose of this study is to express VIM as a new method which is capable of presenting accurate responses with a high convergence speed for both linear and nonlinear equations. VIM has been used in a wide variety of articles. The importance of choosing Lagrange`s multiplier and Initial Function has always been a controversial issue since the proper choice of them can tremendously affect the accuracy and convergence. Improper choice of them, however, can lead to the increase of calculation and decrease of accuracy, which could be considered as a negative point. In previous articles, Lagrange`s multiplier and Initial Function were calculated separately. The simplest function which could satisfy the boundary conditions would be considered as Initial Function. In this article, there is a relationship between these two functions in a way that firstly one of them is chosen and then the next function will be calculated. For this purpose, to solve the linear problem, Laplace transformation is applied. This could be useful not only in determining Initial Function but also in simplifying the governing equations, which in turn can lead to the decrease of calculation remarkably. In the nonlinear state, using linear solution, the relationship between Lagrange`s multiplier and Initial Function is gained. To this end, first this method is described, and then analytical and semi-analytical solutions for the linear vibration of beams under different boundary conditions are presented. Next, the nonlinear vibration of beams is investigated as well as the accuracy and the convergence speed, and finally, the results are compared with those achieved through DQM and FEM.

2- Basic Idea of Variational Iteration Method (VIM)

VIM, which has recently received a great deal of attention in a broad range of engineering fields is a very powerful method for solving a wide variety of linear and nonlinear equations. In this study, this method has been employed so as to calculate the natural frequencies of linear and nonlinear Euler-Bernoulli beam.

In order to describe the fundamental idea of it, consider the following nonlinear equation [28-29].

$$Lu(\zeta)+Nu(\zeta)=g(\zeta), \tag{1}$$

In which L is a linear operator, N is a nonlinear operator and $g(\zeta)$ is a definite function. In this method, the correctional function is defined as below [28-29]:

$$u_{n+1}(\zeta)=u_n(\zeta)+\int_0^{\zeta}\mu\,(\mathrm{L}u_n(\eta)+\mathrm{N}\tilde{u}_n(\eta)-g(\eta))\,d\eta, \tag{2}$$

In which μ is the Lagrange`s multiplier which should be determined by variation theory and $\tilde{u}_k$ is considered as the limited variation. In other words, $\delta\tilde{u}_k=0$. $u_0(\zeta)$ is also the possible Initial Function for the unknown parameters.

The important issue in this method is to determine the proper Initial Function and Lagrange`s multiplier in order to start the trend for solving the problem because the suitable choice of these two elements would effectively affect the convergence speed. It is necessary to explain clearly how to determine the most optimal coefficients. Determining coefficients is a part of the novelty of the present work that will be shown in the following.

3- Vibration Analysis for Linear Euler-Bernoulli Beam

3- 1- Governing Equations

Considering a beam with a uniform cross-section and defining non-dimensional parameters (4), the governing differential equation for the free linear vibration of Euler-Bernoulli will be as (3) [30]:

$$\frac{d^4W}{d\zeta^4}-\lambda^4W=0, \tag{3}$$

$$\zeta=\frac{x}{L}\ ,\ W=\frac{Y}{L}\ ,\ \lambda^4=\frac{\rho SL^4\omega^2}{EI}, \tag{4}$$

Where λ is the non-dimensional natural frequency, ρ is the density, S is the transverse cross section, E is Young's module, I is the moment inertia of the beam cross section, L is the length of the beam, and W is the transverse displacement of the beam.

3- 2- Solving the Linear Equation with VIM

Comparing equations (1) and (3) and using equation (2), the correctional function will be stated as follows [14]:

$$W_{n+1}(\zeta)=W_n(\zeta)+\int_0^{\zeta}\mu\left(\frac{d^4W_n(\eta)}{d\eta^4}-\lambda^4W_n(\eta)\right)d\eta. \tag{5}$$

In order to solve the equation by VIM, it is needed to determine the Lagrange`s multiplier μ and the Initial Function $W_0(\zeta)$ whose calculation trend is seen in the following.

Using integration by parts, equation (5) could be rewritten as below:

$$\begin{aligned}W_{n+1}(\zeta)&=W_n(\zeta)+\left[\mu W_n^{(3)}(\eta)-\mu^{(1)}W_n^{(2)}(\eta)\right.\\&\left.+\mu^{(2)}W_n^{(1)}(\eta)-\mu^{(3)}W_n(\eta)\right]_0^{\zeta}+\int_0^{\zeta}\left[\mu^{(4)}-\lambda^4\mu\right]W_n(\eta)d\eta,\end{aligned} \tag{6}$$

By applying variation to the both sides of equation (6) with respect to W_n the result will be:

$$\begin{aligned}\delta W_{n+1}(\zeta)&=\delta W_n(\zeta)+\mu\big|_{\eta=\zeta}\,\delta W_n^{(3)}(\zeta)\\&-\mu^{(1)}\big|_{\eta=\zeta}\,\delta W_n^{(2)}(\zeta)+\mu^{(2)}\big|_{\eta=\zeta}\,\delta W_n^{(1)}(\zeta)\\&-\mu^{(3)}\big|_{\eta=\zeta}\,\delta W_n(\zeta)+\int_0^{\zeta}\left[\mu^{(4)}-\lambda^4\mu\right]\delta W_n(\eta)d\eta.\end{aligned} \tag{7}$$

If $\delta W_{n+1}(\zeta)=0$, $W_n(\zeta)$ will be minimum or maximum. In other words, the left side of the equation (7) should be zero. In this situation, the stationary conditions obtained from relationship (7) will be as follows [33]:

$$\mu\big|_{\eta=\zeta}=0, \tag{8-a}$$

$$\mu^{(1)}\big|_{\eta=\zeta}=0, \tag{8-b}$$

$$\mu^{(2)}\big|_{\eta=\zeta}=0, \tag{8-c}$$

$$1-\mu^{(3)}\big|_{\eta=\zeta}=0, \tag{8-d}$$

$$\mu^{(4)}-\lambda^4\mu=0. \tag{8-e}$$

Lagrange`s multiplier could be gained through the five above-mentioned equations. If the obtained Lagrange`s multiplier satisfies equations (8-a) to (8-e), an analytical solution would be possible for the equation and this Lagrange`s multiplier is unique, and if it satisfies the first four ones, yet does not satisfy equation (8-e), thus the response will be semi-analytical and this multiplier is not unique.

3- 2- 1- Semi-analytical response

Considering equations (8-a) to (8-d), the obtained Lagrange`s multiplier will be as follows [14]:

$$\mu=\frac{(\eta-\zeta)^3}{6}. \tag{9}$$

Inserting equation (9) into equation (5), the correctional function will be:

$$W_{n+1}(\zeta)=W_n(\zeta)+\int_0^{\zeta}\left(\frac{(\eta-\zeta)^3}{6}\right)\left(\frac{d^4W_n(\eta)}{d\zeta^4}-\lambda^4W_n(\eta)\right)d\eta. \tag{10}$$

In order to start the calculations from equation (10), the Initial Function ($W_0(\zeta)$) is needed. It could be different functions as long as it satisfies the boundary conditions at the beginning of the beam. It is a normal practice to consider the first few terms of Maclaurin Series as the Initial Function [22,23]. Laplace transform is used to simplify and find the proper Initial Function. Applying Laplace transform to both sides of equation (10), and showing transverse displacement W in Laplace space with $\bar{W}$, it could be concluded:

$$\bar{W}_{n+1}(s)=\bar{W}_n(s)-\left(\frac{1}{s^4}\right)\Big[\left(s^4\bar{W}_n(s)-W(0)s^3-W'(0)s^2-W''(0)s-W'''(0)\right)-\lambda^4\bar{W}_n(s)\Big], \tag{11}$$

Simplifying the above relationship, we have

$$\bar{W}_{n+1}(s)=+\frac{W(0)s^3+W'(0)s^2+W''(0)s+W'''(0)}{s^4}+\frac{\lambda^4}{s^4}\bar{W}_n(s), \tag{12}$$

Now, with applying inverse Laplace, we will have:

$$W_{n+1}(\zeta)=\underbrace{W(0)+W'(0)\zeta+W''(0)\frac{\zeta^2}{2}+W'''(0)\frac{\zeta^3}{6}}_{W_0(\zeta)}-\int_0^{\zeta}\frac{(\eta-\zeta)^3}{6}\lambda^4W_n(\eta)d\eta. \tag{13}$$

Therefore, the correctional function will be in the form of equation (14):

$$W_{n+1}(\zeta)=W_0(\zeta)-\int_0^{\zeta}\frac{(\eta-\zeta)^3}{6}\lambda^4W_n(\eta)d\eta, \tag{14}$$

In which $W_0(\zeta)$ is the Initial Function, and is presented by relationship (15).

$$W_0(\zeta)=W(0)+W'(0)\zeta+W''(0)\frac{\zeta^2}{2}+W'''(0)\frac{\zeta^3}{6}, \tag{15}$$

By defining coefficients (c_1 - c_4), relationship (15) can be rewritten as the following relationship. The coefficients will be subsequently calculated through applying boundary conditions to the both ends of the beam.

$$W_0(\zeta)=c_1+c_2\zeta+\frac{c_3}{2}\zeta^2+\frac{c_4}{6}\zeta^3. \tag{16}$$

Consequently, considering relationship (14), it is concluded:

$$\begin{aligned}
W_1(\zeta)&=W_0(\zeta)-\int_0^{\zeta}\frac{(\eta-\zeta)^3}{6}\lambda^4W_0(\eta)d\eta,\\
W_2(\zeta)&=W_0(\zeta)-\int_0^{\zeta}\frac{(\eta-\zeta)^3}{6}\lambda^4W_1(\eta)d\eta,\\
&\dots\\
W_k(\zeta)&=W_0(\zeta)-\int_0^{\zeta}\frac{(\eta-\zeta)^3}{6}\lambda^4W_{k-1}(\eta)d\eta,
\end{aligned} \tag{17}$$

And the final result will be:

$$W(\zeta)=W_k(\zeta), \tag{18}$$

In the latter relationship, the value of k is chosen based on the required accuracy. The accuracy considered for calculating the non-dimensional frequency λ is defined according to the following relationship

$$\left|\lambda_j^k-\lambda_j^{k-1}\right|<\varepsilon, \tag{19}$$

Where k is the number of correctional function iteration, j is the number of the mode, and ε represents the required accuracy.

3- 2- 2- Analytical response

To obtain the analytical response of the equation (3) through VIM, first, the Lagrange`s multiplier is required. Equations (8-a) to (8-e) should be taken into consideration so as to find the Lagrange`s multiplier, which is presented below:

$$\mu = \frac{1}{2\lambda^3}\left[\sinh \lambda(\eta-\zeta) - \sin \lambda(\eta-\zeta)\right]. \tag{20}$$

Substituting relationship (20) for equation (5), the correctional function will be as follows:

$$W_{n+1}(\zeta) = W_n(\zeta) + \int_0^{\zeta}\left[\left(\frac{1}{2\lambda^3}\left[\sinh \lambda(\eta-\zeta) - \sin \lambda(\eta-\zeta)\right]\right)\left(\frac{d^4 W_n(\eta)}{d\zeta^4} - \lambda^4 W_n(\eta)\right)\right]d\eta. \tag{21}$$

Likewise, Laplace transform is used to simplify and find the proper initial function.

$$\bar{W}_{n+1}(s) = \bar{W}_n(s) - \left(\frac{1}{s^4-\lambda^4}\right)\left[\left(s^4\bar{W}_n(s) - W(0)s^3 - W'(0)s^2 - W''(0)s - W'''(0)\right) - \lambda^4 \bar{W}_n(s)\right], \tag{22}$$

Simplifying the above relationship, we have

$$\bar{W}_{n+1}(s) = \frac{W(0)s^3 + W'(0)s^2 + W''(0)s + W'''(0)}{s^4-\lambda^4}, \tag{23}$$

Now, by applying inverse Laplace, it is concluded.

$$\begin{aligned} W_{n+1}(\zeta) = &\frac{\left(W'(0)\lambda^2 - W'''(0)\right)}{2\lambda^3}\sin(\lambda\zeta) \\ &+ \frac{\left(W(0)\lambda^2 - W''(0)\right)}{2\lambda^3}\cos(\lambda\zeta) \\ &+ \frac{\left(W'(0)\lambda^2 + W'''(0)\right)}{2\lambda^3}\sinh(\lambda\zeta) \\ &+ \frac{\left(W(0)\lambda^2 + W''(0)\right)}{2\lambda^3}\cosh(\lambda\zeta), \end{aligned} \tag{24}$$

By deleting the integral from the correctional equation, there is no need for iteration and the obtained function in relationship (25) would be the analytical response for relationship (3), for which the coefficients will be gained through applying boundary conditions to the both ends of the beam.

$$\begin{aligned} W(\zeta) = &\frac{\left(c_2\lambda^2 - c_4\right)}{2\lambda^3}\sin(\lambda\zeta) + \frac{\left(c_1\lambda^2 - c_3\right)}{2\lambda^3}\cos(\lambda\zeta) \\ &+ \frac{\left(c_2\lambda^2 + c_4\right)}{2\lambda^3}\sinh(\lambda\zeta) + \frac{\left(c_1\lambda^2 + c_3\right)}{2\lambda^3}\cosh(\lambda\zeta). \end{aligned} \tag{25}$$

The obtained function ($W(\zeta)$) from relationship (18) and (25) is a function of variables ζ, λ, c_1, c_2, c_3 and c_4. Therefore, it could be presented in the following form:

$$W(\zeta) = \sum_{i=1}^{4} f_i(\zeta,\lambda)c_i, \tag{26}$$

In which f_i is a function of ζ and λ.

Now, considering the four boundary conditions at both ends of the beam, the natural frequencies λ could be found. In the following, the way of calculating the frequencies for clamped-clamped boundary condition is presented. The boundary condition for such a beam is presented below.

$$W(0) = \frac{dW}{d\zeta}(0) = 0, \tag{27}$$

$$W(1) = \frac{dW}{d\zeta}(1) = 0, \tag{28}$$

Using relationship (16) or (25) and the relationship (27), one can obtain:

$$\begin{cases} W(0) = c_1 + c_2(0) + c_3(0) + c_4(0) = 0 \Rightarrow c_1 = 0, \\ \dfrac{dW(0)}{d\zeta} = c_1(0) + c_2 + c_3(0) + c_4(0) = 0 \Rightarrow c_2 = 0. \end{cases} \tag{29}$$

Relationship (26) will consequently be simplified in the following form:

$$W(\zeta) = f_3(\zeta,\lambda)\,c_3 + f_4(\zeta,\lambda)\,c_4. \tag{30}$$

Now, considering the two mentioned conditions in (28) and utilizing (30), it is concluded:

$$\begin{aligned} &\begin{cases} W(1) = f_3(1,\lambda)c_3 + f_4(1,\lambda)c_4 = 0 \\ \dfrac{dW(1)}{d\zeta} = f_3'(1,\lambda)\big|_{\zeta=1}c_3 + f_4'(1,\lambda)\big|_{\zeta=1}c_4 = 0 \end{cases} \\ &\Rightarrow \begin{bmatrix} f_3(1,\lambda) & f_4(1,\lambda) \\ f_3'(1,\lambda)\big|_{\zeta=1} & f_4'(1,\lambda)\big|_{\zeta=1} \end{bmatrix}\begin{Bmatrix} c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \end{aligned} \tag{31}$$

Hence, equaling the obtained matrix determinant zero in relationship (31), the characteristic equation will be gained by:

$$\begin{vmatrix} f_3(1,\lambda) & f_4(1,\lambda) \\ f_3'(1,\lambda)\big|_{\zeta=1} & f_4'(1,\lambda)\big|_{\zeta=1} \end{vmatrix} = 0. \tag{32}$$

For other boundary conditions, the same procedure will be suitable.

4- Vibration Analysis for Nonlinear Euler-Bernoulli beam

4- 1- Governing equations

Consider a uniform beam made by homogenous isotropic material without damping. Figure one shows the geometry of the beam in which u and w represent the axial and transverse displacement of the beam, respectively. Therefore, the strain-displacement relationship and curvature-displacement of the beam can be shown below [30,31]:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \quad , \quad \kappa_x = \frac{\partial^2 w}{\partial x^2}. \tag{33}$$

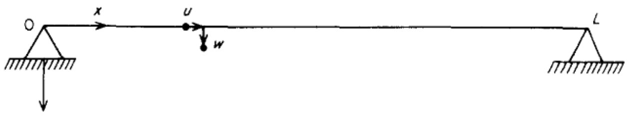

Fig. 1. Geometry of Beam [30]

Considering relationship (33), the axial force could be expressed below [30,31]:

$$N(x,t) = ES\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right], \tag{34}$$

In which E is the Young's module and S is the transverse cross section.
Assuming that the beam does not have any axial displacement at both ends, or in other words, $u(0,t)=u(L,t)=0$, the axial force N is independent of x and only dependent on t. Therefore, it could be written [30-33]:

$$N(x,t) = ES\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]$$
$$\Rightarrow N(t) = \frac{ES}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{35}$$

Considering the above assumption, the Kinetic energy K and strain energy U of the beam will be expressed below [30]:

$$K = \frac{1}{2}\int_0^L \rho S\left(\frac{\partial w}{\partial t}\right)^2 dx, \tag{36}$$

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L \frac{N^2}{ES} dx. \tag{37}$$

Since N is not dependent on x and considering relationship (35), relationship (37) could be rewritten below:

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2}\frac{N^2 L}{ES}$$
$$= \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{ES}{8L}\left(\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right)^2. \tag{38}$$

Considering the Energy Conversation Law, which means that the total energy of kinetic and strain will be constant, we have:

$$K + U = cte. \tag{39}$$

Using equations (36), (38), (39), it could be written:

$$K + U = \frac{1}{2}\int_0^L \rho S\left(\frac{\partial w}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$
$$+ \frac{ES}{8L}\left(\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right)^2 = \text{cte}. \tag{40}$$

Now employing the separation of variables, the displacement would be considered by two functions, one is the function of the time and the other a function of the place which are multiplied

$$w(x,t) = Y(\text{x})\text{T(t)}, \tag{41}$$

Substituting relationship (41) for equation (40), the above relationship will be presented below:

$$K + U = \frac{\rho S}{2}\left(\int_0^L Y^2(x)dx\right)\left(\frac{dT(t)}{dt}\right)^2$$
$$+ \frac{EI}{2}\left(\int_0^L \left(\frac{d^2 Y(x)}{dx^2}\right)^2 dx\right) T^2(\text{t})$$
$$+ \frac{ES}{8L}\left(\int_0^L \left(\frac{\partial Y(x)}{\partial x}\right)^2 dx\right)^2 T^4(t) = cte. \tag{42}$$

In order to make the relationship (42) non-dimensional, these parameters should be considered.

$$\zeta = \frac{x}{L}, \ R = \sqrt{\frac{I}{S}}, W = \frac{Y}{R}, \ t = \tilde{t}\sqrt{\frac{EI}{\rho S L^4}}, \tag{43}$$

Applying the mentioned parameters in relationship (43) to equation (42), it is concluded

$$K + U = \frac{EIR^2}{2L^3}\left(\int_0^1 \mathrm{W}^2(\zeta)d\zeta\right)\dot{T}^2(\tilde{t})$$
$$+ \frac{EIR^2}{2L^3}\left(\int_0^1 \left(\frac{d^2\mathrm{W}(\zeta)}{d\zeta^2}\right)^2 d\zeta\right) T^2(\tilde{\text{t}})$$
$$+ \frac{EIR^2}{8L^3}\left(\int_0^1 \left(\frac{\partial W(\zeta)}{\partial \zeta}\right)^2 d\zeta\right)^2 T^4(\tilde{t}) = \text{cte}. \tag{44}$$

Now equation (44) could be simplified in the form of equation (45)

$$\dot{T}^2(\tilde{t}) + \Lambda_1 T^2(\tilde{\text{t}}) + \Lambda_2 T^4(\tilde{t}) = cte. \tag{45}$$

Parameters Λ_1, Λ_2 in this equation, are expressed below:

$$\Lambda_1 = \frac{\int_0^1 \left(\frac{d^2\mathrm{W}(\zeta)}{d\zeta^2}\right)^2 d\zeta}{\int_0^1 \mathrm{W}^2(\zeta)d\zeta}, \ \Lambda_2 = \frac{\frac{1}{4}\left(\int_0^1 \left(\frac{\partial W(\zeta)}{\partial \zeta}\right)^2 d\zeta\right)^2}{\int_0^1 \mathrm{W}^2(\zeta)d\zeta}. \tag{46}$$

By applying derivatives at the both sides of equation (45) with respect to $\tilde{t}$, it is concluded

$$\ddot{T}(\tilde{t}) + \Lambda_1 T(\tilde{t}) + 2\Lambda_2 T^3(\tilde{t}) = 0. \tag{47}$$

Therefore, equations (47) will represent the governing differential equation for nonlinear Euler-Bernoulli beams [35].
There are two differences in investigating the linear and nonlinear vibration by VIM. The first one is that the governing equation in linear state is investigated according to the position of x, while in that of nonlinear it is written according to the time. The second difference lies in linear equations, the boundary conditions will be applied after VIM whereas in nonlinear ones first the boundary conditions will be applied and then VIM will be employed [14,35].
Considering equation (41), the function $Y(x)$, or the non-dimensional form of it $W(x)$, should have the characteristics of mode shapes. Thus, $W(x)$ should be chosen in a way that satisfies the boundary conditions at both ends of the beam. Then, the numerical value of the coefficients will be gained by relationship (46).
The boundary condition for the nonlinear Euler-Bernoulli beam with simply supported ends will be expressed below:

$$\begin{aligned} W(0) &= \frac{d^2W}{d\zeta^2}(0) = 0, \\ W(1) &= \frac{d^2W}{d\zeta^2}(1) = 0. \end{aligned} \tag{48}$$

There are different functions that can satisfy the above-mentioned boundary conditions in relationship (48). In this study, one of the simplest functions used in equation (49) is employed.

$$W = \sin(\pi\zeta). \tag{49}$$

This function could also be obtained from the linear mode shape mentioned in relationship (25). Now using equation (49) and the relationship (46), the parameters Λ_1, Λ_2 will be gained for a beam with simply supported ends.

$$\Lambda_1 = \pi^4 \quad , \quad \Lambda_2 = \frac{\pi^4}{8}. \tag{50}$$

Similarly, for nonlinear Euler-Bernoulli beams with clamped-clamped boundary conditions, we have

$$\begin{aligned} W(0) &= \frac{dW}{d\zeta}(0) = 0, \\ W(1) &= \frac{dW}{d\zeta}(1) = 0. \end{aligned} \tag{51}$$

The function stated in relationship (52) satisfies the above-mentioned boundary conditions

$$W = \frac{1}{2}\left(1 - \cos(2\pi\zeta)\right), \tag{52}$$

Now using function (52) and relationship (46), the parameters Λ_1, Λ_2 for a clamped-clamped beam will be obtained below

$$\Lambda_1 = \frac{16\pi^4}{3} \quad , \quad \Lambda_2 = \frac{\pi^4}{6}. \tag{53}$$

4- 2- Solving the nonlinear equation using VIM

Comparing relationships (1) and (47) and according to relationship (2), the correctional function will be stated as follows:

$$\begin{aligned} T_{n+1}(\tilde{t}) &= T_n(\tilde{t}) \\ &+ \int_0^{\tilde{t}} \mu \left(\frac{d^2\ddot{T}_n(\eta)}{d\eta^2} + \Lambda_1 T_n(\eta) + 2\Lambda_2 T_n^3(\eta) \right) d\eta. \end{aligned} \tag{54}$$

To solve the equation through VIM, Initial Function, and Lagrange`s multiplier will be necessary for this purpose. The boundary condition will be considered below.

$$T(0) = A \quad , \quad \dot{T}(0) = 0, \tag{55}$$

Therefore, the Initial Function which satisfies relationship (55) will be considered as follows:

$$T_0 = A\cos(\bar{\lambda}\,\tilde{t}). \tag{56}$$

To find Lagrange`s multiplier in nonlinear problems, the linear terms are just considered. Therefore, applying by part integration to the linear terms of the relationship (54), it could be rewritten:

$$\begin{aligned} T_{n+1}(\tilde{t}) &= T_n(\tilde{t}) + \left[\mu T_n^{(1)}(\eta) - \mu^{(1)} T_n(\eta)\right]_0^{\tilde{t}} \\ &+ \int_0^{\tilde{t}} \left[\mu^{(2)} + \Lambda_1\mu\right] T_n(\eta)\, d\eta, \end{aligned} \tag{57}$$

Applying the variation to both sides of the equations (57) with respect to $T_n(\tilde{t})$, it results as follows:

$$\begin{aligned} \delta T_{n+1}(\tilde{t}) &= \delta T_n(\tilde{t}) + \mu\big|_{\eta=\tilde{t}}\, \delta T_n^{(1)}(\tilde{t}) - \mu^{(1)}\big|_{\eta=\tilde{t}}\, \delta T_n(\tilde{t}) \\ &+ \int_0^{\tilde{t}} \left[\mu^{(2)} + \Lambda_1\mu\right] \delta T_n(\eta)\, d\eta, \end{aligned} \tag{58}$$

In order to put $T_n(\tilde{t})$ in minimum or maximum condition, it is needed that $\delta T_{n+1}(\tilde{t})=0$. In other words, the left side of equation (58) should equal zero. Therefore, the obtained stationary conditions from equation (58) will be:

$$\mu\big|_{\eta=\tilde{t}} = 0, \tag{59-a}$$

$$1 - \mu^{(1)}\big|_{\eta=\tilde{t}} = 0, \tag{59-b}$$

$$\mu^{(2)} + \Lambda_1\mu = 0. \tag{59-c}$$

Considering the stated conditions in relationships (59-a) to (59-c), the Lagrange`s multiplier will be [28]:

$$\mu = \frac{1}{\sqrt{\Lambda_1}} \sin\left(\sqrt{\Lambda_1}(\eta - \tilde{t})\right). \tag{60}$$

In order to find the relationship between the Lagrange`s multiplier and Initial Function, the linear terms of equation (47) are considered, it results in the following

$$\ddot{T}(\tilde{t}) + \Lambda_1 T(\tilde{t}) = 0 \Rightarrow \ddot{T}(\tilde{t}) = -\Lambda_1 T(\tilde{t}). \tag{61}$$

Substituting equation (56) for equation (61), we will have:

$$\begin{aligned} &\ddot{T}(\tilde{t}) = -\Lambda_1 A\cos(\bar{\lambda}\tilde{t}) \\ &\xrightarrow{\int} \dot{T}_1(\tilde{t}) = \left[\frac{\Lambda_1 A\sin(\bar{\lambda}\tilde{t})}{\bar{\lambda}}\right]_0^{\tilde{t}} = \frac{\Lambda_1 A\sin(\bar{\lambda}\tilde{t})}{\bar{\lambda}} \\ &\xrightarrow{\int} T_1(\tilde{t}) = \left[\frac{\Lambda_1 A\cos(\bar{\lambda}\tilde{t})}{\bar{\lambda}^2}\right]_0^{\tilde{t}} = \frac{-\Lambda_1 A}{\bar{\lambda}} + \frac{\Lambda_1 A\cos(\bar{\lambda}\tilde{t})}{\bar{\lambda}}, \end{aligned} \tag{62}$$

In order to avoid the creation of secular terms in the next iteration, the first term of $T_1(t)$ is neglected and by putting the coefficients of $\cos(\lambda t)$ in equation (56) and (62) equal to zero, the following relation is found [35]:

$$\Lambda_1 = \bar{\lambda}^2. \tag{63}$$

Putting (63) in (60), Lagrange`s multiplier will be:

$$\mu = \frac{1}{\bar{\lambda}}\sin(\bar{\lambda}(\eta - \tilde{t})). \tag{64}$$

Using relationships (54, 64), the correctional function is:

$$\begin{aligned} &T_{n+1}(\tilde{t}) = T_n(\tilde{t}) + \int_0^{\tilde{t}}\left(\frac{1}{\bar{\lambda}}\sin(\bar{\lambda}(\eta - \tilde{t}))\right) \\ &\left(\frac{d^2\ddot{T}_n(\eta)}{d\eta^2} + \Lambda_1 T_n(\eta) + 2\Lambda_2 T_n^3(\eta)\right)d\eta. \end{aligned} \tag{65}$$

Considering (56) as Initial Function and employing (65), the second function will be gained:

$$\begin{aligned} &T_2(\tilde{t}) = A\cos(\bar{\lambda}\tilde{t}) + \int_0^{\tilde{t}}\left(\frac{1}{\bar{\lambda}}\sin(\bar{\lambda}(\eta - \tilde{t})).\right) \\ &\left(\frac{d^2\ddot{T}_n(\eta)}{d\eta^2} + \Lambda_1 T_n(\eta) + 2\Lambda_2 T_n^3(\eta)\right)d\eta \Rightarrow \\ &T_2(\tilde{t}) = A\cos(\bar{\lambda}\tilde{t}) - \frac{2\Lambda_2 A^3\cos(\bar{\lambda}\tilde{t}) - 2\Lambda_2 A^3\cos(3\bar{\lambda}\tilde{t})}{32\bar{\lambda}^2} \\ &+ \frac{\left(16\bar{\lambda}^2 - 16\Lambda_1 - 24\Lambda_2 A^2\right)A\bar{\lambda}\tilde{t}\sin(\bar{\lambda}\tilde{t})}{32\bar{\lambda}^2}, \end{aligned} \tag{66}$$

Equaling the secular term made in (66) zero, the vibration frequency is calculated:

$$\bar{\lambda} = \sqrt{\Lambda_1 + 1.5\Lambda_2 A^2}. \tag{67}$$

Using the same procedure, considering the stated correctional function in (65), and using $T_2(\tilde{t})$, $T_3(\tilde{t})$ are obtained. This trend should be followed to reach the required accuracy.
It should be noted that in order to find the response with the required accuracy there is no need to iterate a lot and one or two iterations would be enough.

5- Results and Conclusions

In this article, VIM, as a new method, is used in order to investigate the linear and nonlinear vibration of Euler-Bernoulli beams and the accuracy and convergence speed are investigated, as well.
First, in order to investigate the accuracy and convergence speed in linear problems, semi-analytical responses are used. Since there is a total agreement between linear analytical responses and other analytical responses available in other methods, the issue of investigating the accuracy is illogical. Moreover, since in linear analytical solution using VIM the response will be achieved through only one iteration, the issue of investigating convergence speed will be irrational as well. Table 1 illustrates the non-dimensional natural frequency obtained from semi-analytical response for Euler-Bernoulli beams with clamped-clamped boundary conditions for different iterations of the correctional equation.
As can be seen in Table 1, by increasing the number of iterations, the result will converge to real values so that the first non-dimensional natural frequency will have 0.0001 accuracy after the fourth iteration. Also, second, third and fourth frequencies will have the same accuracy after the sixth, eighth and tenth iteration, respectively. Therefore the convergence is highly suitable in this method.

Table 1. The first four frequencies for Euler-Bernoulli beam with clamped-clamped boundary conditions for different iterations of correctional equation

k	λ_1	λ_2	λ_3	λ_4
2	4.7778	6.1931		
3	4.7305	7.3281		
4	4.7300	7.8081		
5	4.7300	7.8525		
6	4.7300	7.8532	11.0856	12.0215
7	4.7300	7.8532	10.9968	13.4014
8	4.7300	7.8532	10.9956	14.0519
9	4.7300	7.8532	10.9956	14.1352
10	4.7300	7.8532	10.9956	14.1371
11	4.7300	7.8532	10.9956	14.1371

Tables 2 and 3 will show the proportion of non-dimensional linear frequency to that of nonlinear one for two different boundary conditions.
Table 2 demonstrates the results of this proportion for different values of maximum vibration amplitude for simply supported boundary condition and the results are compared with the closed form results of DQM and FEM, which shows an acceptable accuracy of the results obtained from VIM. Table 3 presents the same results for the beam with clamped-clamped boundary conditions.
As in nonlinear problem, the optimized Lagrange`s multiplier and Initial Function are used, the maximum extent of convergence is expected. In all nonlinear responses presented here, two iterations are used because the second iteration will bring about the required convergence and its difference with the third will be extremely small.

Table 2. The proportion of nonlinear frequency to linear frequency for simply supported Euler-Bernoulli beam

A	VIM	Exact [36]	DQM[3] [37]	DQ[2] [31]	FEM[1] [38]
0.1	1.0009	1.0009	1.0009	1.0010	1.0009
0.2	1.0037	1.0037	1.0037	1.0043	1.0037
0.4	1.0149	1.0149	1.0149	1.0170	1.0148
0.6	1.0331	1.0332	1.0332	1.0384	1.0339
0.8	1.0580	1.0583	1.0583	1.0673	1.0578
1.0	1.0892	1.0897	1.0897	1.1030	1.0889

Table 3. The proportion of nonlinear frequency to linear frequency for clamped-clamped Euler-Bernoulli beam

A	VIM	GFEM [34]	ASM[4] [39]	DQ [31]	FEM [38]
0.1	1.0003	1.0003	1.0003	1.0003	1.0003
0.2	1.0012	1.0012	1.0012	1.0011	1.0012
0.4	1.0048	1.0048	1.0048	1.0044	1.0048
0.6	1.0107	1.0107	1.0107	1.0100	1.0107
0.8	1.0190	1.0190	1.0190	1.0178	1.0190
1.0	1.0296	1.0295	1.0296	1.0278	1.0295

6- Conclusions

In this article, the function of VIM in analyzing the linear and nonlinear vibration of the beam is investigated, and it is seen that this method has the proper accuracy and convergence. VIM is capable of presenting analytical and semi-analytical solutions for linear problems.

The proper choice of Lagrange`s multiplier and Initial Function will play a key role in this method. Semi-analytical results presented in the linear state show that the obtained values will converge to the accurate values quickly in a way that the first and second frequencies will reach the accuracy of four decimal after four and six iterations, respectively. In order to investigate the ability of this method in nonlinear problems, the nonlinear vibration of Euler-Bernoulli beams is also studied for two different boundary conditions. VIM presents semi-analytical responses to nonlinear problems. In this article, although the Lagrange`s multiplier satisfies all the exploited stationary conditions, since the employed mode shapes are related to the linear equations, the responses will be semi-analytical. Finally, the results are compared with some those of other methods, including DQM and FEM and the proper accuracy is presented, as well.

References

[1] J. He, Variational iteration method for delay differential equations, *Communications in Nonlinear Science and Numerical Simulations*, 2 (1997) 235-236.

[2] J.-H. He, Variational iteration method–a kind of non-linear analytical technique: some examples, *International journal of non-linear mechanics*, 34(4) (1999) 699-708.

[3] J.-H. He, Variational iteration method for autonomous ordinary differential systems, *Applied Mathematics and Computation*, 114(2-3) (2000) 115-123.

[4] J.-H. He, Variational iteration method—some recent results and new interpretations, *Journal of computational and applied mathematics*, 207(1) (2007) 3-17.

[5] S.-Q. Wang, J.-H. He, Variational iteration method for solving integro-differential equations, *Physics Letters A*, 367(3) (2007) 188-191.

[6] M. Dehghan, M. Tatari, The use of He's variational iteration method for solving a Fokker–Planck equation, *Physica Scripta*, 74(3) (2006) 310.

[7] S. Abbasbandy, A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, *Journal of Computational and Applied Mathematics*, 207(1) (2007) 59-63.

[8] M. Miansari, D. Ganji, M. Miansari, Application of He's variational iteration method to nonlinear heat transfer equations, *Physics Letters A*, 372(6) (2008) 779-785.

[9] H. Tari, D. Ganji, H. Babazadeh, The application of He's variational iteration method to nonlinear equations arising in heat transfer, *Physics Letters A*, 363(3) (2007) 213-217.

[10] M. Tatari, M. Dehghan, Solution of problems in calculus of variations via He's variational iteration method, *Physics Letters A*, 362(5-6) (2007) 401-406.

[11] J. Biazar, H. Ghazvini, He's variational iteration method for fourth-order parabolic equations, *Computers & Mathematics with Applications*, 54(7-8) (2007) 1047-1054.

[12] A. Hemeda, Variational iteration method for solving wave equation, *Computers & Mathematics with Applications*, 56(8) (2008) 1948-1953.

[13] M.A. Noor, K.I. Noor, S.T. Mohyud-Din, Modified variational iteration technique for solving singular fourth-order parabolic partial differential equations, Nonlinear Analysis: Theory, *Methods & Applications*, 71(12) (2009) e630-e640.

[14] Y. Liu, C.S. Gurram, The use of He's variational iteration method for obtaining the free vibration of an Euler–Bernoulli beam, *Mathematical and Computer Modelling*, 50(11-12) (2009) 1545-1552.

[15] J.-H. He, Variational approach for nonlinear oscillators, Chaos, *Solitons & Fractals*, 34(5) (2007) 1430-1439.

[16] J.-H. He, Hamiltonian approach to nonlinear oscillators, *Physics Letters A*, 374(23) (2010) 2312-2314.

[17] M. Baghani, M. Fattahi, A. Amjadian, Application of the variational iteration method for nonlinear free vibration of conservative oscillators, *Scientia Iranica*, 19(3) (2012) 513-518.

[18] Y.-J. Huang, H.-K. Liu, A new modification of the variational iteration method for van der Pol equations, *Applied Mathematical Modelling*, 37(16-17) (2013) 8118-8130.

[19] S.S. Siddiqi, M. Iftikhar, Variational iteration method for the solution of seventh order boundary value problems using He's polynomials, *Journal of the Association of Arab Universities for Basic and Applied Sciences*, 18(1)

1 Finite Element method
2 Spline-based differential quadrature method
3 Differential quadrature method
4 Assumed Space Mode method

(2015) 60-65.

[20] H. Jafari, A comparison between the variational iteration method and the successive approximations method, *Applied Mathematics Letters*, 32 (2014) 1-5.

[21] A.J. Al-Sawoor, M.O. Al-Amr, A new modification of variational iteration method for solving reaction–diffusion system with fast reversible reaction, *Journal of the Egyptian Mathematical Society*, 22(3) (2014) 396-401.

[22] H. Ghaneai, M. Hosseini, Variational iteration method with an auxiliary parameter for solving wave-like and heat-like equations in large domains, *Computers & Mathematics with Applications*, 69(5) (2015) 363-373.

[23] Y. Chen, J. Zhang, H. Zhang, X. Li, J. Zhou, Re-examination of natural frequencies of marine risers by variational iteration method, *Ocean Engineering*, 94 (2015) 132-139.

[24] M. Daeichi, M. Ahmadian, Application of variational iteration method to large vibration analysis of slenderness beams considering mid-plane stretching, Scientia Iranica. Transaction B, *Mechanical Engineering*, 22(5) (2015) 1911.

[25] O. Martin, A modified variational iteration method for the analysis of viscoelastic beams, *Applied Mathematical Modelling*, 40(17-18) (2016) 7988-7995.

[26] S. Khuri, A. Sayfy, Generalizing the variational iteration method for BVPs: Proper setting of the correction functional, *Applied Mathematics Letters*, 68 (2017) 68-75.

[27] A. Ghorbani, M. Bakherad, A variational iteration method for solving nonlinear Lane–Emden problems, *New Astronomy*, 54 (2017) 1-6.

[28] K. Torabi, M. Ghassabi, M. Heidari-Rarani, D. Sharifi, Variational Iteration Method for Free Vibration Analysis of a Timoshenko Beam under Various Boundary Conditions, *International Journal of Engineering-Transactions A: Basics*, 30(10) (2017) 1565-1572.

[29] K. Torabi, D. Sharifi, M. Ghassabi, Nonlinear vibration analysis of a Timoshenko beam with concentrated mass using variational iteration method, *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 39(12) (2017) 4887-4894.

[30] A.-M. Wazwaz, *Partial differential equations and solitary waves theory*, Springer Science & Business Media, 2010.

[31] W. Abdul-Majid, *Linear and Non linear Integral Equations: Methods and Applications*, in, Springer Heidelberg Dordrecht, London. New York, 2011.

[32] K. Low, On the methods to derive frequency equations of beams carrying multiple masses, *International Journal of Mechanical Sciences*, 43(3) (2001) 871-881.

[33] K. Torabi, D. Sharifi, M. Ghassabi, A. Mohebbi, Semi-analytical Solution for Nonlinear Transverse Vibration Analysis of an Euler–Bernoulli Beam with Multiple Concentrated Masses Using Variational Iteration Method, *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering*, (2018) 1-16.

[34] Y. Feng, C. Bert, Application of the quadrature method to flexural vibration analysis of a geometrically nonlinear beam, *Nonlinear Dynamics*, 3(1) (1992) 13-18.

[35] M.A. Foda, Influence of shear deformation and rotary inertia on nonlinear free vibration of a beam with pinned ends, *Computers & structures*, 71(6) (1999) 663-670.

[36] M. Asghari, M. Kahrobaiyan, M. Ahmadian, A nonlinear Timoshenko beam formulation based on the modified couple stress theory, *International Journal of Engineering Science*, 48(12) (2010) 1749-1761.

[37] G. Bhashyam, G. Prathap, Galerkin finite element method for non-linear beam vibrations, *Journal of Sound and Vibration*, 72(2) (1980) 191-203.

[38] A. Barari, H.D. Kaliji, M. Ghadimi, G. Domairry, Non-linear vibration of Euler-Bernoulli beams, *Latin American Journal of Solids and Structures*, 8(2) (2011) 139-148.

[39] G. Singh, A. Sharma, G.V. Rao, Large-amplitude free vibrations of beams—a discussion on various formulations and assumptions, *Journal of Sound and Vibration*, 142(1) (1990) 77-85.

[40] Q. Guo, H. Zhong, Non-linear vibration analysis of beams by a spline-based differential quadrature method, *Journal of Sound and Vibration*, 1(269) (2004) 413-420.

[41] C. Mei, Finite element displacement method for large amplitude free flexural vibrations of beams and plates, *Computers & Structures*, 3(1) (1973) 163-174.

[42] D.A. Evensen, Nonlinear vibrations of beams with various boundary conditions, *AIAA journal*, 6(2) (1968) 370-372.

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