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Nonlinear Free Vibration of Buckled Size-Dependent Functionally Graded Nanobeams Using Homotopy Perturbation Method

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ABSTRACT: The present study aims at investigating nonlinear free vibration of thermally buckled functionally graded nanobeam. The nonlocal nonlinear Euler-Bernoulli beam theory as well as linear eigenmodes of a functionally graded nanobeam vibrating around the first buckling configuration are employed to derive a system of ordinary differential equations via the Galerkin method. Semi-analytical solutions are obtained based on both the homotopy perturbation method and the variational iteration method. Results show that the difference between nonlinear and linear frequencies increases with a rise in the maximum lateral initial deflection, small scale parameter value, and index of the power law. Investigating the effect of the ratio of length to thickness on the variance between the nonlinear and linear frequencies shows that the aspect ratio makes no difference on the classical ratio of nonlinear to linear frequencies although the difference between the nonlocal nonlinear and linear frequencies decreases with a rise in the aspect ratio. In contrast to the ratio of the first nonlinear frequency to the first linear one which will decrease if compressive axial load increases, the values of the compressive axial load which are beyond the load bearing capacity of the functionally graded nanobeam do not affect the ratio of the second nonlinear to linear frequencies.

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1- Introduction

The unique characteristics of Functionally Graded Materials (FGM) resulting from the smooth and continuous variation of properties along certain dimensions have made them the notable materials which can be used in many engineering application fields [1]. The growing development of technology provides the possibility of using FGM thin beams in micro/nano-electro-mechanical systems, such as electrically actuated devices and atomic force microscopes [2], thus the study of mechanical behavior of Functionally Graded (FG) micro-/nano-structures has recently become a topic of interest to researchers.

The importance of incorporating the size effect into continuum mechanics in order to investigate the mechanical behavior of micro- or nano-scale devices, is well known and higher-order continuum theories containing additional material constants have been developed to this end [3, 4]. Strain gradient theory, modified strain gradient theory, couple stress theory, modified couple stress theory, nonlocal elasticity theory, surface elasticity theory, and nonlocal-strain gradient theory are some of the famous higher-order continuum theories employed by researchers to investigate mechanical behavior of size-dependent structures.

On the basis of the strain gradient Timoshenko beam theory, the free vibration characteristics of functionally graded microbeams were investigated by Ansari et al. [5]. Ansari et al. [6] also combined the most general strain gradient elasticity theory containing five additional material length scale parameters with the classical Timoshenko beam theory to investigate the bending and buckling of functionally graded microbeams. Using the strain gradient theory, Setoodeh and Afrahim [7] incorporated the size effect into nonlinear Euler–Bernoulli beam theory and studied the size dependent nonlinear vibration behavior of functionally graded micro-pipes conveying fluid. Ghorbani Shenas et al. [8] investigated the effects of the geometrical design parameters of pre-twisted microbeams in thermal environment based on modified strain gradient theory.

Ghorbanpour Arani et al. [9] studied the nonlinear vibration of a nanobeam coupled with a piezoelectric nanobeam based on the strain gradient theory.

Reddy [10] employed a modified couple stress theory to derive the nonlinear non-classical Timoshenko and Euler-Bernoulli beam theories to study static bending, free vibration, and buckling of FG hinged micro-beams. These theories were used to investigate the nonlinear bending response of clamped FG micro-beams under mechanical loadings by Arbin and Reddy [11] as well.

Ansari et al. [5] compared two different beam models on the basis of the modified couple stress theory and the strain gradient theory in predicting the natural frequencies of functionally graded microbeams. They showed that the value of natural frequency predicted by strain gradient theory is higher than that predicted by the modified couple stress theory. The differences in predicting load-bearing capacity and lateral deflection of functionally graded microbeams between various beam theories derived according to modified couple stress theory, strain gradient theory and modified strain gradient theory are studied by Ansari et al. [6] as well. The comparison studies between the strain gradient theory and the couple stress theory done by Setoodeh and Afrahim [7] showed that the former induces a higher stiffness.

According to the nonlocal elasticity theory hypothesis, Eltaher et al. [12, 13] and Şimşek and Yurtcu [2]

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independently employed nonlocal Timoshenko beam theory [12, 2] and nonlocal Euler-Bernoulli beam theory [13, 2] to study the static bending and buckling of FG nanobeams with different boundary conditions. Eltaher et al. [14] studied free vibration of FG nanobeams based upon nonlocal Euler-Bernoulli beam theory and finite element method. The effects of neutral axis location on linear natural frequencies of FG macro-/nanobeams were investigated by Eltaher et al. [15, 16] as well. Uymaz [17] used generalized beam theory and the nonlocal elasticity to present forced vibration of FG nanobeams. Nonlinear free vibration of FG nanobeams was studied by Nazemnezhad and Hosseini-Hashemi [18] based on nonlocal Euler-Bernoulli beam theory and multiple scale method. Using nonlocal Timoshenko beam theory, Rahmani and Pedram [19] investigated the effects of gradient index and geometrical dimensions on linear free vibration of FG nanobeams. He's variational method and nonlocal Euler–Bernoulli beam theory were used to study the large amplitude free vibration of FG nanobeams resting on nonlinear elastic foundation by Niknam and Aghdam [20]. Kiani [21] proposed a mathematical model to investigate the vibration and instability of moving FG nanobeams based on nonlocal Rayleigh beam theory. Obtained results clearly showed that the value of small scale parameter is an important factor to accurately estimate dynamic responses of FG nanobeams although boundary conditions, order of the mode of vibration, and geometrical dimensions can affect the role of the small scale parameter in simulating dynamic responses of FG nanobeams. Ziaee's study on the effect of small scale parameter on linear vibration of thermally buckled FG nanobeams showed the important role of compressive axial force exerted on FG nanobeams in nonlocal behavior of vibrating FG nanobeams [22]. Ebrahimi and Salari [23] employed nonlocal Euler-Bernoulli beam theory to investigate the effect of different parameters such as small scale parameter effects, different material compositions, mode number, and the ratio of length to thickness on the normalized natural frequencies of the simply supported FG nanobeams. They also used nonlocal Timoshenko beam theory [24] and nonlocal Euler-Bernoulli beam theory [25, 26] to investigate free vibration of functionally graded nanobeams subjected to an in-plane thermal loading. They showed that thermal effect and boundary conditions affect the vibration behavior of FG nanobeams significantly. Gourbanpour Arani et al. [27] used the nonlocal nonlinear Timoshenko beam theory to study the nonlinear vibration of embedded single-walled boron nitride nanotube under imposed electric potential and thermal loading. They found that the magnitudes of nonlinear frequency are more than the magnitudes of linear one. Also, they showed that the effects of Winkler elastic parameter and Pasternak shear modulus on the fundamental frequency are approximately the same. The nonlinear vibration of functionally graded nanobeams resting on a nonlinear elastic foundation has been simulated by Trabelssi et al. [28] on the basis of nonlocal Euler-Bernoulli beam theory. Based on similar theory, Lv and Liu [29] studied the nanomaterial uncertainties on linear vibration and static stability of functionally graded nanobeams in thermal environment.

Because the surface to volume ratio of nano-structures is high, Hosseini-Hashemi et al. [30] studied nonlinear free vibration of simply-supported functionally graded nanobeams in the presence of the surface effects via nonlocal nonlinear Euler-Bernoulli beam theory. Their investigations clearly revealed that surface effects and nonlocal parameter have an opposite impact on the ratio of fundamental frequencies. Zhang et al. [31] investigated the influence of surface and thermal effects as well as small scale parameter on the flexural wave propagation of piezoelectric functionally graded nanobeam. Their study showed that the temperature effects on the phase velocity and group velocity are remarkable for wave number. Barati [32] employed He's variational method to investigate different parameters such as temperature change and elastic foundation on nonlinear frequency of a flexoelectric nanobeam incorporating surface elasticity.

After introducing the higher-order nonlocal strain gradient theory by Lim et al. [33], the researchers investigated the nonlinear vibration behavior of size-dependent functionally graded beam in the framework of simplified nonlocal strain gradient theory [34, 35].

Although it is well known that proper values of the nonlocal parameter must be used if the accurate study of mechanical behavior of micro/nanostructures is desired, a thorough research has not been done to estimate the value of small scale parameter corresponding to mechanical response of functionally graded micro-/nanobeams so far [18]. Hence, all researchers who used nonlocal continuum theories to simulate size-dependent mechanical behavior of FG nanobeams investigated the effects of small scale parameter on mechanical behavior of FG nanobeams by changing the value of the small scale parameter [14-18].

As it is known, thermal stress due to the temperature rise in micro/nanobeams with immovable ends produces compressive axial force which can lead to buckling the beams if its value increases over the critical value [36]. Therefore, the investigation of dynamical behaviour of thermally buckled micro/nanobeams, especially FGM micro-/nano- beams to improve the thermal resistance of micro-/nanobeams, is of great importance [22].

Sun et al. [37] used the multiple scaled Lindsted-Poincare method to study the free vibration of simply supported nanobeam around its buckled configuration. They estimated the first nonlinear frequency and evaluated the effects of the magnetic field and initial vibration amplitude on the nonlinear frequency [37]. Sahmani and Aghdam [38] simulated the nonlinear vibration of postbuckled multilayered functionally graded simply-supported nanobeams. To this end, they incorporate the nonlocal-strain gradient theory into third order shear deformable beam theory to obtain governing equations of motion and employed an improved perturbation method in conjunction with Galerkin method to solve the equations.

Based upon the author's knowledge, there is no notable study on nonlinear vibration of thermally buckled fully fixed FG nanobeams. So, the investigation of the effects of small scale parameter on nonlinear frequencies of fully fixed FG nanobeams is the main purpose of this article. The variation of material property graduated in the thickness direction is modeled according to the simple power-law distribution. Euler-Bernoulli beam theory, von-Karman geometric nonlinearity and Eringen's nonlocal elasticity theory are employed to derive the partial differential equation of motion. The Homotopy Perturbation Method (HPM) and variational iteration method are employed to find the first and the second approximation of nonlinear frequencies as well as response of FG nanobeams. In the parametric studies of this work, due to lack of information, small scale parameter (e_0a) is varied between 0 to 2 nm to investigate the effects of small scale parameter on response of vibrating buckled FG nanobeams.

2- Equation of Motion

The governing equation of nonlinear vibration of fixed-fixed FG nanobeams with length L, width b, thickness h and immovable ends can be obtained as follows [22] (see appendix A for details):

$$\begin{aligned}
&\left(1-\frac{(e_0a)^2 N_T}{D_{xx}}+\frac{(e_0a)^2 A_{xx} r^2}{2D_{xx}L^2}\int_0^1\left(\frac{d\bar{W}_s}{d\bar{x}}\right)^2 d\bar{x}\right)\frac{\partial^4 V}{\partial\bar{x}^4}\\
&+\left(\frac{(e_0a)^2 A_{xx} r^2}{D_{xx}L^2}\int_0^1\frac{d\bar{W}_s}{d\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{d^4\bar{W}_s}{d\bar{x}^4}\\
&+\left(\frac{N_T L^2}{D_{xx}}-\frac{A_{xx} r^2}{2D_{xx}}\int_0^1\left(\frac{d\bar{W}_s}{d\bar{x}}\right)^2 d\bar{x}\right)\frac{\partial^2 V}{\partial\bar{x}^2}\\
&-\left(\frac{A_{xx} r^2}{D_{xx}}\int_0^1\frac{d\bar{W}_s}{d\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{d^2\bar{W}_s}{d\bar{x}^2}+\frac{\partial^2 V}{\partial\bar{t}^2}-\frac{(e_0a)^2}{L^2}\frac{\partial^4 V}{\partial\bar{x}^2\partial\bar{t}^2}+\\
&+\left(\frac{(e_0a)^2 A_{xx} r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2 d\bar{x}\right)\frac{d^4\bar{W}_s}{d\bar{x}^4}\\
&+\left(\frac{(e_0a)^2 A_{xx} r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2 d\bar{x}+\frac{(e_0a)^2 A_{xx} r^2}{D_{xx}L^2}\int_0^1\frac{d\bar{W}_s}{d\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^4 V}{\partial\bar{x}^4}\\
&+\left(-\frac{A_{xx} r^2}{2D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2 d\bar{x}\right)\frac{d^2\bar{W}_s}{d\bar{x}^2}\\
&-\left(\frac{A_{xx} r^2}{2D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2 d\bar{x}+\frac{A_{xx} r^2}{D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)\left(\frac{d\bar{W}_s}{d\bar{x}}\right)d\bar{x}\right)\frac{\partial^2 V}{\partial\bar{x}^2}=0
\end{aligned}\tag{1}$$

in which

$$A_{xx}=\int_A E(z)dA,\; D_{xx}=\int_A (z-z_0)^2 E(z)dA,\, z_0=\int_A zE(z)dA\Big/\int_A E(z)dA \tag{2-a}$$

$$N_T=\int_A E(z)\alpha(z)\Delta T(z)dA,\; I_0=\int_A \rho(z)dA \tag{2-b}$$

$$E(z)=E_1+(E_2-E_1)\left(\frac{2z+h}{2h}\right)^n \tag{2-c}$$

$$\rho(z)=\rho_1+(\rho_2-\rho_1)\left(\frac{2z+h}{2h}\right)^n \tag{2-d}$$

$$\alpha(z)=\alpha_1+(\alpha_2-\alpha_1)\left(\frac{2z+h}{2h}\right)^n \tag{2-e}$$

$$\bar{x}=\frac{x}{L},\bar{t}=t\sqrt{D_{xx}\big/I_0L^4}, \tag{3}$$

where $\bar{W}_s(x)$ and $V=V(\bar{x},\bar{t})$ are the buckling configuration of nanobeam and the dynamic disturbance, respectively. ρ_i , E_i and α_i (i=1, 2) are mass density, Young modulus and coefficient of thermal expansion of the two materials used in construction of FG beam, respectively. n represents the power-law index. The cross section area and the gyration radius of the cross section of the beam are shown by A and r, respectively. z_0 denotes the distance of the neutral surface of the FG nanobeam from the mid-plane of the FG nanobeam. t is time. Also; e_0a represents the material length scale parameter including material constant and internal characteristic length. The temperature change is shown by $\Delta T(z)$.

The buckling mode shape of fixed-fixed FG nanobeams was reported by Ziaee [22]. The buckling mode shape corresponding to the smallest value of buckling load is [22]:

$$\bar{W}_s=\pm\sqrt{\frac{\left(N_TL^2/D_{xx}\right)+\left((e_0a)^2N_T/D_{xx}-1\right)(2\pi)^2}{\left(A_{xx}r^2/4D_{xx}\right)(2\pi)^2+\left((e_0a)^2A_{xx}r^2/4D_{xx}L^2\right)(2\pi)^4}}\left(1-\cos(2\pi\bar{x})\right) \tag{4}$$

It is worth mentioning that if one sets the expression under the square root in Eq. (4) equal to zero, the critical value of N_T i.e. the thermal buckling load (N_{Tcr}) will be obtained [22]. To find the nonlinear response of vibrating buckled FG nanobeam, one can express the solution of Eq. (1) in terms of the linear free oscillation modes ($\psi_i(\bar{x})$) of buckled FG nanobeams as follows (see Appendix B for details):

$$V(\bar{x},\bar{t})=\sum_{i=1}^{n}q_i(\bar{t})\psi_i(\bar{x}) \tag{5}$$

If the nonlinear terms of Eq. (1) are taken equal to zero, the eigenvalue problem that governs linear free oscillation modes can be found [22]:

$$\begin{aligned}
&\left(1-\frac{(e_0a)^2 N_T}{D_{xx}}+\frac{(e_0a)^2 A_{xx} r^2}{2D_{xx}L^2}\int_0^1\left(\frac{d\bar{W}_s}{d\bar{x}}\right)^2 d\bar{x}\right)\frac{\partial^4 V}{\partial\bar{x}^4}\\
&+\left(\frac{(e_0a)^2 A_{xx} r^2}{D_{xx}L^2}\int_0^1\frac{d\bar{W}_s}{d\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{d^4\bar{W}_s}{d\bar{x}^4}\\
&+\left(\frac{N_T L^2}{D_{xx}}-\frac{A_{xx} r^2}{2D_{xx}}\int_0^1\left(\frac{d\bar{W}_s}{d\bar{x}}\right)^2 d\bar{x}\right)\frac{\partial^2 V}{\partial\bar{x}^2}\\
&-\left(\frac{A_{xx} r^2}{D_{xx}}\int_0^1\frac{d\bar{W}_s}{d\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{d^2\bar{W}_s}{d\bar{x}^2}+\frac{\partial^2 V}{\partial\bar{t}^2}-\frac{(e_0a)^2}{L^2}\frac{\partial^4 V}{\partial\bar{x}^2\partial\bar{t}^2}=0
\end{aligned}\tag{6}$$

According to Galerkin method, one can obtain the system of ordinary differential equations of motion as follows:

$$\sum_{i=1}^{n}M_{ji}\ddot{q}_i+\sum_{i=1}^{n}K_{ji}q_i+\sum_{i,p=1}^{n}\Gamma_{jip}q_iq_p+\sum_{i,p,q=1}^{n}\Lambda_{jipk}q_iq_pq_k=0 \tag{7}$$

Because of the orthogonality of linear free oscillation modes of FG nanobeams, Eq. (7) is simplified to Eq. (8):

$$\ddot{q}_j+\bar{\omega}_j^2q_j+\sum_{i,p=1}^{n}\Gamma_{jip}q_iq_p+\sum_{i,p,k=1}^{n}\Lambda_{jipk}q_iq_pq_k=0 \tag{8}$$

in which

$$\bar{\omega}_{j}^{2}=\frac{K_{jj}}{M_{jj}} \tag{9-a}$$

$$M_{ji}=\int_0^1 \psi_i \psi_j d\bar{x}-\frac{(e_0 a)^2}{L^2}\int_0^1 \frac{d^2\psi_i}{d\bar{x}^2}\psi_j d\bar{x} \tag{9-b}$$

$$\begin{aligned}K_{ji}=&\left(-\frac{A_{xx}r^2}{D_{xx}}\int_0^1 \frac{d\psi_i}{d\bar{x}}\frac{d\bar{W}_s}{d\bar{x}}d\bar{x}\right)\left(\int_0^1 \frac{d^2\bar{W}_s}{d\bar{x}^2}\psi_j d\bar{x}\right)\\&+\left(\frac{N_T L^2}{D_{xx}}-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{d\bar{W}_s}{d\bar{x}}\right)^2 d\bar{x}\right)\left(\int_0^1 \frac{d^2\psi_i}{d\bar{x}^2}\psi_j d\bar{x}\right)+\\&\left(\frac{(e_0a)^2 A_{xx}r^2}{D_{xx}L^2}\int_0^1 \frac{d\psi_i}{d\bar{x}}\frac{d\bar{W}_s}{d\bar{x}}d\bar{x}\right)\left(\int_0^1 \frac{d^4\bar{W}_s}{d\bar{x}^4}\psi_j d\bar{x}\right)+\\&\left(1-\frac{(e_0a)^2 N_T}{D_{xx}}+\frac{(e_0a)^2 A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{d\bar{W}_s}{d\bar{x}}\right)^2 d\bar{x}\right)\left(\int_0^1 \frac{d^4\psi_i}{d\bar{x}^4}\psi_j d\bar{x}\right)\end{aligned} \tag{9-c}$$

$$\begin{aligned}\Gamma_{jip}=&\frac{1}{M_{jj}}\left[\left(\frac{(e_0a)^2 A_{xx}r^2}{D_{xx}L^2}\int_0^1 \frac{d\psi_i}{d\bar{x}}\frac{d\bar{W}_s}{d\bar{x}}d\bar{x}\right)\left(\int_0^1 \frac{d^4\psi_p}{d\bar{x}^4}\psi_j d\bar{x}\right)\right]+\\&\frac{1}{M_{jj}}\left[\left(-\frac{A_{xx}r^2}{D_{xx}}\int_0^1 \frac{d\psi_i}{d\bar{x}}\frac{d\bar{W}_s}{d\bar{x}}d\bar{x}\right)\left(\int_0^1 \frac{d^2\psi_p}{d\bar{x}^2}\psi_j d\bar{x}\right)\right]+\\&\frac{1}{M_{jj}}\left[\left(-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1 \frac{d\psi_i}{d\bar{x}}\frac{d\psi_p}{d\bar{x}}d\bar{x}\right)\left(\int_0^1 \frac{d^2\bar{W}_s}{d\bar{x}^4}\psi_j d\bar{x}\right)\right]+\\&\frac{1}{M_{jj}}\left[\left(\frac{(e_0a)^2 A_{xx}r^2}{2D_{xx}L^2}\int_0^1 \frac{d\psi_i}{d\bar{x}}\frac{d\psi_p}{d\bar{x}}d\bar{x}\right)\left(\int_0^1 \frac{d^4\bar{W}_s}{d\bar{x}^4}\psi_j d\bar{x}\right)\right]\end{aligned} \tag{9-d}$$

$$\begin{aligned}\Lambda_{jipk}=&\frac{1}{M_{jj}}\left[\left(\frac{(e_0a)^2 A_{xx}r^2}{2D_{xx}L^2}\int_0^1 \frac{d\psi_i}{d\bar{x}}\frac{d\psi_p}{d\bar{x}}d\bar{x}\right)\left(\int_0^1 \frac{d^4\psi_k}{d\bar{x}^4}\psi_j d\bar{x}\right)\right]+\\&\frac{1}{M_{jj}}\left[\left(-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1 \frac{d\psi_k}{d\bar{x}}\frac{d\psi_p}{d\bar{x}}d\bar{x}\right)\left(\int_0^1 \frac{d^2\psi_i}{d\bar{x}^2}\psi_j d\bar{x}\right)\right]\end{aligned} \tag{9-e}$$

M_{jj} and K_{jj} can be found based on Eq. (9b) and Eq. (9c) by equating $i=j$, respectively. $\bar{\omega}_j$ represents the dimensionless frequency.

3- Homotopy Perturbation Method

To solve the system of ordinary differential equations shown by Eq. (8), HPM is employed. Based on the concept of HPM, a homotopy is constructed as follows [39]:

$$\begin{aligned}&(1-P)\left(\ddot{q}_j+\bar{\omega}_j^2 q_j\right)\\&+P\left(\ddot{q}_j+\bar{\omega}_j^2 q_j+\sum_{i,p=1}^{n}\Gamma_{jip}q_i q_p+\sum_{i,p,k=1}^{n}\Lambda_{jipk}q_i q_p q_k\right)=0\end{aligned} \tag{10}$$

where $P\in[0,1]$. It is also assumed that the solution of Eq. (10) and the frequency $\bar{\omega}_j^2$ can be expressed as a power series of the homotopy perturbation parameter P:

$$q_j=q_{j0}+Pq_{j1}+P^2q_{j2}+\ldots \tag{11-a}$$

$$\bar{\omega}_j^2=\bar{\omega}_{j0}^2+P\bar{\omega}_{j1}^2+P^2\bar{\omega}_{j2}^2+\ldots \tag{11-b}$$

Substituting Eq. (11) into Eq. (10) and equating coefficients like power of P, one can obtain the following equations:

$$P^0:\ \ddot{q}_{j0}+\bar{\omega}_{j0}^2 q_{j0}=0,\quad q_{j0}(0)=A_j,\frac{dq_{j0}}{d\bar{t}}(0)=0 \tag{12-a}$$

$$\begin{aligned}P^1:\ &\ddot{q}_{j1}+\bar{\omega}_{j0}^2 q_{j1}=-\bar{\omega}_{j1}^2 q_{j0}\\&-\sum_{i,p=1}^{n}\Gamma_{jip}q_{i0}q_{p0}-\sum_{i,p,k=1}^{n}\Lambda_{jipk}q_{i0}q_{p0}q_{k0},\quad q_{j1}(0)=0,\\&\frac{dq_{j1}}{d\bar{t}}(0)=0\end{aligned} \tag{12-b}$$

$$\begin{aligned}P^2:\ &\ddot{q}_{j2}+\bar{\omega}_{j0}^2 q_{j2}=-\bar{\omega}_{j_1}^2 q_{j1}-\bar{\omega}_{j2}^2 q_{j0}-\sum_{i,p=1}^{n}\Gamma_{jip}\left(q_{i0}q_{p1}+q_{i1}q_{p0}\right)\\&-\sum_{i,p,k=1}^{n}\Lambda_{jipk}\left(q_{i0}q_{p0}q_{k1}+q_{i1}q_{p0}q_{k0}+q_{i0}q_{p1}q_{k0}\right),\quad q_{j2}(0)=0,\frac{dq_{j2}}{d\bar{t}}(0)=0\end{aligned} \tag{12-c}$$

The solution of Eq. (12a) can be written in the following form:

$$q_{j0}=\frac{1}{2}A_j\left(\exp\left(\bar{i}\,\bar{\omega}_{j0}\bar{t}\right)+CC\right),\bar{i}=\sqrt{-1} \tag{13}$$

where CC stands for complex conjugate of the preceding terms.
Substituting Eq. (13) into Eq. (12b) yields:

$$\begin{aligned}&\ddot{q}_{j1}+\bar{\omega}_{j0}^2 q_{j1}=-\bar{\omega}_{j1}^2\frac{A_j}{2}\exp\left(\bar{i}\,\bar{\omega}_{j0}\bar{t}\right)\\&-\frac{1}{4}\sum_{i,p=1}^{n}\Gamma_{jip}A_iA_p\left[\exp\left(\bar{i}\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}\right)\bar{t}\right)+\exp\left(\bar{i}\left(\bar{\omega}_{i0}-\bar{\omega}_{p0}\right)\bar{t}\right)\right]+\\&-\frac{1}{8}\sum_{i,p,k=1}^{n}\Lambda_{jipk}A_iA_pA_k\left[\exp\left(\bar{i}\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)\bar{t}\right)+\exp\left(\bar{i}\left(-\bar{\omega}_{i0}+\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)\bar{t}\right)\right.\\&\left.+\exp\left(\bar{i}\left(\bar{\omega}_{i0}-\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)\bar{t}\right)+\exp\left(\bar{i}\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}-\bar{\omega}_{k0}\right)\bar{t}\right)\right]+CC,\bar{i}=\sqrt{-1}\end{aligned} \tag{14}$$

To eliminate the secular terms of a particular solution of Eq. (14), the coefficient of ($i\bar{\omega}_{j0}\bar{t}$) must be taken equal to zero. If there is no internal resonance, the condition of eliminating secular terms leads to

$$\bar{\omega}_{j1}^2=-\frac{2}{4}\sum_{i=1,i\neq j}^{n}A_i^2\left(\Lambda_{jjii}+\Lambda_{jiji}+\Lambda_{jiij}\right)-\frac{3}{4}A_j^2\Lambda_{jjjj} \tag{15}$$

Substituting Eq. (15) into Eq. (11b) and equating the homotopy perturbation parameter with 1, the first approximate solution of 'j^{th}' nonlinear frequency of buckled FG nanobeam is obtained:

$$\bar{\omega}_{j0}^2=\bar{\omega}_j^2+\frac{1}{2}\sum_{i=1,i\neq j}^{n}A_i^2\left(\Lambda_{jjii}+\Lambda_{jiji}+\Lambda_{jiij}\right)+\frac{3}{4}A_j^2\Lambda_{jjjj} \tag{16}$$

After eliminating the secular terms of Eq. (14), the solution of Eq. (14) makes the first-order approximate solution of lateral displacement of FG nanobeam:

$$\begin{aligned} q_j &= \left(C_j + A_j\right)\cos\left(\bar{\omega}_{j0}\bar{t}\right) + \\ &-\frac{1}{2}\sum_{i,p=1}^{N} A_i A_p \Gamma_{jip}\left(\frac{\cos\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}\right)\bar{t}}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}\right)^2}+\frac{\cos\left(\bar{\omega}_{i0}-\bar{\omega}_{p0}\right)\bar{t}}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}-\bar{\omega}_{p0}\right)^2}\right) \\ &-\frac{1}{4}\sum_{i,p,k=1^*}^{N} A_i A_p A_k \Lambda_{jipk}\left[\frac{\cos\left(\bar{\omega}_{i0}-\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)\bar{t}}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}-\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)^2}\right]+ \\ &-\frac{1}{4}\sum_{i,p,k=1^*}^{N} A_i A_p A_k \Lambda_{jipk}\left[\frac{\cos\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}-\bar{\omega}_{k0}\right)\bar{t}}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}-\bar{\omega}_{k0}\right)^2}\right]+ \\ &-\frac{1}{4}\sum_{i,p,k=1^*}^{N} A_i A_p A_k \Lambda_{jipk}\left[\frac{\cos\left(-\bar{\omega}_{i0}+\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)\bar{t}}{\bar{\omega}_{j0}^2-\left(-\bar{\omega}_{i0}+\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)^2}\right]+ \\ &-\frac{1}{4}\sum_{i,p,k=1}^{N} A_i A_p A_k \Lambda_{jipk}\left(\frac{\cos\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)\bar{t}}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)^2}\right) \end{aligned} \tag{17}$$

if two indexes are equal, the third index is not equal to j

where:

$$\begin{aligned} C_j &= \frac{1}{2}\sum_{i,p=1}^{N} A_i A_p \Gamma_{jip}\left(\frac{1}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}\right)^2}+\frac{1}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}-\bar{\omega}_{p0}\right)^2}\right) \\ &+\frac{1}{4}\sum_{i,p,k=1^*}^{N} A_i A_p A_k \Lambda_{jipk}\left[\frac{1}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}-\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)^2}+\right. \\ &\left.+\frac{1}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}-\bar{\omega}_{k0}\right)^2}+\frac{1}{\bar{\omega}_{j0}^2-\left(-\bar{\omega}_{i0}+\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)^2}\right]+ \\ &+\frac{1}{4}\sum_{i,p,k=1}^{N} A_i A_p A_k \Lambda_{jipk}\left(\frac{1}{\bar{\omega}_{j0}^2-\left(\bar{\omega}_{i0}+\bar{\omega}_{p0}+\bar{\omega}_{k0}\right)^2}\right) \end{aligned} \tag{18}$$

if two indexes are equal, the third index is not equal to j

It should be noted that the singular terms must be eliminated from Eqs. (17) and (18) by selecting proper values of *i*, *p*, and *k*.

If the 'j^{th}' mode shape of vibrating buckled beam is excited (i.e. $A_j \neq 0$ and $A_m = 0$ where $m \neq j$), Eqs. (16) and (17) will be simplified as follows:

$$\bar{\omega}_{j0}^2 = \bar{\omega}_j^2 + \frac{3}{4}A_j^2\Lambda_{jjjj} \tag{19}$$

$$\begin{aligned} q_j &= \left(A_j + \frac{A_j^2\Gamma_{jjj}}{3\bar{\omega}_{j0}^2} - \frac{A_j^3\Lambda_{jjjj}}{32\bar{\omega}_{j0}^2}\right)\cos\left(\bar{\omega}_{j0}\bar{t}\right) \\ &-\frac{A_j^2\Gamma_{jjj}}{\omega_{j0}^2}\left(\frac{1}{2}-\frac{\cos\left(2\bar{\omega}_{j0}\bar{t}\right)}{6}\right)+\frac{A_j^3\Lambda_{jjjj}}{32\omega_{j0}^2}\cos\left(3\bar{\omega}_{j0}t\right) \end{aligned} \tag{20}$$

and

$$\bar{\omega}_{m0}^2 = \bar{\omega}_j^2 + \frac{1}{2}A_j^2\left(\Lambda_{mmjj}+\Lambda_{mjmj}+\Lambda_{mjjm}\right) \tag{21}$$

$$\begin{aligned} q_m &= \left(C_m\right)\cos\left(\bar{\omega}_{m0}\bar{t}\right) - \frac{1}{2}A_j^2\Gamma_{mjj}\left(\frac{\cos\left(2\bar{\omega}_{j0}\right)\bar{t}}{\bar{\omega}_{m0}^2-\left(2\bar{\omega}_{j0}\right)^2}+\frac{1}{\bar{\omega}_{m0}^2}\right) \\ &-\frac{3}{4}A_j^3\Lambda_{mjjj}\frac{\cos\left(\bar{\omega}_{j0}\right)\bar{t}}{\bar{\omega}_{m0}^2-\bar{\omega}_{j0}^2}-\frac{1}{4}A_j^3\Lambda_{mjjj}\frac{\cos\left(3\bar{\omega}_{j0}\right)\bar{t}}{\bar{\omega}_{m0}^2-\left(3\bar{\omega}_{j0}\right)^2} \end{aligned} \tag{22}$$

$$\begin{aligned} \left(C_m\right) &= \frac{1}{2}A_j^2\Gamma_{mjj}\left(\frac{1}{\bar{\omega}_{m0}^2-\left(2\bar{\omega}_{j0}\right)^2}+\frac{1}{\bar{\omega}_{m0}^2}\right) \\ &+\frac{3}{4}A_j^3\Lambda_{mjjj}\frac{1}{\bar{\omega}_{m0}^2-\bar{\omega}_{j0}^2}-\frac{1}{4}A_j^3\Lambda_{mjjj}\frac{1}{\bar{\omega}_{m0}^2-\left(3\bar{\omega}_{j0}\right)^2} \end{aligned} \tag{23}$$

To obtain the second approximation of nonlinear frequencies of buckled FG nanobeams, after substituting the solution of Eq. (14) into Eq. (12c), the secular terms of a particular solution of Eq. (12c) must be eliminated.

This second-order approximate solution of a buckled FG nanobeam whose response is estimated by the first mode shape of its linear vibration is:

$$\bar{\omega}_{10}^2 = -\left(\bar{\omega}_{11}^2-\bar{\omega}_1^2\right)-\sqrt{\left(\bar{\omega}_{11}^2-\bar{\omega}_1^2\right)^2+4\beta_1} \tag{24}$$

where

$$\begin{aligned} \beta_1 &= \bar{\omega}_{11}^2\left(\frac{A_1\Gamma_{111}}{3}-\frac{A_1^2\Lambda_{1111}}{32}\right)-\frac{5A_1^2\Gamma_{111}^2}{6} \\ &+\frac{3A_1^3\Gamma_{111}\Lambda_{1111}}{4}-\frac{3A_1^4\Lambda_{1111}^2}{64} \end{aligned} \tag{25-a}$$

$$\bar{\omega}_{11}^2 = -\frac{3}{4}A_1^2\Lambda_{1111} \tag{25-b}$$

4- Variational Iteration Method

The variational iteration method is also used to estimate nonlinear frequencies of buckled FG nanobeams. To this purpose, Eq. (8) is arranged as follows [39]:

$$\ddot{q}_j + \bar{\Omega}_j^2 q_j = F_j\left(q_1,\ldots,q_n\right) \tag{26}$$

where

$$\begin{aligned} F_j\left(q_1,\ldots,q_n\right) &= \left(\bar{\Omega}_j^2-\bar{\omega}_j^2\right)q_j \\ &-\sum_{i,p=1}^{n}\Gamma_{jip}q_i q_p-\sum_{i,p,k=1}^{n}\Lambda_{jipk}q_i q_p q_k \end{aligned} \tag{27}$$

Based on the assumptions of variational iteration method, the initial approximation of $q_j^{(0)}$ is (if $q_j^{(0)}(0)=A_j$, $\dot{q}_j^{(0)}(0)=0$):

$$q_j^{(0)} = \frac{1}{2} A_j \left(\exp\left(\bar{i}\,\bar{\Omega}_j \bar{t} \right) + CC \right), \quad j = 1,\dots,n \ \ and \ \ \bar{i} = \sqrt{-1} \tag{28}$$

and the formula which governs the 'm^{th}' iteration is:

$$\begin{aligned} q_j^{(m+1)} &= q_j^{(m)} \\ &- \frac{\bar{i}}{2\bar{\Omega}_j} \int_0^t \exp\left(\bar{i}\,\bar{\Omega}_j \left(s - t \right) \right) F_j \left(q_1^{(m)},\dots,q_n^{(m)} \right) ds \\ &+ \frac{\bar{i}}{2\bar{\Omega}_j} \int_0^t \exp\left(\bar{i}\,\bar{\Omega}_j \left(t - s \right) \right) F_j \left(q_1^{(m)},\dots,q_n^{(m)} \right) ds \\ &, \bar{i} = \sqrt{-1} \end{aligned} \tag{29}$$

Substituting Eq. (28) into Eq. (27) to find $F_j(q_1^{(0)}, \dots, q_n^{(0)})$ yields:

$$\begin{aligned} F_j^0 &= \frac{\left(\bar{\Omega}_j^2 - \bar{\omega}_j^2 \right)}{2} A_j \exp\left(\bar{i}\,\bar{\Omega}_j \bar{t} \right) \\ &- \frac{1}{4} \sum_{i,p=1}^{n} \Gamma_{jip} A_i A_p \left[\exp\left(\bar{i} \left(\bar{\Omega}_i + \bar{\Omega}_p \right) \right) + \exp\left(\bar{i} \left(\bar{\Omega}_i - \bar{\Omega}_p \right) \right) \right] \\ &- \frac{1}{8} \sum_{i,p,k=1}^{n} \Lambda_{jipk} A_i A_p A_k \left[\exp\left(\bar{i} \left(\bar{\Omega}_i + \bar{\Omega}_p + \bar{\Omega}_k \right) \right) + \right. \\ &\exp\left(\bar{i} \left(-\bar{\Omega}_i + \bar{\Omega}_p + \bar{\Omega}_k \right) \right) + \exp\left(\bar{i} \left(\bar{\Omega}_i - \bar{\Omega}_p + \bar{\Omega}_k \right) \right) \\ &\left. + \exp\left(\bar{i} \left(\bar{\Omega}_i + \bar{\Omega}_p - \bar{\Omega}_k \right) \right) \right] + CC, \ \bar{i} = \sqrt{-1} \end{aligned} \tag{30}$$

To eliminate secular terms from $q_j^{(1)}$, the coefficient of $\exp(i\bar{\Omega}_j \bar{t})$ in Eq. (30) must be taken equal to zero:

$$\frac{\left(\bar{\Omega}_j^2 - \bar{\omega}_j^2 \right)}{2} A_j - \frac{1}{4} \sum_{i=1, i \neq j}^{n} A_j A_i^2 \left(\Lambda_{jjii} + \Lambda_{jiji} + \Lambda_{jiij} \right) - \frac{3}{8} A_j^3 \Lambda_{jjjj} = 0 \tag{31}$$

The solution of Eq. (31) is the first-order approximate solution of nonlinear frequencies of FG nanobeams.

$$\bar{\Omega}_j^2 = \bar{\omega}_j^2 + \frac{1}{2} \sum_{i=1, i \neq j}^{n} A_i^2 \left(\Lambda_{jjii} + \Lambda_{jiji} + \Lambda_{jiij} \right) + \frac{3}{4} A_j^2 \Lambda_{jjjj} \tag{32}$$

On the basis of iteration formula of variational method (Eq. (29)) [39], the solution of $q_j^{(1)}$ can be obtained:

$$\begin{aligned} q_j^{(1)} &= A_j \cos\left(\bar{\Omega}_j t \right) - \frac{1}{2} \sum_{i,p=1}^{N} A_i A_p \Gamma_{jip} \left(\frac{\cos\left(\bar{\Omega}_i + \bar{\Omega}_p \right) \bar{t}}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i + \bar{\Omega}_p \right)^2} + \frac{\cos\left(\bar{\Omega}_i - \bar{\Omega}_p \right) \bar{t}}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i - \bar{\Omega}_p \right)^2} \right) \\ &- \frac{1}{4} \sum_{i,p,k=1^*}^{N} A_i A_p A_k \Lambda_{jipk} \left[\frac{\cos\left(\bar{\Omega}_i - \bar{\Omega}_p + \bar{\Omega}_k \right) \bar{t}}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i - \bar{\Omega}_p + \bar{\Omega}_k \right)^2} + \frac{\cos\left(\bar{\Omega}_i + \bar{\Omega}_p - \bar{\Omega}_k \right) \bar{t}}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i + \bar{\Omega}_p - \bar{\Omega}_k \right)^2} + \right. \\ &\left. \frac{\cos\left(-\bar{\Omega}_i + \bar{\Omega}_p + \bar{\Omega}_k \right) \bar{t}}{\bar{\Omega}_j^2 - \left(-\bar{\Omega}_i + \bar{\Omega}_p + \bar{\Omega}_k \right)^2} \right] - \frac{1}{4} \sum_{i,p,k=1}^{N} A_i A_p A_k \Lambda_{jipk} \left(\frac{\cos\left(\bar{\Omega}_i + \bar{\Omega}_p + \bar{\Omega}_k \right) \bar{t}}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i + \bar{\Omega}_p + \bar{\Omega}_k \right)^2} \right) + \\ &+ \left(\frac{1}{2} \sum_{i,p=1}^{N} A_i A_p \Gamma_{jip} \left(\frac{1}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i + \bar{\Omega}_p \right)^2} + \frac{1}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i - \bar{\Omega}_p \right)^2} \right) \right) \cos\left(\bar{\Omega}_j t \right) \\ &+ \left(\frac{1}{4} \sum_{i,p,k=1^*}^{N} A_i A_p A_k \Lambda_{jipk} \left[\frac{1}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i - \bar{\Omega}_p \ \bar{=} \ \bar{}_k \right)^2} + \frac{1}{\bar{}_j^2 \left(\bar{}_i \ \bar{}_p \ \bar{}_k \right)^2} + \right. \right. \\ &\left. \left. \frac{1}{\bar{\Omega}_j^2 - \left(-\bar{\Omega}_i + \bar{\Omega}_p + \bar{\Omega}_k \right)^2} \right] \right) \cos\left(\bar{\Omega}_j t \right) \\ &+ \left(\frac{1}{4} \sum_{i,p,k=1}^{N} A_i A_p A_k \Lambda_{jipk} \left(\frac{1}{\bar{\Omega}_j^2 - \left(\bar{\Omega}_i + \bar{\Omega}_p + \bar{\Omega}_k \right)^2} \right) \right) \cos\left(\bar{\Omega}_j t \right) \end{aligned} \tag{33}$$

if two indexes are equal, the third index is not equal to j

It should be noted that the singular terms must be eliminated from Eq. (33) by selecting proper values of *i*, *p*, and *k*.
It can be seen that Eqs. (32) and (33) conform to Eqs. (16) and (17) respectively.
If the 'j^{th}' vibrating mode shape of buckled beam is excited (i.e. $A_j \neq 0$ and $A_m = 0$ where $m \neq j$), Eqs. (32) and (33) will be simplified as follows:

$$\bar{\Omega}_j^2 = \bar{\omega}_j^2 + \frac{3}{4} A_j^2 \Lambda_{jjjj} \tag{34}$$

$$\begin{aligned} q_j^{(1)} &= \left(A_j + \frac{A_j^2 \Gamma_{jjj}}{3\bar{\omega}_{j0}^2} - \frac{A_j^3 \Lambda_{jjjj}}{32\bar{\omega}_{j0}^2} \right) \cos\left(\bar{\Omega}_j \bar{t} \right) \\ &- \frac{A_j^2 \Gamma_{jjj}}{\omega_{j0}^2} \left(\frac{1}{2} - \frac{\cos\left(2\bar{\Omega}_j \bar{t} \right)}{6} \right) + \frac{A_j^3 \Lambda_{jjjj}}{32\omega_{j0}^2} \cos\left(3\bar{\Omega}_j t \right) \end{aligned} \tag{35}$$

and

$$\bar{\Omega}_m^2 = \bar{\omega}_m^2 + \frac{1}{2} A_j^2 \left(\Lambda_{mmjj} + \Lambda_{mjmj} + \Lambda_{mjjm} \right) \tag{36}$$

$$\begin{aligned} q_m &= \left(C_m \right) \cos\left(\bar{\Omega}_m t \right) - \frac{1}{2} A_j^2 \Gamma_{mjj} \left(\frac{\cos\left(2\bar{\Omega}_j \right) \bar{t}}{\bar{\Omega}_m^2 - \left(2\bar{\Omega}_j \right)^2} + \frac{1}{\bar{\Omega}_m^2} \right) \\ &- \frac{3}{4} A_j^3 \Lambda_{mjjj} \frac{\cos\left(\bar{\Omega}_j \right) \bar{t}}{\bar{\Omega}_m^2 - \bar{\Omega}_j^2} - \frac{1}{4} A_j^3 \Lambda_{mjjj} \frac{\cos\left(3\bar{\Omega}_j \right) \bar{t}}{\bar{\Omega}_m^2 - \left(3\bar{\Omega}_j \right)^2} \end{aligned} \tag{37}$$

$$\begin{aligned} \left(C_m \right) &= \frac{1}{2} A_j^2 \Gamma_{mjj} \left(\frac{1}{\bar{\Omega}_m^2 - \left(2\bar{\Omega}_j \right)^2} + \frac{1}{\bar{\Omega}_m^2} \right) \\ &+ \frac{3}{4} A_j^3 \Lambda_{mjjj} \frac{1}{\bar{\Omega}_m^2 - \bar{\Omega}_j^2} - \frac{1}{4} A_j^3 \Lambda_{mjjj} \frac{1}{\bar{\Omega}_m^2 - \left(3\bar{\Omega}_j \right)^2} \end{aligned} \tag{38}$$

Eliminating secular terms of F_j^1, one can find the second-order approximate solution of nonlinear frequencies.
The second-order approximate solution of a buckled FG nanobeam whose response is estimated by the first mode shape of its linear vibration is:

$$Z_0 \bar{\Omega}_1^8 + Z_1 \bar{\Omega}_1^6 + Z_2 \bar{\Omega}_1^4 + Z_3 \bar{\Omega}_1^2 + Z_4 = 0 \tag{39}$$

where

$$Z_0 = A_1 / 2 \tag{40-a}$$

$$Z_1 = \left(\frac{A_1^2\Gamma_{111}}{6} - \frac{A_1^3\Lambda_{1111}}{64}\right) - \frac{A_1}{2}\bar{\omega}_1^2 - \frac{3}{8}A_1^3\Lambda_{1111} \quad \text{(40-b)}$$

$$Z_2 = -\bar{\omega}_1^2\left(\frac{A_1^2\Gamma_{111}}{6} - \frac{A_1^3\Lambda_{1111}}{64}\right) + \frac{5}{12}\Gamma_{111}^2 A_1^3 - \Lambda_{1111}\left(\frac{9}{24}A_j^4\Gamma_{111} - \frac{6}{256}A_j^5\Lambda_{1111}\right) \quad \text{(40-c)}$$

$$Z_3 = -\Gamma_{111}\left(-5\frac{A_1^4\Gamma_{111}^2}{36} + \frac{A_1^5\Gamma_{111}\Lambda_{1111}}{64}\right) - \Lambda_{1111}\left(\frac{9}{8}A_j\left(\frac{A_1^2\Gamma_{111}}{3} - \frac{A_1^3\Lambda_{1111}}{32}\right)^2 + \frac{39}{144}A_1^5\Gamma^2{}_{111}\right) - \frac{1}{128}A_1^6\Gamma_{111}\Lambda^2{}_{1111} \quad \text{(40-d)}$$

$$Z_4 = -\frac{39A_1^4\Gamma_{111}^2\Lambda_{1111}}{144}\left(\frac{A_1^2\Gamma_{111}}{3} - \frac{A_1^3\Lambda_{1111}}{32}\right) - \frac{3\Lambda_{1111}}{8}\left(\frac{A_1^2\Gamma_{111}}{3} - \frac{A_1^3\Lambda_{1111}}{32}\right)^3 - \frac{3A_1^6\Lambda_{1111}^3}{4096}\left(\frac{A_1^2\Gamma_{111}}{3} - \frac{A_1^3\Lambda_{1111}}{32}\right) - \frac{3A_1^3\Lambda^2{}_{1111}}{256}\left(\frac{A_1^2\Gamma_{111}}{3} - \frac{A_1^3\Lambda_{1111}}{32}\right)^2 + \frac{21}{9216}A_1^7\Gamma_{111}^2\Lambda^2{}_{1111} \quad \text{(40-e)}$$

5- Validation

Sedighi and Daneshmand [40] formulated the first-order and the second-order approximate solution of the first nonlinear frequency of buckled beam based on homotopy perturbation method with an auxiliary term. To use the results available in [40], the index of power law and small scale parameter are taken equal to zero. The obtained formulae for the first nonlinear frequency of buckled beam are in good agreement with those reported by Sedighi et al. [40]. The presented formulae will estimate the pre-buckling nonlinear frequencies of functionally graded nanobeams if thermal load is less than the critical one. The ratio of the first nonlinear frequency to the first linear one of isotropic beam reported by Fallah and Aghdam [41], Pirbodaghi et al. [42], Qaisi [43] and Azrar et al. [44] who used different methods to estimate nonlinear frequencies of isotropic and/or functionally graded beams, are compared with the presented data in Table 1. To make this comparison, the small scale parameter value, the index of power law and the axial load are taken equal to zero. As can be observed, there is good agreement between the results. The obtained ratio of the first nonlinear to linear frequencies of functionally graded beams are compared with those reported by Fallah and Aghdam [41] as well and the obtained maximum percentage difference is less than 1%.

6- Results and Discussion

In order to investigate the effects of different parameters such as the aspect ratio, the small scale parameter and the index of power-law on the vibratory behavior of the buckled FG nanobeams, the governing algorithms are written in MATLAT.

In the present study, nonlocal nonlinear Euler-Bernoulli beam theory is used to simulate the nonlinear vibration of buckled long and narrow (L/h>20) FG nanobeams because according to Aydogdu's study [45], the nonlocal Euler-Bernoulli beam theory can predict mechanical behavior of nanobeams accurately when length/thickness ratio of nanobeams is more than 20 (L/h>20). Also, thermo-mechanical properties of silicon nitride (Si_3N_4) and stainless steel-grade 304 (SUS304) are also used in this section. It is assumed that the nanobeam is constructed of pure metal when the power-law index (n) is zero and with an increase in "n", the volume fraction of silicon nitride gradually increases in nanobeam.

As mentioned before, the first and/or the second approximation of nonlinear dimensionless frequency of buckled FG nanobeam represented by Eq. (16) or Eq. (32) is not usable unless there are not any internal resonances. The study on variation of natural frequencies of buckled beam [46] and buckled FG nanobeam [22] with compressive axial force showed that the one-to-one and three-to-one internal resonances can be seen at the certain values of compressive axial force. Ziaee [22] showed that the values of compressive axial force in which the interaction of the modes may exist change with the material composition of FG nanobeam as well as the small scale parameter. Hence, Eq. (16) or Eq. (32) is applicable except for special cases. The variation of the ratio of thermal load to critical axial load in which one-to-one internal resonances may occur with small scale parameter, index of power-law and the ratio of the length to the thickness is listed in Table 2. The results shown in Table 2 clearly reveal that thermal load to critical axial load ratio in which one-to-one internal resonances may occur is independent of aspect ratio and index of power law although it changes with small scale parameter.

Previous studies [22, 45] also demonstrated that regardless of the value of index of power law and small scale parameter, no interaction of the modes exists as long as compressive axial load is less than the value in which the one-to-one internal resonance may occur. Hence, this study is limited to loading domain mentioned above.

According to Eqs. (16) and (17), one can conclude that

Table 1. Comparison between the obtained ratio of nonlinear to linear frequencies and available data. "H" and "V" stand for Hemotopy perturbation method and Variational iteration method respectively.

$V(0.5,0)$ (W_s=0)	Present (First-order approximate solution (H and V))	Present(Second-order approximate solution (H))	Present (Second-order approximate solution (V))	Ref. [41]	Ref. [42]	Ref. [43]	Ref. [44]
1	1.05696	1.05672	1.05672	1.055	1.0572	1.0628	1.0221
2	1.21190	1.20932	1.20939	1.2056	1.2125	1.2140	1.0856
3	1.43339	1.42542	1.42577	1.4214	1.4344	1.3904	1.1831
4	1.69555	1.68018	1.68105	1.6776	1.6171	1.5635	1.3064

Table 2. The variation of the ratio of thermal load to critical axial load (N_T / N_{Tcr}) in which one-to-one internal resonances may occur

	n=0			*n*=0.3			*n*=0.6		
	e_0a=0	e_0a=1	e_0a=2	e_0a=0	e_0a=1	e_0a=2	e_0a=0	e_0a=1	e_0a=2
L/h=30	3.4125	3.197	2.68	3.4125	3.197	2.68	3.4125	3.197	2.68
L/h=40	3.4125	3.287	2.96	3.4125	3.287	2.96	3.4125	3.287	2.96
L/h=50	3.4125	3.33	3.11	3.4125	3.33	3.11	3.4125	3.33	3.11
L/h=60	3.4125	3.355	3.195	3.4125	3.355	3.195	3.4125	3.355	3.195
L/h=70	3.4125	3.37	3.25	3.4125	3.37	3.25	3.4125	3.37	3.25
L/h=80	3.4125	3.38	3.287	3.4125	3.38	3.287	3.4125	3.38	3.287

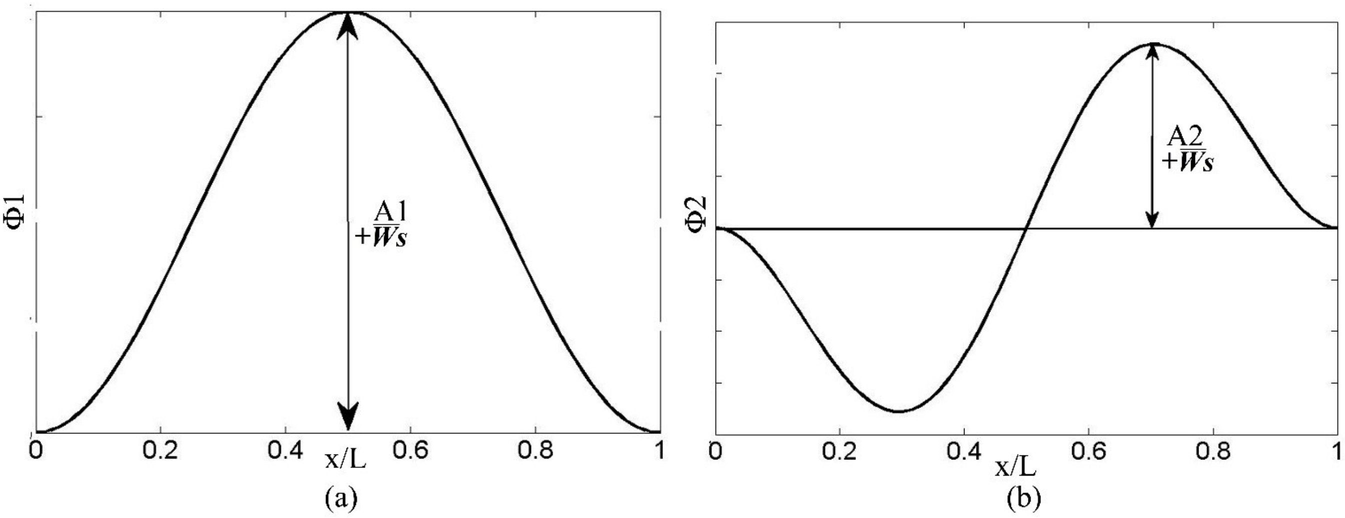

Fig. 1. Initial lateral deflection of buckled FG nanobeam

initial lateral deformation ($\overline{V}(x,0)$) which can be expressed as $\overline{V}(x,0)=\sum_{j=1}^{N} A_j \psi_j$ significantly affects nonlinear frequency and lateral response of buckled FG nanobeam. In this section, to examine the influence of the initial lateral deformation on the first and the second nonlinear frequency and the response of buckled beam, two different initial excitations which can be stated by the first or the second linear mode shape of vibrating buckled beam are used (Fig. 1). As it is observed in Fig. 1, A_1 and A_2 introduce the maximum lateral initial deflection which must be measured from buckling configuration of FG nanobeam.

Eq. (19) or Eq. (34) clearly shows the direct relationship between the maximum lateral initial deflection and the nonlinear dimensionless frequency. Based on those equations, not only the ratio of the nonlinear dimensionless frequency to the linear one tends to 1, if the value of A_j tends to zero, but also one can observe the increasing the ratio of nonlinear dimensionless frequency to the linear one by increasing A_j. The dependency of Λ_{jjjj} (see Eq. (9e)) to the ratio of tension stiffness to flexural stiffness (i.e. A_{xx} / D_{xx}) shows that an increase in the index of power-law leads to increasing the ratio of the nonlinear to the linear dimensionless frequencies as a result of increasing the ratio of A_{xx}/D_{xx} due to increasing the volume fraction of stiffer material. On the other hand, the value of Λ_{jjjj} (see Eq. (9e)) has an inverse relationship with L^2, therefore, it is expected to observe increasing the ratio of the nonlinear dimensionless frequency to linear one with a decrease in the length of nanobeam.

The variation of the ratio of the first nonlinear dimensionless frequencies to the first linear ones with the maximum lateral initial dimensionless deflection is shown in Fig. 2. In this Figure, the initial dimensionless deflection is represented by $\overline{V}(x,0)=A_1\psi_1$. The influence of the ratio of the thermal load to critical one, small scale parameter value, index of power-law and the ratio of length to thickness on the ratio of the first nonlinear dimensionless frequency to the first linear one is indicated in Fig. 2 as well. Fig. 2 clearly reveals that the difference between first nonlinear and linear dimensionless frequency increases with a rise in the maximum lateral initial dimensionless deflection, small scale parameter value, and index of power law, although by increasing compressive axial load, the first nonlinear dimensionless frequency becomes close to the linear one (Fig. 3).

The comparison between Figs. 2 (a) and 2 (c) (Fig. 2 (b) and 2 (d)) shows that regardless of the value of axial force (index of power law), with a decrease in the ratio of length to thickness, the effects of small scale parameter value on the ratio of the first nonlinear dimensionless frequency to the first linear one increase.

Fig. 3 clearly reveals that with an increase in small scale parameter value and/or a decrease in index of power law (i.e. an increase in the volume fraction of softer material), load bearing capacity of FG nanobeam decreases. It is also concluded that at the fixed value of compressive axial load, the ratio of the nonlinear to linear dimensionless frequencies increases with a decrease in small scale parameter value and/ or a rise in index of power law.

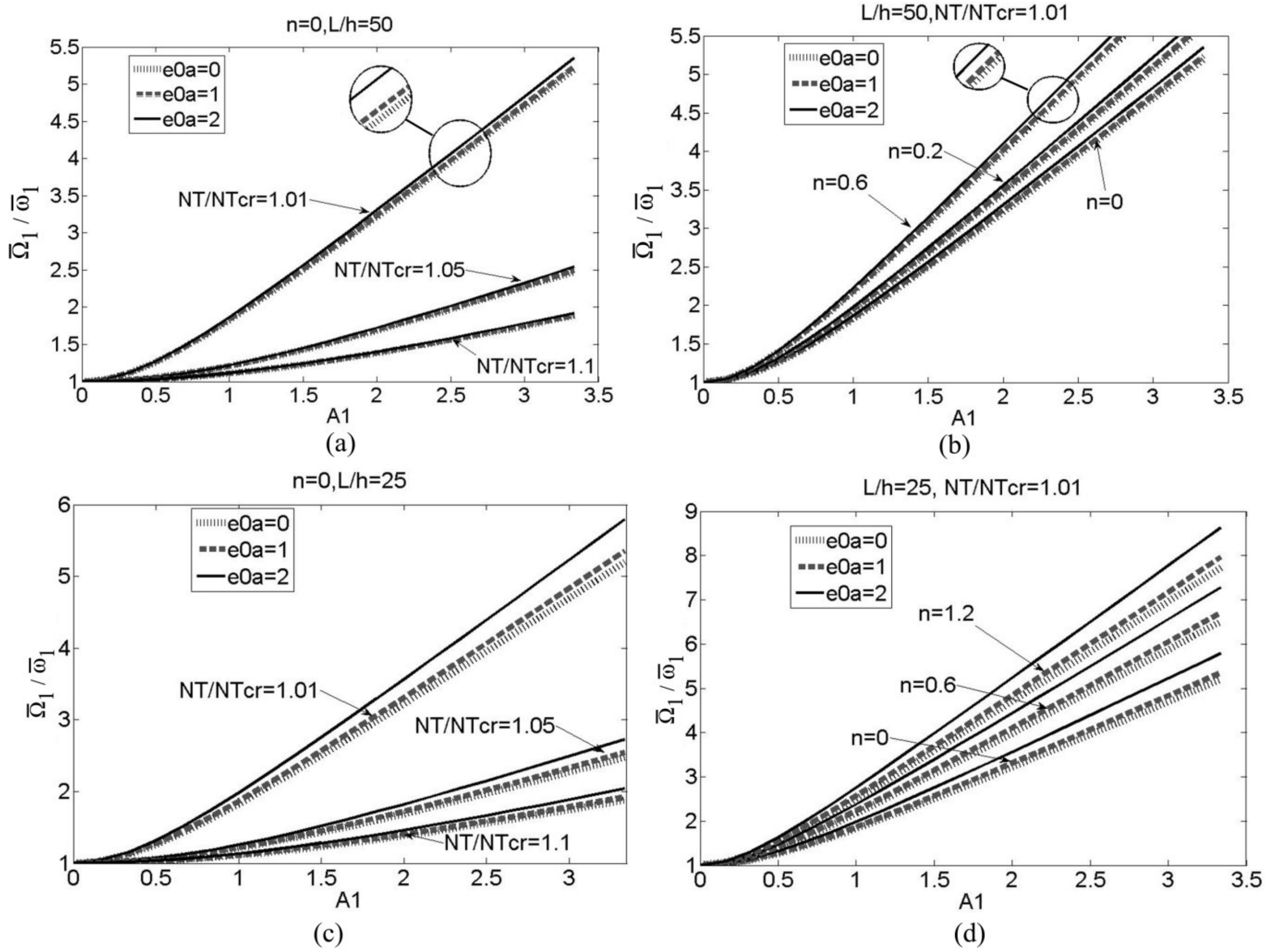

Fig. 2. The effect of initial deflection ($V(\bar{x},0)=A_1\psi_1$) on the ratio of the first nonlinear dimensionless frequency to the first linear one with an increase in a, c) axial load; b, d) index of power law

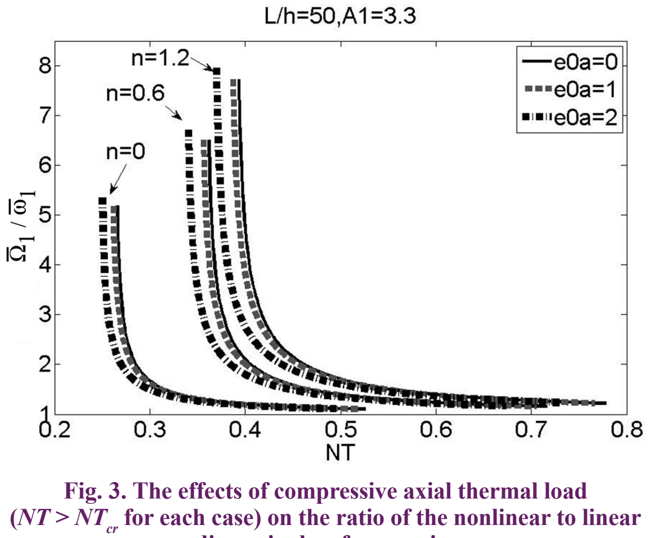

Fig. 3. The effects of compressive axial thermal load ($NT > NT_{cr}$ for each case) on the ratio of the nonlinear to linear dimensionless frequencies

Fig. 3 represents that a nanobeam with a smaller value of small scale parameter experiences the static instability later than a nanobeam with a larger one. Then, the maximum deflection of a nanobeam under a fixed compressive axial load at postbuckling configuration rises as the value of small scale parameter increases. As a result, the value of K_{jj} (Eq. (9c), when $i=j$) rises due to the direct proportionality between K_{jj} and $\bar{W}_s$ which leads to increasing the natural dimensionless frequencies of buckled nanobeams that may explain the reason of decreasing the ratio of the nonlinear to linear dimensionless frequencies with an increase in the small scale parameter value in a fixed value of compressive thermal load. According to Fig. 4, one can deduce that the ratio of length to thickness can affect the ratio of the frequencies obtained based on nonlocal continuum mechanics significantly. If the small scale parameter tends to zero and/or the aspect ratio increases, the influence of the ratio of length to thickness on the variance between nonlinear and linear dimensionless frequencies will decrease. As it is shown, the ratio of the length to thickness has no effect on the ratio of classical nonlinear dimensionless frequencies to classical linear ones because the classical involved parameters (see Eqs. (9b), (9c) and (9a) when $e_0a=0$) are independent on the length of nanobeam.

Based on initial deflection expressed as $V(\bar{x},0)=A_1\psi_1$, the impacts of small scale parameter and aspect ratio combined with initial dimensionless deflection (Table 3), index of power law (Table 4), and compressive axial force (Table 5) on

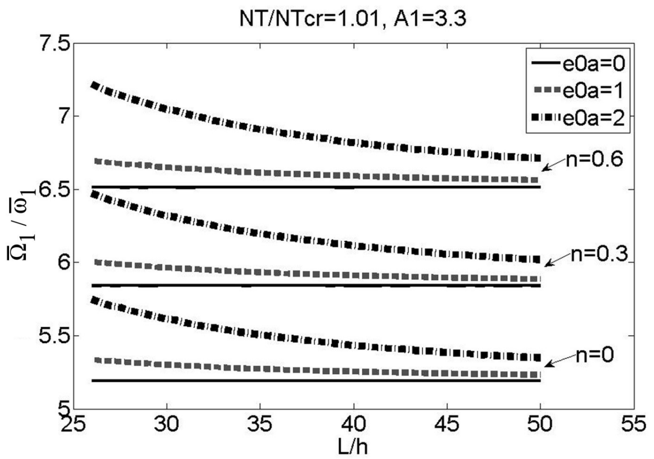

Fig. 4. The effect of aspect ratio on the ratio of nonlinear to linear dimensionless frequencies

the second nonlinear dimensionless frequency to the second linear one of FG nanobeams are listed in Tables 3 to 5. As it is expected, the values of aspect ratio do not affect the second classical nonlinear to linear dimensionless frequencies ratio (i.e. $e_0a = 0$) regardless of the initial dimensionless deflection of nanobeam, material compositions and axial force. If the small scale parameter value rises, the influence of aspect ratio on the second nonlinear to linear dimensionless frequencies ratio will be seen. With an increase in the initial deflection of FG nanobeam, index of power law, and small scale parameter value, the difference between the second nonlinear and linear dimensionless frequencies increases, although a rise in aspect ratio decreases the difference between the second nonclassical nonlinear and linear dimensionless frequencies. According to Table 5, a rise in axial compressive force which is beyond the load bearing capacity of FG nanobeam has no effect on the nonlinear to linear dimensionless frequency ratio.

Fig. 5 shows how the number of linear mode shapes used to estimate lateral dimensionless deflection of vibrating FG

Table 3. The variation of the ratio of the second nonlinear dimensionless frequency to the second linear one with the initial deflection. Initial dimensionless deflection is expressed as $V(\bar{x},0)=A_1\psi_1$, $n = 0$, $N_T/N_{Tcr}=1.01$

	$A_1=0.834$			$A_1=1.67$			$A_1=2.5$		
	$e_0a=0$	$e_0a=1$	$e_0a=2$	$e_0a=0$	$e_0a=1$	$e_0a=2$	$e_0a=0$	$e_0a=1$	$e_0a=2$
$L/h=30$	1.033	1.037	1.052	1.126	1.143	1.197	1.266	1.3	1.4
$L/h=40$	1.033	1.035	1.043	1.126	1.135	1.165	1.266	1.285	1.343
$L/h=50$	1.033	1.034	1.039	1.126	1.132	1.150	1.266	1.278	1.315
$L/h=60$	1.033	1.034	1.037	1.126	1.130	1.143	1.266	1.275	1.300
$L/h=70$	1.033	1.033	1.036	1.126	1.129	1.138	1.266	1.272	1.291
$L/h=80$	1.033	1.033	1.035	1.126	1.128	1.135	1.266	1.271	1.285

Table 4. The variation of the ratio of the second nonlinear dimensionless frequency to the second linear one with index of power law. Initial dimensionless deflection is expressed as $V(\bar{x},0)=A_1\psi_1$, $A_1 = 2.5$, $N_T/N_{Tcr}=1.01$

	$n=0$			$n=0.3$			$n=0.6$		
	$e_0a=0$	$e_0a=1$	$e_0a=2$	$e_0a=0$	$e_0a=1$	$e_0a=2$	$e_0a=0$	$e_0a=1$	$e_0a=2$
$L/h=30$	1.266	1.300	1.406	1.330	1.371	1.498	1.401	1.449	1.599
$L/h=40$	1.266	1.285	1.343	1.330	1.353	1.423	1.401	1.428	1.510
$L/h=50$	1.266	1.278	1.315	1.330	1.345	1.389	1.401	1.418	1.470
$L/h=60$	1.266	1.275	1.300	1.330	1.340	1.371	1.401	1.413	1.449
$L/h=70$	1.266	1.272	1.291	1.330	1.338	1.360	1.401	1.410	1.436
$L/h=80$	1.266	1.271	1.285	1.330	1.336	1.353	1.401	1.408	1.428

Table 5. The variation of the ratio of the second nonlinear dimensionless frequency to the second linear one with axial load. Initial dimensionless deflection is expressed as $V(\bar{x},0)=A_1\psi_1$, $A_1 = 2.5$, $n = 0$

	$N_T/N_{Tcr}=1.01$			$N_T/N_{Tcr}=1.1$			$N_T/N_{Tcr}=1.5$		
	$e_0a=0$	$e_0a=1$	$e_0a=2$	$e_0a=0$	$e_0a=1$	$e_0a=2$	$e_0a=0$	$e_0a=1$	$e_0a=2$
$L/h=30$	1.266	1.300	1.406	1.266	1.300	1.406	1.266	1.300	1.406
$L/h=40$	1.266	1.285	1.343	1.266	1.285	1.343	1.266	1.285	1.343
$L/h=50$	1.266	1.278	1.315	1.266	1.278	1.315	1.266	1.278	1.315
$L/h=60$	1.266	1.275	1.300	1.266	1.275	1.300	1.266	1.275	1.300
$L/h=70$	1.266	1.272	1.291	1.266	1.272	1.291	1.266	1.272	1.291
$L/h=80$	1.266	1.271	1.285	1.266	1.271	1.285	1.266	1.271	1.285

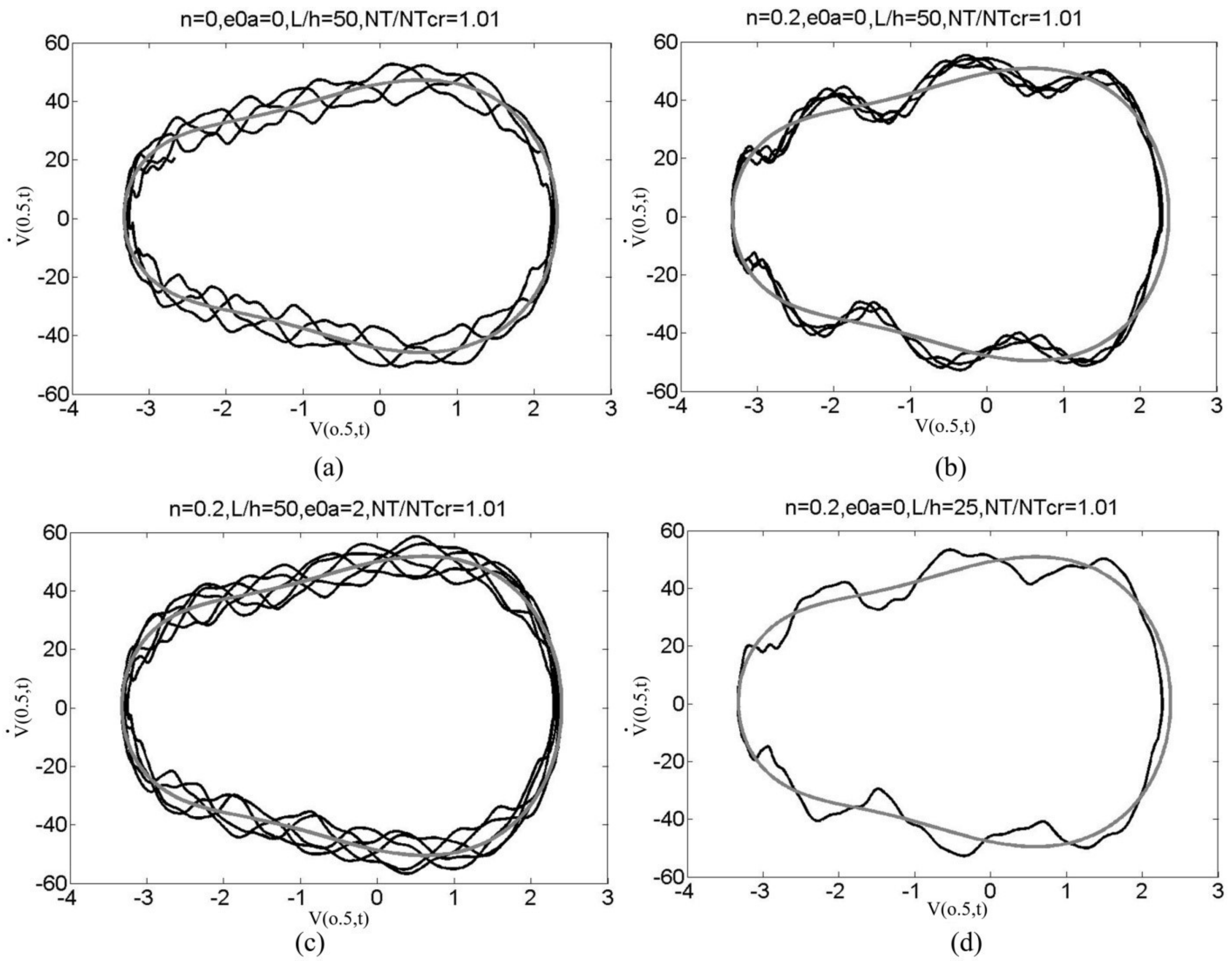

Fig. 5. Phase plane of vibrating FG nanobeam around the first buckling mode.

nanobeam around the first buckling configuration affects the estimation of lateral vibration of FG nanobeam. It is assumed that the initial deflection is expressed as $V(\bar{x},0)=A_1\psi_1$. The gray and the black lines introduce the phase planes estimated by the first linear mode shape and the first eight linear eigenmodes of vibrating FG nanobeam respectively. As it is seen, although the value of small scale parameter, index of power law and aspect ratio impact on the phase plane estimated by the first eight linear eigenmode of vibrating FG nanobeam, the greatest difference between two estimations made by the first eigen-mode and the first eight eigenmodes belongs to the velocity of lateral vibration of buckled FG nanobeam.

The impact of initial deflection upon estimating nonlinear dimensionless frequencies is also investigated. Fig. 6 illustrates the variation of the first two nonlinear to linear dimensionless frequency of buckled FG nanobeams excited by two different initial dimensionless deflection represented by $V(\bar{x},0)=A_1\psi_1$ and $V(\bar{x},0)=A_2\psi_2$ respectively. It is clearly seen that the difference between the first nonlinear dimensionless frequency and the first linear one will be more if the initial dimensionless deflection of FG nanobeam is similar to the first linear eigenmode of FG nanobeam. Similar conclusion can be obtained when the variance between second nonlinear and linear frequency is investigated.

7- Conclusions

This study aims at investigating the nonlinear free vibration of thermally buckled functionally graded nanobeam. It is assumed that material properties are gradually graded in thickness direction. Nonlocal nonlinear Euler-Bernoulli beam theory is used to derive nonlocal governing equation of motion. Linear eigen-modes of FG nanobeam vibrating around the first buckling configuration are employed to change partial differential equation of motion to a system of ordinary differential equations which is solved based on homotopy perturbation method and variational iteration method. The obtained formula that governs the nonlinear frequencies clearly states that initial lateral deformation ($V(\bar{x},0)$) significantly affects nonlinear frequency and lateral response of buckled FG nanobeam. Results show that the difference between nonlinear and linear frequency increases with a rise in the maximum lateral initial deflection, small scale parameter value, and index of power law. Investigating the effect of the ratio of length to thickness on variance between nonlinear and linear dimensionless frequencies shows that aspect ratio has no effect on difference on the ratio of the classical nonlinear to linear dimensionless frequencies,

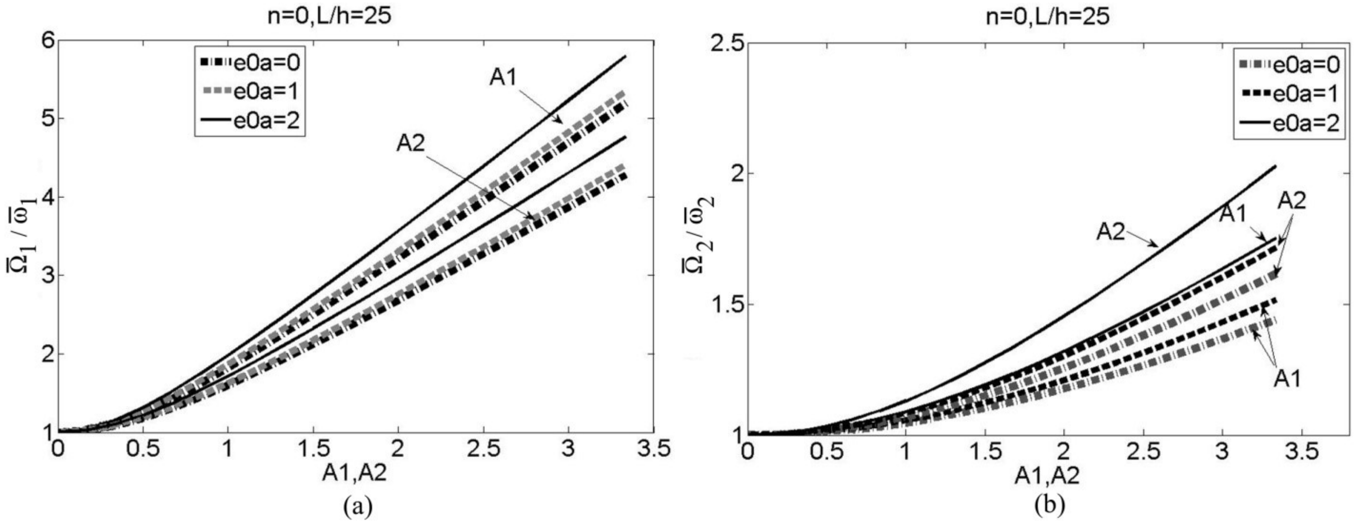

Fig. 6. The impact of initial deflection upon estimating nonlinear frequencies

although the difference between nonlocal nonlinear and linear dimensionless frequencies decreases with a rise in aspect ratio. In contrast to the ratio of the first nonlinear dimensionless frequency to the first linear one which will decrease if compressive axial load increases, the values of compressive axial load which are beyond the load bearing capacity of FG nanobeam do not affect the ratio of the second nonlinear dimensionless frequency to the second linear one. It is also concluded that the difference between the first (second) nonlinear dimensionless frequency and the first (second) linear one will be more if the initial deflection of FG nanobeam is similar to the first (second) linear eigen-mode of FG nanobeam. The results also show how the number of linear mode shape used to estimate the lateral deflection of vibrating FG nanobeam around the first buckling configuration affects the estimation of the lateral vibration of buckled FG nanobeam.

Appendix A

Based on Euler-Bernoulli theory together with von-Karman geometrical nonlinearity in conjunction with nonlocal elasticity theory, one can obtain the governing equation of nonlinear free lateral vibration of FG nanobeams (see Fig. A1 for geometrical details) under pre compressive axial thermal force as:

$$-(e_0a)^2[(-\frac{A_{xx}}{2L}\int_0^L\left(\frac{\partial W}{\partial x}\right)^2 dx+N_T)\frac{\partial^4 W}{\partial x^4}+I_0\frac{\partial^4 W}{\partial x^2\partial t^2}]$$
$$-\left(\frac{A_{xx}}{2L}\int_0^L\left(\frac{\partial W}{\partial x}\right)^2 dx-N_T\right)\frac{\partial^2 W}{\partial x^2} \qquad \text{(A1)}$$
$$+D_{xx}\frac{\partial^4 W}{\partial x^4}+I_0\frac{\partial^2 W}{\partial t^2}=0$$

where z_0 is the distance of the neutral surface of the FG nanobeam from the mid-plane of the FG nanobeam. Also, $W=W(x,t)$ is the transverse displacement of any point on the mid-plane of beam element. $A=(b\times h)$ represents the cross section area of nanobeam. $\rho(z)$, $E(z)$, and $\alpha(z)$ are mass density, Young modulus and coefficient of thermal expansion

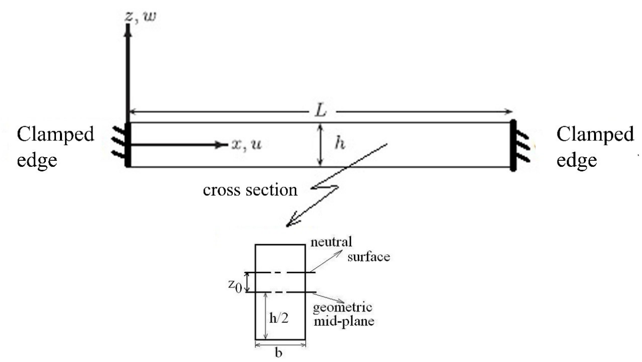

Fig. A1. Geometrical details of FG nanobeam under consideration

respectively which are functionally graded in the thickness direction (Eqs. (2c) to (2e)). The thermal load is shown by N_T. A_{xx} and D_{xx} denote the flexural stiffness and the tensile stiffness, respectively (see Eqs. (2a) and (2b)). I_0 is defined by the second part of Eq. (2b).. Also, the small scale parameter is shown by e_0a.

The following dimensionless variables can be defined to simplify the parametric studies [22]:

$$\bar{x}=\frac{x}{L}, \bar{W}=\frac{W}{r}, \bar{t}=t\sqrt{D_{xx}/I_0L^4}, \qquad \text{(A2)}$$

where L and r are the length of the nanobeam and the gyration radius of the cross section of the beam, respectively.

Then Eq. (A1) can be rewritten as:

$$\frac{\partial^4\bar{W}}{\partial\bar{x}^4}-\left((e_0a)^2 N_T/D_{xx}\right)\frac{\partial^4\bar{W}}{\partial\bar{x}^4}+$$
$$\left((e_0a)^2 A_{xx}r^2/2D_{xx}L^2\right)\left(\int_0^1\left(\partial\bar{W}/\partial\bar{x}\right)^2 d\bar{x}\right)\frac{\partial^4\bar{W}}{\partial\bar{x}^4}+$$
$$\left(\left(N_T L^2/D_{xx}\right)-\left(A_{xx}r^2/2D_{xx}\right)\int_0^1\left(\partial\bar{W}/\partial\bar{x}\right)^2 d\bar{x}\right)\frac{\partial^2\bar{W}}{\partial\bar{x}^2} \qquad \text{(A3)}$$
$$+\left(\frac{\partial^2\bar{W}}{\partial\bar{t}^2}-\frac{(e_0a)^2}{L^2}\frac{\partial^4\bar{W}}{\partial\bar{t}^2\partial\bar{x}^2}\right)=0$$

If the inertia effects are eliminated from Eq. (A3), one can find the governing equation of buckling behavior of FG

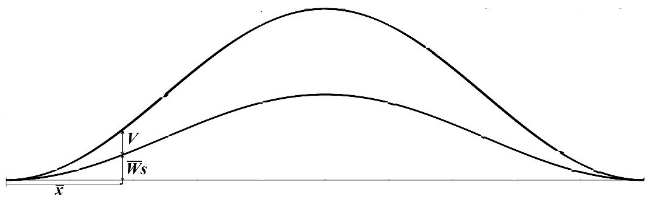

Fig. A2. Buckling configuration (first mode) as well as dynamic disturbance (first mode)

nanobeams as:

$$\begin{aligned}&\left(1-\frac{(e_0a)^2N_T}{D_{xx}}+\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^4\bar{W}_s}{\partial\bar{x}^4}\\&+\left(\frac{N_TL^2}{D_{xx}}-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^2\bar{W}_s}{\partial\bar{x}^2}=0\end{aligned}\qquad\text{(A4)}$$

in which $\bar{W}_s(\bar{x})$ is buckling configuration of nanobeam, which dynamic disturbance ($V(\bar{t},\bar{x})$) occurs around (for example, see Fig. A2).

Substituting $\bar{W}=\bar{W}_s(\bar{x})+V(\bar{t},\bar{x})$ (see Fig. A2) into Eq. (A1), the nonlinear vibration of buckled nanobeam is obtained as:

$$\begin{aligned}&\left(1-\frac{(e_0a)^2N_T}{D_{xx}}+\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^4V}{\partial\bar{x}^4}\\&+\left(\frac{(e_0a)^2A_{xx}r^2}{D_{xx}L^2}\int_0^1\frac{\partial\bar{W}_s}{\partial\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^4\bar{W}_s}{\partial\bar{x}^4}\\&+\left(\frac{N_TL^2}{D_{xx}}-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^2V}{\partial\bar{x}^2}\\&-\left(\frac{A_{xx}r^2}{D_{xx}}\int_0^1\frac{\partial\bar{W}_s}{\partial\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^2\bar{W}_s}{\partial\bar{x}^2}+\frac{\partial^2V}{\partial\bar{t}^2}\\&-\frac{(e_0a)^2}{L^2}\frac{\partial^4V}{\partial\bar{x}^2\partial\bar{t}^2}+\left(\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^4\bar{W}_s}{\partial\bar{x}^4}\\&+\left(\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2d\bar{x}+\frac{(e_0a)^2A_{xx}r^2}{D_{xx}L^2}\int_0^1\frac{\partial\bar{W}_s}{\partial\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^4V}{\partial\bar{x}^4}\\&-\left(\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^2\bar{W}_s}{\partial\bar{x}^2}\\&-\left(\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2d\bar{x}+\frac{A_{xx}r^2}{D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)d\bar{x}\right)\frac{\partial^2V}{\partial\bar{x}^2}\\&+\left(1-\frac{(e_0a)^2N_T}{D_{xx}}+\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^4\bar{W}_s}{\partial\bar{x}^4}\\&+\left(\frac{N_TL^2}{D_{xx}}-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^2\bar{W}_s}{\partial\bar{x}^2}=0\end{aligned}\qquad\text{(A5)}$$

According to Eq. (A4), the summation of the last two terms of Eq. (A5) is equal to zero. Then Eq. (A5) can be rewritten as:

$$\begin{aligned}&\left(1-\frac{(e_0a)^2N_T}{D_{xx}}+\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^4V}{\partial\bar{x}^4}\\&+\left(\frac{(e_0a)^2A_{xx}r^2}{D_{xx}L^2}\int_0^1\frac{\partial\bar{W}_s}{\partial\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^4\bar{W}_s}{\partial\bar{x}^4}\\&+\left(\frac{N_TL^2}{D_{xx}}-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^2V}{\partial\bar{x}^2}\\&-\left(\frac{A_{xx}r^2}{D_{xx}}\int_0^1\frac{\partial\bar{W}_s}{\partial\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^2\bar{W}_s}{\partial\bar{x}^2}+\frac{\partial^2V}{\partial\bar{t}^2}\\&-\frac{(e_0a)^2}{L^2}\frac{\partial^4V}{\partial\bar{x}^2\partial\bar{t}^2}+\left(\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^4\bar{W}_s}{\partial\bar{x}^4}\\&+\left(\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2d\bar{x}+\frac{(e_0a)^2A_{xx}r^2}{D_{xx}L^2}\int_0^1\frac{\partial\bar{W}_s}{\partial\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^4V}{\partial\bar{x}^4}+\\&\left(-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^2\bar{W}_s}{\partial\bar{x}^2}\\&-\left(\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)^2d\bar{x}+\frac{A_{xx}r^2}{D_{xx}}\int_0^1\left(\frac{\partial V}{\partial\bar{x}}\right)\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)d\bar{x}\right)\frac{\partial^2V}{\partial\bar{x}^2}=0\end{aligned}\qquad\text{(A6)}$$

Eq. (A6) is known as nonlinear vibration equation of buckled nanobeam.

Appendix B

Dropping the quadratic and cubic nonlinear terms from Eq. (A6), the linear vibration of thermally buckled FG nanobeams can be obtained as [22]:

$$\begin{aligned}&\left(1-\frac{(e_0a)^2N_T}{D_{xx}}+\frac{(e_0a)^2A_{xx}r^2}{2D_{xx}L^2}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^4V}{\partial\bar{x}^4}\\&+\left(\frac{(e_0a)^2A_{xx}r^2}{D_{xx}L^2}\int_0^1\frac{\partial\bar{W}_s}{\partial\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^4\bar{W}_s}{\partial\bar{x}^4}\\&+\left(\frac{N_TL^2}{D_{xx}}-\frac{A_{xx}r^2}{2D_{xx}}\int_0^1\left(\frac{\partial\bar{W}_s}{\partial\bar{x}}\right)^2d\bar{x}\right)\frac{\partial^2V}{\partial\bar{x}^2}\\&-\left(\frac{A_{xx}r^2}{D_{xx}}\int_0^1\frac{\partial\bar{W}_s}{\partial\bar{x}}\frac{\partial V}{\partial\bar{x}}d\bar{x}\right)\frac{\partial^2\bar{W}_s}{\partial\bar{x}^2}+\frac{\partial^2V}{\partial\bar{t}^2}\\&-\frac{(e_0a)^2}{L^2}\frac{\partial^4V}{\partial\bar{x}^2\partial\bar{t}^2}=0\end{aligned}\qquad\text{(B1)}$$

$e^{-\bar{\omega}i\bar{t}}\phi(\bar{x})$ for $V(\bar{t},\bar{x})$ can be substituted in Eq. (B1) due to its linear nature and ordinary differential equation governing mode shape of linear vibration ($\phi(\bar{x})$) can be obtained as follows:

$$\begin{aligned}&\frac{d^4\varphi(\bar{x})}{d\bar{x}^4}+\left(\lambda_0^2+\bar{\omega}^2\lambda_3\right)\frac{d^2\varphi(\bar{x})}{d\bar{x}^2}-\bar{\omega}^2\lambda_4\varphi(\bar{x})=\\&-\left(\int_0^1\frac{d\bar{W}_s}{d\bar{x}}\frac{d\varphi(\bar{x})}{d\bar{x}}d\bar{x}\right)\left(\lambda_1\frac{d^4\bar{W}_s}{d\bar{x}^4}-\lambda_2\frac{d^2\bar{W}_s}{d\bar{x}^2}\right)\end{aligned}\qquad\text{(B2)}$$

where $\bar{\omega}=\omega\sqrt{I_0L^4/Dxx}$ is dimensionless natural frequency of buckled beam and [22]

$$\lambda_0^2 = \frac{\left(N_T L^2/D_{xx}\right) - \left(A_{xx} r^2/2D_{xx}\right)\int_0^1 \left(\partial \bar{W}_s/\partial \bar{x}\right)^2 d\bar{x}}{1-\left(\left(e_0 a\right)^2 N_T/D_{xx}\right)+\left(\left(e_0 a\right)^2 A_{xx} r^2/2D_{xx}L^2\right)\int_0^1 \left(\partial \bar{W}_s/\partial \bar{x}\right)^2 d\bar{x}} \quad \text{(B3)}$$

$$\lambda_1 = \frac{\left(\left(e_0 a\right)^2 A_{xx} r^2/L^2 D_{xx}\right)}{1-\left(\left(e_0 a\right)^2 N_T/D_{xx}\right)+\left(\left(e_0 a\right)^2 A_{xx} r^2/2D_{xx}L^2\right)\int_0^1 \left(\partial \bar{W}_s/\partial \bar{x}\right)^2 d\bar{x}} \quad \text{(B4)}$$

$$\lambda_2 = \frac{\left(A_{xx} r^2/D_{xx}\right)}{1-\left(\left(e_0 a\right)^2 N_T/D_{xx}\right)+\left(\left(e_0 a\right)^2 A_{xx} r^2/2D_{xx}L^2\right)\int_0^1 \left(\partial \bar{W}_s/\partial \bar{x}\right)^2 d\bar{x}} \quad \text{(B5)}$$

$$\lambda_3 = \frac{\left(\left(e_0 a\right)^2/L^2\right)}{1-\left(\left(e_0 a\right)^2 N_T/D_{xx}\right)+\left(\left(e_0 a\right)^2 A_{xx} r^2/2D_{xx}L^2\right)\int_0^1 \left(\partial \bar{W}_s/\partial \bar{x}\right)^2 d\bar{x}} \quad \text{(B6)}$$

$$\lambda_4 = \frac{(1)}{1-\left(\left(e_0 a\right)^2 N_T/D_{xx}\right)+\left(\left(e_0 a\right)^2 A_{xx} r^2/2D_{xx}L^2\right)\int_0^1 \left(\partial \bar{W}_s/\partial \bar{x}\right)^2 d\bar{x}} \quad \text{(B7)}$$

Based on the Polynomial-based Differential Quadrature method (PDQ) [47], a non-uniform mesh can be used to divide computational domain $0 \le \bar{x} \le 1$ into (N-1) intervals. The mesh points are placed at the shifted Chebyshev-Gauss-Lobatto points [48],

$$\bar{x}_i = 0.5\left[1-\cos\left(\pi\left(i-1\right)/N-1\right)\right],\ i=1,2,...,N \quad \text{(B8)}$$

Quan and Chang's Approach [47] is used to compute weighting coefficients for the first order derivative $(d\phi/d\bar{x})(\bar{x})$ at any grid point as

$$\frac{d\varphi}{d\bar{x}}(\bar{x}_i) = \sum_{j=1}^{N} a_{ij}\varphi(\bar{x}_i),\ or\ \left\{\frac{d\varphi}{d\mathbf{x}}\right\} = \mathbf{A}\{\varphi\} \quad \text{(B9)}$$

where

$$a_{ij} == \frac{1}{\bar{x}_j - \bar{x}_i}\prod_{k=1,k\ne i,j}^{N}\frac{\bar{x}_i - \bar{x}_k}{\bar{x}_j - \bar{x}_k},\quad i \ne j \quad \text{(B10)}$$

$$a_{ii} = \sum_{k=1,k\ne i}^{N}\frac{1}{\bar{x}_i - \bar{x}_k} \quad \text{(B11)}$$

To implement boundary conditions, the modification of the weighting coefficient matrices method [47, 48] is employed. According to this method, the first and the last rows of the matrix $\mathbf{A}=[a_{ij}]$ must be replaced with zero to satisfy derivative conditions ($d\phi/d\bar{x}|\bar{x}=0$ and $d\phi/d\bar{x}|\bar{x}=1$). The new matrix is named $\tilde{\mathbf{A}}$. Using $\mathbf{A}$ and $\tilde{\mathbf{A}}$, higher order derivatives are defined as follows [47]:

$$\left\{\frac{d^2\varphi}{d\mathbf{x}^2}\right\} = \mathbf{A}\tilde{\mathbf{A}}\{\varphi\},\ or\ \left\{\frac{d^2\varphi}{d\mathbf{x}^2}\right\} = \tilde{\mathbf{B}}\{\varphi\} \quad \text{(B12)}$$

$$\left\{\frac{d^4\varphi}{d\mathbf{x}^4}\right\} = \left(\mathbf{A}\mathbf{A}\right)\tilde{\mathbf{B}}\varphi,\ or\ \left\{\frac{d^4\varphi}{d\mathbf{x}^4}\right\} = \tilde{\mathbf{D}}\{\varphi\} \quad \text{(B13)}$$

Substituting Eqs. (B8) to (B13) and post-buckling configuration (Eq. (4) into Eq. (B2) and applying the remainder of the boundary conditions ($\phi(0)=0$ and $\phi(1)=0$), the following discretized equation can be obtained as:

$$\left(\sum_{j=2}^{N-1}\left(\tilde{d}_{ij} + \lambda_0^2\tilde{b}_{ij} + q_{ij}\right)\varphi_j\right) = \bar{\varpi}^2\left(\sum_{j=2}^{N-1}\left(\lambda_4\delta_{ij} - \lambda_3\tilde{b}_{ij}\right)\varphi_j\right) \quad \text{(B14)}$$

where $\tilde{b}_{ij}$, $\tilde{d}_{ij}$ and δ_{ij} are the components of $\tilde{\mathbf{B}}$, $\tilde{\mathbf{D}}$ and Kronecker delta respectively and

$$q_{ij} = -\left(\lambda_2 + 2\pi\lambda_1\right)\frac{d^2\bar{W}_s}{d\bar{x}^2}(\bar{x}_i)S_j \quad \text{(B15)}$$

where

$$S_j = \sum_{i=1}^{N-1}\left(\frac{d^2\bar{W}_s}{d\bar{x}^2}(\bar{x}_i)\tilde{a}_{ij} + \frac{d^2\bar{W}_s}{d\bar{x}^2}(\bar{x}_{i+1})\tilde{a}_{(i+1)j}\right)\frac{\bar{x}_{i+1} - \bar{x}_i}{2} \quad \text{(B15)}$$

By solving the eigen-frequency equation (Eq. (B14)), one can obtain the dimensionless linear frequencies of buckled beam ($\bar{\varpi}^2$) as well as corresponding mode shapes ($\boldsymbol{\psi}^T=\{0\ ,\ \boldsymbol{\varphi}^T\ ,\ 0\}$).

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