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On Sobolev spaces and density theorems on Finsler manifolds

Behroz Bidabad*[a], Alireza Shahi[b]

[a] *Department of Mathematics and Computer Science, Amirkabir University of Technology, 424, Hafez Ave., Tehran 15914, Iran*
[b] *Department of Mathematics and Computer Science, Amirkabir University of Technology, 424, Hafez Ave., Tehran 15914, Iran*

ABSTRACT: Here, a natural extension of Sobolev spaces is defined for a Finsler structure F and it is shown that the set of all real C^∞ functions with compact support on a forward geodesically complete Finsler manifold (M, F), is dense in the extended Sobolev space $H_1^p(M)$. As a consequence, the weak solutions u of the Dirichlet equation $\Delta u = f$ can be approximated by C^∞ functions with compact support on M. Moreover, let $W \subset M$ be a regular domain with the C^r boundary ∂W, then the set of all real functions in $C^r(W) \cap C^0(\overline{W})$ is dense in $H_k^p(W)$, where $k \leq r$. Finally, several examples are illustrated and sharpness of the inequality $k \leq r$ is shown.

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1. Introduction

A Sobolev space is a vector space of functions endowed with a norm which is a combination of L^p-norm of the function itself and its derivatives up to a certain order. Its objectives are to deal with some basic PDE problems on manifolds equipped with Riemannian metrics. For instance, the Yamabe problem asserts that for any compact Riemannian manifold (M, g) of dimension $n \geq 3$, there always exists a metric with constant scalar curvature. Clearly, solutions of Yamabe elliptic equation on Riemannian manifolds are laying in the Sobolev space $H_1^2(M)$, see for instance [3]. Currently, the question of elliptic equations on some natural extensions of Riemannian spaces, particularly on Finsler manifolds are extensively studied, see for instance [4, 9, 13, 14, 15]. Another natural question is to wonder whether a function can be approximated by another one with better properties. Density problems permit to investigate this question and find conditions under which a function on a Sobolev space can be approximated by smooth functions with compact support.

Historically, one of the significant density theorems is proved by S. B. Myers [17] in 1954 for compact Riemannian manifolds and then in 1959 by M. Nakai for finite-dimensional Riemannian manifolds. Next, in 1976, T. Aubin has investigated density theorems on Riemannian manifolds, cf. [3]. Y. Ge and Z. Shen [11] defined a canonical energy functional on Sobolev spaces and investigated the eigenvalues and eigenfunctions related to this functional, on a compact reversible Finsler manifold. Next, Y. Yang has defined a Sobolev space on a reversible Finsler manifold (M, F) by using the osculating Riemannian metric and the corresponding Levi-Civita connection on the underlying manifold M, cf., [21]. In 2011 the Myers-Nakai theorem is extended to the Finsler manifolds of class C^k, where $k \in N \cup \{\infty\}$, cf., [13]. Lately, S. Ohta has studied many Sobolev inequalities in Finsler spaces, see [18].

Recently the present authors have studied some natural extensions of Riemannian results, more or less linked to this question see for instance [7, 8, 10].

In the present work, a natural extension of Riemannian metrics is considered, and solutions to the above questions are studied. More intuitively, a Sobolev space is defined by considering a Riemannian metric on the sphere bundle

**Corresponding author.*
E-mail addresses: bidabad@aut.ac.ir, alirezashahi@aut.ac.ir

SM, rather than the manifold M, and following Theorems are proved. Denote by $D(M)$ the set of all real C^∞ functions with compact support on M and let $\overset{\circ}{H}{}_1^p(M)$ be the closure of $D(M)$ in $H_1^p(M)$, where (M, F) is an $n-$dimensional C^∞ Finsler manifold, $p \geq 1$ a real number, k a non-negative integer and $H_k^p(M)$ certain Sobolev space determined by the Finsler structure F.

Theorem 1.1. Let (M, F) be a forward (or backward) geodesically complete Finsler manifold, then $\overset{\circ}{H}{}_1^p(M) = H_1^p(M)$.

As an application for a real function $f : M \longrightarrow I\!R$ on a compact C^∞, reversible Finsler manifold for which $\int_M f dv_F = 0$, the weak solutions u of the Dirichlet equation $\Delta u = f$ can be approximated by C^∞ functions with compact support on M.

We provide also some examples.

Let $W \subset M$ be an s-dimensional regular domain with C^r boundary ∂W, then $(\overline{W}, F)$ is called a Finsler manifold with C^r boundary.

Theorem 1.2. Let $(\overline{W}, F)$ be a compact Finsler manifold with C^r boundary. Then $C^r(\overline{W})$ is dense in $H_k^p(W)$, for $k \leq r$.

Next, using a counterexample we show that the assumption $k \leq r$, in Theorem 1.2 is sharp and can not be omitted. As a consequence of the above density theorems, we can approximate solutions of partial differential equations on a Sobolev space determined by F, with C^∞ or C^r functions on (M, F) and hence study some recent problems on Finsler geometry, for instance, Ricci flow, Yamabe flow, etc.

It should be recalled that the new definition of Sobolev space in Finsler geometry introduced in the present work, is a more general definition and has the following advantages.

- This definition of Sobolev space, reduces to that of Ge and Shen, in the case $k = 1$ and $p = 2$, provided the underlying manifold is closed and the Finsler structure is reversible.
- In this approach, the reversibility condition on the Finslerian structure is not required.
- The present definition applies also to the geometric objects defined on TM.
- This approach makes possible to study the Sobolev norms of horizontal curvature tensor and its covariant derivatives up to the certain order k.
- This approach permits to generalize Theorem1.1 for $H_k^p(M)$, where $k \geq 2$.

We adopt here the notations and terminologies of [2, 5], and [19] and recall that all the Finsler manifolds in the present work are assumed to satisfy in Remark 3.1.

2. Preliminaries and terminologies

Let M be a connected differentiable manifold of dimension n. Denote the bundle of tangent vectors of M by $\pi_1 : TM \longrightarrow M$, the fiber bundle of non-zero tangent vectors of M by $\pi : TM_0 \longrightarrow M$ and the pulled-back tangent bundle by π^*TM. A point of TM_0 is denoted by $z = (x, y)$, where $x = \pi z \in M$ and $y \in T_{\pi z}M$. Let (x, U) be a local chart on M and (x^i, y^i) the induced local coordinates on $\pi^{-1}(U)$, where $\mathbf{y} = y^i \frac{\partial}{\partial x^i} \in T_{\pi z}M$, and i running over the range $1, 2, ..., n$. A (globally defined) *Finsler structure* on M is a function $F : TM \longrightarrow [0, \infty)$ with the following properties; F is C^∞ on the entire slit tangent bundle $TM\backslash 0$; $F(x, \lambda y) = \lambda F(x, y)$ $\forall \lambda > 0$; the $n \times n$ Hessian matrix $(g_{ij}) = \frac{1}{2}([F^2]_{y^i y^j})$ is positive-definite at every point of TM_0. The pair (M, F) is called a *Finsler manifold*. Given the induced coordinates (x^i, y^i) on TM, coefficients of spray vector field are defined by $G^i = 1/4 g^{ih}(\frac{\partial^2 F^2}{\partial y^h \partial x^j} y^j - \frac{\partial F^2}{\partial x^h})$. One can observe that the pair $\{\delta/\delta x^i, \partial/\partial y^i\}$ forms a horizontal and vertical frame for TTM, where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}$, $G_i^j := \frac{\partial G^j}{\partial y^i}$. Denote by SM the sphere bundle, where $SM := \bigcup_{x \in M} S_x M$ and $S_x M := \{y \in T_x M | F(y) = 1\}$. The *Sasakian metric* on SM is defined by

$$\hat{g} = \delta_{ab} w^a \otimes w^b + \delta_{\alpha\beta} w^{n+\alpha} \otimes w^{n+\beta}, \tag{2.1}$$

where $a, b = 1, .., n$ and $\alpha, \beta = 1, ..., n-1$, and $\{w^a, w^{n+\alpha}\}$ is an ordered orthonormal coframe on SM, cf., [6] The volume element dV_{SM} of SM with respect to the Sasakian metric $\hat{g}$ is

$$\begin{aligned} dV_{SM} &= w^1 \wedge ... \wedge w^{2n-1} \\ &= \sqrt{\det(g_{ij})} dx^1 \wedge ... \wedge dx^n \wedge w^{n+1} \wedge ... \wedge w^{2n-1}. \end{aligned}$$

This volume element can be rewritten as follows,

$$dV_{SM} = \Omega d\tau \wedge dx, \tag{2.2}$$

where, $\Omega = \det(\frac{g_{ij}}{F})$, $d\tau = \sum_{k=1}^{n} (-1)^{k-1} y^k dy^1 \wedge ...\hat{dy^k} \wedge ... \wedge dy^n$ and dx is the n-form $dx = dx^1 \wedge ... \wedge dx^n$, cf. [12]. We have a volume form on (M, F)

$$dV_F = (\frac{1}{c_{n-1}} \int_{S_x M} \Omega d\tau) dx, \tag{2.3}$$

where c_{n-1} denotes the volume of the unit Euclidean sphere S^{n-1}, cf. [12].

Let $\sigma : [a, b] \longrightarrow M$ a piecewise C^∞ curve with the velocity $\frac{d\sigma}{dt} = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i} \in T_{\sigma(t)}(M)$. Its integral length is defined by $L(\sigma) = \int_a^b F(\sigma, \frac{d\sigma}{dt}) dt$. For $x_1, x_2 \in M$ denote by $\Gamma(x_1, x_2)$ the collection of all piecewise C^∞ curves $\sigma : [a, b] \longrightarrow M$ with $\sigma(a) = x_1$ and $\sigma(b) = x_2$ and by $d(x_1, x_2)$ the metric distance from x_1 to x_2,

$$d(x_1, x_2) = \inf_{\Gamma(x_1, x_2)} L(\sigma). \tag{2.4}$$

Lemma 2.1. [5] Let (M,F) be a Finsler manifold. At any point $x \in M$, there exists a local coordinate system (ϕ, U) such that the closure of U is compact, $\phi : \overline{U} \longrightarrow I\!R^n$, $\phi(x) = 0$ and ϕ maps U diffeomorphically onto an open ball of $I\!R^n$.

A Finsler manifold is said to be forward (resp. backward) geodesically complete if every geodesic $\gamma(t)$, $a \leq t < b$, parameterized to have constant Finslerian speed, can be extended to a geodesic on $a \leq t < \infty$. (resp. $-\infty < t \leq b$). If the Finsler structure F is reversible, then d is symmetric. In this case, forward completeness is equivalent to backward completeness. Compact Finsler manifolds at the same time are both forward and backward complete, whether d is symmetric or not.

3. A Sobolev space on Finsler manifolds

Let (M, F) be a C^∞ Finsler manifold. For any real function u on M, we denote again $u \circ \pi$ by u. The jth covariant derivative of u is denoted by $\nabla^j u$, where ∇ is a horizontal covariant derivative of Cartan connection, j is a nonnegative integer, hence $\nabla^0 u = u$. Let us denote the inner scalar product on SM with respect to the Sasakian metric (2.1) by $(.,.)$ and $|\nabla^j u|^2 = (\nabla^j u, \nabla^j u)$. We denote by $C_k^p(M)$ the space of smooth functions $u \in C^\infty(M)$ such that $|\nabla^j u| \in L^p(SM)$ for any j run over the range $0, 1, ..., k$ and $p \geq 1$, that is

$$\begin{aligned} C_k^p(M) = \; & \{u \in C^\infty(M) : \forall j = 0, 1, ..., k, \\ & \textstyle\int_{SM} [(\nabla^j u, \nabla^j u)]^{\frac{p}{2}} dV_{SM} < \infty\}. \end{aligned}$$

Remark 3.1. It is well known that S^{n-1} is diffeomorphic to the $S_x M$, where $x \in M$. Let A be the radial projection from $S_x M = \{(y^i) \in I\!R^n : F(x, y^i \frac{\partial}{\partial x^i}) = 1\} \subset I\!R^n$ onto the unit sphere $S^{n-1} \subset I\!R^n$ and $(\det J(A))$ determinant of its Jacobian. Everywhere in this paper, we assume that there exists a positive real number $R > 0$ such that $(\det J(A)) \sqrt{\det g_{ij}} \geq \frac{R}{c_{n-1}}$, where c_{n-1} is the volume of S^{n-1}. Due to the dependence of Cartan covariant derivatives to the direction on Finsler cases we have to consider the above inequality. Note that every compact Finsler manifold satisfies the preceding inequality, due to the compactness of SM.

Proposition 3.1. [16] Let (x, Ω) be a local coordinate chart on M and $f : SM \subset TM_0 \to I\!R$ an integrable function with support in $\pi^{-1}(\Omega)$. Then we have

$$\int_{SM} f(x, y) dV_{SM} = \int_\Omega (\int_{S^{n-1}} f(x, \frac{y}{F}) \frac{det(g_{ij})}{F^n} d\sigma) dx, \tag{3.1}$$

where $d\sigma$ is the standard volume form on S^{n-1}, $dx = dx^1 \wedge .... \wedge dx^n$, $y = y(x,\theta)$ for $(x,\theta) \in \Omega \times S^{n-1}$ and $\theta = (\theta^1, ..., \theta^n)$ are local coordinate on S^{n-1}.

Definition 3.1. The Sobolev space $H_k^p(M)$ is the completion of $C_k^p(M)$ with respect to the norm

$$\| u \|_{H_k^p(M)} = \sum_{j=0}^{k} (\int_{SM} [(\nabla^j u, \nabla^j u)]^{\frac{p}{2}} dV_{SM})^{\frac{1}{p}},$$

where $p \geq 1$ is a real number.

Let $f : M \to I\!R$ be a real function, then using the volume form dV_F defined by (2.3) we can consider the space of L^p−norm as follows

$$L^p(M) = \{f : M \to I\!R \ \ is\ measurable : \int_M f^p dV_F < \infty\}. \tag{3.2}$$

Lemma 3.1. The Finslerian Sobolev space $H_k^p(M)$ is a subspace of $L^p(M)$.

Proof. For all $u \in H_k^p(M)$, by Definition 3.1 we have $\int_{SM} |u|^p dV_{SM} < \infty$. An appropriate choice of $\{\Omega_i\}_{i=1}^{\infty}$, together with Proposition 3.1 and relations (2.2) and (2.3) leads to

$$\begin{aligned}\int_{SM} |u|^p dV_{SM} &= \sum_{i=1}^{\infty} \int_{\pi^{-1}(\Omega_i)} |u|^p dV_{SM} \\ &= \sum_{i=1}^{\infty} \int_{\Omega_i} \int_{S^{n-1}} |u|^p \frac{det(g_{ij})}{F^n} d\sigma dx \\ &= \sum_{i=1}^{\infty} \int_{\Omega_i} \int_{S_x M} |u|^p \frac{det(g_{ij})}{F^n} (\det J(A)) \\ &dV_{S_x M} dx \\ &= \sum_{i=1}^{\infty} \int_{\Omega_i} \int_{S_x M} |u|^p \frac{det(g_{ij})}{F^n} (\det J(A)) \\ &\sqrt{\det g_{ij}} d\tau dx.\end{aligned}$$

By assumption, there exists $R > 0$ such that $(\det J(A))\sqrt{\det g_{ij}} \geq \frac{R}{c_{n-1}}$, hence

$$\begin{aligned}\sum_{i=1}^{\infty} \int_{\Omega_i} \int_{S_x M} |u|^p \frac{det(g_{ij})}{F^n} (\det J(A)) \ \sqrt{\det g_{ij}} d\tau dx \\ \geq R \sum_{i=1}^{\infty} \int_{\Omega_i} |u|^p (\frac{1}{c_{n-1}} \int_{S_x M} \Omega d\tau) dx \\ = R \sum_{i=1}^{\infty} \int_{\Omega_i} |u|^p dV_F \\ = R \int_M |u|^p dV_F.\end{aligned}$$

Therefore, $u \in L^p(M)$ and the proof is complete. □

The following example shows that the assumption $(\det J(A)) \sqrt{\det g_{ij}} \geq \frac{R}{c_{n-1}}$ is essential and can not be dropped.

Example 1. Let $M = I\!R^2$, hence $SM \simeq I\!R^2 \times S^1$. Consider the function $f : I\!R^2 \to I\!R$ defined by

$$f(x,y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & o.w. \end{cases}.$$

Choose a metric on M, such that the fibers $S_{(x,y)}M$ of SM have the radius $r_x = e^{-x^2}$. Let $U = I\!R \times (0,1)$ which leads to

$$\int_{SM} f dV_{SM} = \int_{U\times S_{(x,y)}M} d\theta dxdy = \int_U (\int_{S_{(x,y)}M} d\theta)dxdy$$
$$= \int_0^1 \int_{-\infty}^{+\infty} (2\pi e^{-x^2})dxdy < \infty.$$

On the other hand

$$\int_M f d\mu = \int_0^1 \int_{-\infty}^{+\infty} 1 dxdy = \infty.$$

Therefore $f \notin L^1(I\!R^2)$.

Remark 3.2. Ge and Shen in [11], defined a norm of Sobolev spaces on a closed reversible Finsler manifold, for $k = 1$ and $p = 2$ as follows

$$\| u \|_{GS} = (\int_M u^2 dV_F)^{\frac{1}{2}} + (\int_M (F(\nabla u))^2 dV_F)^{\frac{1}{2}}, \tag{3.3}$$

where $(\nabla u)(x)$ is the gradient of u in $x \in M$. A similar argument as in the proof of Lemma 3.1, shows that the definition of Sobolev space given in the present work, reduces to that of [11], for $k = 1$ and $p = 2$, on a closed reversible Finsler manifold.

Remark 3.3. In Definition 3.1 we use an inner product on SM to define a Sobolev space and naturally it has a structure of vector space, while the definition given in [11] is not a vector space on complete Finsler manifolds in general. In fact, Kristály and Rudas show that on $(B^n(1), F)$, where $B^n(1)$ is an $n-$dimensional unit ball of $I\!R^n$ and F is a Funk metric, the Sobolev space defined in [11] has no more structure of a vector space, due to the irreversibility of F, cf., [14].

Let J be a nonnegative, real-valued function, in the space of C^∞ functions with compact support on $I\!R^n$, denoted by $C_0^\infty(I\!R^n)$ and having properties :
(i)$J(x) = 0$ if $|x| \geq 1$.
(ii)$\int_{I\!R^n} J(x)dx = 1$.
Consider the function $J_\epsilon(x) = \epsilon^{-n} J(\frac{x}{\epsilon})$ which is nonnegative in $C_0^\infty(I\!R^n)$ and satisfies :
(i) $J_\epsilon(x) = 0$ if $|x| \geq \epsilon$,
(ii) $\int_{I\!R^n} J_\epsilon(x)dx = 1$.
J_ϵ is called a *mollifier* and the convolution $J_\epsilon * u(x) := \int_{I\!R^n} J_\epsilon(x-y)u(y)dy$, defined for the function u is called a *regularization* of u.

Lemma 3.2. [1] Let u be a function defined on $\Omega \subset I\!R^n$ and vanishes identically outside the domain Ω :
(a) If $u \in L^1_{Loc}(\overline{\Omega})$ then $J_\epsilon * u \in C_0^\infty(I\!R^n)$.
(b) If $supp(u) \subset \Omega$, then $J_\epsilon * u \in C_0^\infty(\Omega)$ provided $\epsilon < dist(supp(u), \partial\Omega)$.
(c) If $u \in L^p(\Omega)$ where $1 \leq p < \infty$ then $J_\epsilon * u \in L^p(\Omega)$. Moreover,
$\| J_\epsilon * u \|_p \leq \| u \|_p$ and $\lim_{\epsilon\to 0^+} \| J_\epsilon * u - u \|_p = 0$

Lemma 3.3. [1] Let $u \in H_k^p(\Omega)$ and $1 \leq p < \infty$. If $\Omega' \subset\subset \Omega \subset I\!R^n$, that is $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is a compact subset of $I\!R^n$, then $\lim_{\epsilon\to o^+} J_\epsilon * u = u$ in $H_k^p(\Omega')$.

4. Density theorems on Finsler manifolds

In the previous section, we set necessary tools on SM which permits to use Aubin's techniques in Finsler geometry, cf. [3]. Let (M, F) be a Finsler manifold and $D(M)$ the space of C^∞ functions with compact support on M. In this section, we use Hopf-Rinow's theorem to introduce the first density theorem on boundaryless Finsler manifolds and investigate another density theorem for Finslerian manifolds with C^r boundary.

4.1. Case of manifolds without boundary

Proposition 4.1. Let (M, F) be a forward geodesically complete Finsler manifold, then any function $\phi \in H_1^p(M)$ can be approximated by functions with compact support on M.

Proof. Let $\phi \in H_1^p(M)$, then by Lemma 3.1 we have $\phi \in L^p(M)$ and hence it can be approximated by C^∞ functions. Therefore $C^\infty(M) \cap H_1^p(M)$ is dense in $H_1^p(M)$. To prove Proposition 4.1 it suffices to show the assertion for $\phi \in C^\infty(M) \cap H_1^p(M)$. Let $\phi \in C^\infty(M) \cap H_1^p(M)$ and fix a point $x_0 \in M$. By Hopf-Rinow's theorem for forward geodesically complete Finsler manifolds, every pair of points in M, containing x_0 can be joined by a minimizing geodesic emanating from x_0. Let us consider the Finslerian distance function (2.4), and the sequence of functions $\phi_j(x) = \phi(x) f(d(x_0, x) - j)$, where $f : I\!R \longrightarrow I\!R$ is defined by

$$f(t) = \begin{cases} 1 & t \leq 0 \\ 1-t & 0 < t < 1 \\ 0 & t \geq 1 \end{cases} . \tag{4.1}$$

Clearly, f is a continuous and decreasing function and it is differentiable almost everywhere on $I\!R$. We should prove, each $\phi_j(x)$ has a compact support. If $d(x_0, x) \leq j$ then $d(x_o, x) - j \leq 0$ or $f(d(x_o, x) - j) = 1$ hence $\phi_j(x) = \phi(x)$. If $d(x_0, x) \geq j + 1$ then $d(x_o, x) - j \geq 1$ or $\phi_j(x) = 0$. Thus each ϕ_j has a compact support and obviously $\lim_{j\to\infty} \phi_j(x) = \phi(x)$. This completes the proof . □

Proposition 4.2. Let (M, F) be a forward geodesically complete Finsler manifold, then any function $\phi \in H_1^p(M)$ can be approximated by functions with compact support in $H_1^p(M)$.

Proof. Let $\phi \in H_1^p(M) \cap C^\infty(M)$ be an arbitrary function and $\{\phi_j\}_{j=1}^\infty$ the sequence of related functions given in the proof of Proposition 4.1. We show that $\| \phi_j - \phi \|_{H_1^p(M)}$ tends to zero as $j \to \infty$. Since ϕ is a C^∞ real function on M and $\phi(x) \in H_1^p(M)$, $(\nabla\phi)_{(x,y)}$ exists and bounded almost everywhere on $(x, y) \in SM$. It is well known on forward geodesically complete Finsler manifolds, the Finslerian distance function d is C^∞ out of a small neighborhood of x_0 and it is C^1 in a punctured neighborhood of x_0 cf. [20]. By definition of ϕ_j, $|\phi_j(x)| \leq |\phi(x)|$, where $\phi(x)$ is a C^∞ function lying in $L^p(SM)$. Therefore by the Lebesgue dominated convergence theorem ϕ_j and $\phi_j - \phi$ are also in $L^p(SM)$. Recall that

$$\begin{aligned} &\| \phi_j - \phi \|_{H_1^p(M)} \\ &= \sum_{k=0}^{1} (\textstyle\int_{SM} [(\nabla^k(\phi_j - \phi), \nabla^k(\phi_j - \phi))]^{\frac{p}{2}} dV_{SM})^{\frac{1}{p}}. \end{aligned} \tag{4.2}$$

We show that equation (4.2) tends to zero as j tends to infinity. First, assume support of $\phi_j - \phi$ is contained in a local chart (x, Ω) , then by Proposition 3.1 we have

$$\begin{aligned} &\textstyle\int_{SM} |\phi_j - \phi|^p dV_{SM} \\ &= \textstyle\int_\Omega (\int_{S^{n-1}} |\phi_j - \phi|^p_{(x,\frac{y}{F})} \frac{det(g_{ij})}{F^n} d\sigma) dx. \end{aligned}$$

Consider forward metric ball $B^+_{x_0}(j) = \{x \in \Omega : d(x_0, x) < j\}$, we have

$$\begin{aligned} &\int_{SM} |\phi_j - \phi|^p dV_{SM} \\ &\leq \int_{B^+_{x_0}(j)} (\int_{S^{n-1}} |\phi_j - \phi|^p_{(x,\frac{y}{F})} \frac{det(g_{ij})}{F^n} d\sigma) dx + \int_{\Omega \setminus B^+_{x_0}(j)} (\int_{S^{n-1}} |\phi_j - \phi|^p_{(x,\frac{y}{F})} \frac{det(g_{ij})}{F^n} d\sigma) dx. \end{aligned} \tag{4.3}$$

When j tends to infinity $\Omega \setminus B^+_{x_0}(j) = \emptyset$, therefore

$$\begin{aligned} &\int_{SM} |\phi_j - \phi|^p dV_{SM} \\ &\leq \int_{B^+_{x_0}(j)} (\int_{S^{n-1}} |\phi_j - \phi|^p_{(x,\frac{y}{F})} \frac{det(g_{ij})}{F^n} d\sigma) dx. \end{aligned}$$

On $B^+_{x_0}(j)$ we have $d(x_0, x) < j$ or $\phi_j(x) = \phi(x)$, hence

$$\| \phi_j - \phi \|_p \leq \left(\int_{SM} |\phi_j - \phi|^p dV_{SM} \right)^{\frac{1}{p}} \longrightarrow 0. \tag{4.4}$$

Next if the support of $\phi_j - \phi$ is not contained in a local chart, then similar to the proof of Lemma 3.1 by an appropriate choice of sequence $\{\Omega_i\}_{i=1}^{\infty}$ we can show that $\| \phi_j - \phi \|_p \rightarrow 0$. We prove $\| \phi_j - \phi \|_{H_1^p(M)}$ converges to zero. To this end, it suffices to show that $\| \nabla\phi_j - \nabla\phi \|_p$ or $\| \nabla(\phi_j - \phi) \|_p$ converges to zero. By means of Leibnitz's formula for $\phi_j(x) = \phi(x)f(d(x_0, x) - j)$, and triangle inequality we obtain

$$|\nabla\phi_j| \leq |\nabla\phi| + |\phi| \sup_{t\in[0,1]} |f'(t)|. \tag{4.5}$$

Again with Lebesgue dominated theorem, we have $|\nabla\phi_j| \in L^p(SM)$ and hence $\phi_j(x) \in H_1^p(M)$. Repeating above steps for $|\nabla(\phi_j - \phi)|$ instead of $|(\phi_j - \phi)|$ leads to

$$\begin{aligned} &\| \nabla\phi_j - \nabla\phi \|_p \\ &= \| \nabla(\phi_j - \phi) \|_p \leq \left(\textstyle\int_{SM} |\nabla(\phi_j - \phi)|^p dV_{SM} \right)^{\frac{1}{p}} \longrightarrow 0. \end{aligned} \tag{4.6}$$

Therefore, by the relations (4.4) and (4.6), we obtain

$$\| \phi_j - \phi \|_{H_1^p(M)} = \| \phi_j - \phi \|_p + \| \nabla(\phi_j - \phi) \|_p \longrightarrow 0.$$

Thus ϕ_j converges to ϕ in $H_1^p(M)$.

□

Proof of Theorem 1.1 is an application of Propositions 4.1 and 4.2, similar to that in Riemannian geometry.

Proof of Theorem 1.1. To prove Theorem 1.1, by means of Propositions 4.1 and 4.2, it remains to approximate each ϕ_j by functions in $D(M)$. Let j be a fixed index for which ϕ_j has a compact support. Let K be the compact support of ϕ_j and $\{V_i\}_{i=1}^m$ a finite covering of K such that by means of Lemma 2.1, for fix index i, V_i is homeomorphic to the open unit ball B of $I\!R^n$. Let (V_i, ψ_i) be the corresponding chart, we complete the proof by means of partition of unity. More intuitively, let $\{\alpha_i\}$ be a partition of unity subordinate to the covering $\{V_i\}_{i=1}^m$. For approximating ϕ_j by C^∞ functions with compact support in $H_1^p(M)$, it remains to approximate each $\alpha_i\phi_j$ for $1 \leq i \leq m$. For fixed i, ψ_i is a homeomorphic map between V_i and the unit ball B. Consider the functions $(\alpha_i\phi_j) \circ \psi_i^{-1}$ which have their support in B. Let us denote $u := (\alpha_i\phi_j) \circ \psi_i^{-1}$ to be consistent with notations of Lemmas 3.2 and 3.3. Consider the convolution $J_\epsilon * u$ with $\lim_{\epsilon\to o^+} J_\epsilon * u = u$. Let $h'_\epsilon := J_\epsilon * u \in C^\infty(I\!R^n)$, then h'_ϵ has a compact support, that is $h'_\epsilon \in D(B)$. We approximate u by $h_k := h'_{\frac{\epsilon}{k}}$. More precisely

$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} h'_{\frac{\epsilon}{k}} = \lim_{\epsilon\to o^+} h'_\epsilon = u = (\alpha_i\phi_j) \circ \psi_i^{-1}.$$

Moreover, h_k is C^∞. Hence h_k converges to u in $H_1^p(B)$. Now $h_k \circ \psi_i$ converges to $\alpha_i\phi_j$ in $H_1^p(V_i)$ and $h_k \circ \psi_i \in D(V_i)$. Thus we have approximated each $\alpha_i\phi_j$ by functions in $D(V_i)$. This completes the proof. □

Similar proof can be repeated for backward geodesically complete spaces.

Example 2. Let (M, F) be a Compact Finsler manifold. It is forward geodesically complete, hence by Theorem 1.1, $D(M)$ is dense in $H_1^p(M)$.

Corollary 4.1. Let (M, F) be a compact, connected, C^∞, reversible Finsler manifold and $f : M \longrightarrow I\!R$ a real function for which $\int_M f dv_F = 0$, then the weak solution u of the Dirichlet equation $\Delta u = f$ can be approximated by C^∞ functions with compact support on M.

Proof. Theorem 1.1 can be used to approximate weak solutions of Dirichlet problem on Finsler manifolds. Indeed in complete Finsler manifolds with certain conditions the Dirichlet problem $\Delta u = f$ has a unique solution which lies in Sobolev space $H_1^2(M)$, for similar proof one can refer to [3] and [21]. Hence by Theorem 1.1 we can approximate these weak solutions by C^∞ functions with compact support on M. □

4.2. Case of manifolds with C^r boundary

In proof of Theorem 1.2 we use the technic applied in proof of the following theorem on half-spaces on $I\!R^n$.

Theorem 4.1. [3] $C^\infty(\overline{E})$ is dense in $H^p_k(E)$, where E is a half-space $E = \{x \in I\!R^n : x_1 < 0\}$ and $C^\infty(\overline{E})$ is the set of functions that are restriction to $\overline{E}$ of C^∞ functions on $I\!R^n$.

Proof of Theorem 1.2. Let ϕ be a real C^∞ function on the Sobolev space $H^p_k(W)$, that is, $\phi \in C^\infty(W) \cap H^p_k(W)$. Here we approximate ϕ by the functions in $C^r(\overline{W})$. Since $\overline{W}$ is compact, we can consider $(V_i, \psi_i), i = 1, \cdots, N$ as a finite C^r atlas on $\overline{W}$. Each V_i depending on $V_i \subset W$ or V_i has intersection with ∂W, is homeomorphic either to the unit ball of $I\!R^n$ or a half-ball $D = B \cap \overline{E}$, respectively, where $E = \{x \in I\!R^n : x_1 < 0\}$. Let $\{\alpha_i\}$ be a C^∞ partition of unity subordinate to the finite covering $\{V_i\}_{i=1}^m$ of $\overline{W}$. By properties of partition of unity, it remains to show that each $\alpha_i\phi$, supported in V_i, can be approximated by functions in $C^r(V_i)$. Each V_i is homeomorphic either to the unit ball B or a half-ball D. First, let V_i be homeomorphic to B, then by the relation $V_i \subset W$ we have $\alpha_i\phi \in C^\infty(V_i)$, therefore $\alpha_i\phi \in C^r(V_i)$.
Now let V_i be homeomorphic to $D = B \cap \overline{E}$ and ψ_i a homeomorphism between V_i and D. Consider the sequence of functions h_m as restricted to D of $((\alpha_i\phi) \circ \psi_i^{-1})(x_1 - \frac{1}{m}, x_2, x_3, \cdots, x_s)$. Let $\phi \in H^p_k(W) \cap C^\infty(W)$, by appropriate choice of $\{V_i\}$ and $\{\alpha_i\}$, the restriction of $((\alpha_i\phi) \circ \psi_i^{-1})(x_1 - \frac{1}{m}, x_2, x_3, \cdots, x_s)$ to D and its derivative up to order r converge to $((\alpha_i\phi) \circ \psi_i^{-1})$ in $H^p_k(D)$, where D has the Euclidean metric. Therefore, by Proposition 3.1, we obtain

$$(\int_{SM} |\nabla^t(h_m \circ \psi_i - \alpha_i\phi)|^p dV_{SM}) = \sum_{i=1}^{\infty} \int_{V_i} (\int_{S^{n-1}} |\nabla^t(h_m \circ \psi_i - \alpha_i\phi)|^p_{(x, \frac{y}{F(x,y)})} \frac{det(g_{ij}(x,y))}{F(x,y)^n} d\sigma) dx \to 0,$$

where $0 \le t \le k$ is an integer. Hence $h_m \circ \psi_i \longrightarrow \alpha_i\phi$ in $H^p_k(V_i)$, $k \le r$, $h_m \circ \psi_i \in C^r(\overline{W})$ and proof is complete. □

In the following example, we show that the assumption $k \le r$ in Theorem 1.2 is sharp and can not be omitted.

Example 3. Let $\overline{W} = [-1, 1] \times [0, 1]$ be a manifold with boundary of class C^0. Denote the points of W by $x = (x^1, x^2)$ and the points of $T_x(W)$ by $y = (y^1, y^2)$. Let $F(x, y) = \sqrt{g_x(y, y)}$ be a Finsler structure defined by $g = g_{ij}\, dx^i \otimes dx^j = (dx^1)^2 + (dx^2)^2$ on W. Define, $u \in H^p_1(W)$ by $u : W \to I\!R$ and

$$u(x^1, x^2) = \begin{cases} 1 & x^1 > 0 \\ 0 & x^1 \le 0 \end{cases}. \tag{4.7}$$

We claim that for sufficiently small ε, there is no $\phi \in C^1(\overline{W})$ such that $\| u - \phi \|_{H^p_1(W)} < \varepsilon$. Assume for a while that our assumption is not true and the function ϕ exists. Let $S = \{(x^1, x^2) : -1 \le x^1 \le 0\ , 0 \le x^2 \le 1\}$ and $K = \{(x^1, x^2) : 0 < x^1 \le 1\ , 0 < x^2 \le 1\}$, then $\overline{W} = S \cup K$. On S, $u(x^1, x^2) = 0$, hence $\| 0 - \phi \|_{H^p_1(S)} < \varepsilon$ or $\| \phi \|_1 + \| \nabla\phi \|_1 < \varepsilon$, therefore $\| \phi \|_1 < \varepsilon$. On K, $u(x^1, x^2) = 1$ thus $\| 1 - \phi \|_1 < \varepsilon$ or $\| \phi \|_1 > 1 - \varepsilon$. Put $\psi(x^1) = \int_0^1 \phi(x^1, x^2) dx^2$, then there exist the real numbers a and b with $-1 \le a \le 0$ and $0 < b \le 1$ such that $\psi(a) < \varepsilon$ and $\psi(b) > 1 - \varepsilon$. Thus

$$\begin{aligned} 1 - 2\varepsilon < \psi(b) - \psi(a) &= \int_a^b \psi'(x^1) dx^1 \\ &\le \int_{\overline{W}} |D_{x^1}\phi(x^1, x^2)| dx^1 dx^2 \\ &\le (\int_{\overline{W}} 1^{p'} dx^1 dx^2)^{\frac{1}{p'}} (\int_{\overline{W}} |D_{x^1}\phi(x^1, x^2)|^p dx^1 dx^2)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p'}} \| D_{x^1}\phi(x^1, x^2) \|_{L^p(\overline{W})} < 2^{\frac{1}{p'}} \varepsilon, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Hence $1 < (2 + 2^{\frac{1}{p'}})\varepsilon$ which is not possible for small ε. This contradict our provisional assumption and prove the statement.

For some other Sobolev inequalities in Finsler geometry one can refer to [18].

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