AUT Journal of Mechanical Engineering

AUT J. Mech. Eng., 3(2) (2019) 173-186
DOI: 10.22060/ajme.2018.14240.5740

Reduced Order Model for Boundary Instigation of Burgers Equation of Turbulence Using Direct and Indirect Control Approaches

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ABSTRACT: In this paper, a reduced order model is reconstructed for boundary control and excitation of the unsteady viscous Burgers equation. First, the standard reduced order proper orthogonal decomposition model, which has been extracted from the governing equations without control inputs, was evaluated and illustrated the satisfactory results in short time period. Two approaches are used to imply the effects of boundaries excitations and the related control routines. In the first, a source term was added to the governing equation of the reduced order dynamical system and was contributed as an expansion of the proper orthogonal decomposition modes without control input. For removing the inhomogeneities on the boundaries, the boundaries values are subtracted from all of the snapshots with an appropriate control input. The other approach is based on the rewriting of the diffusion term as an expanded form which contains the effect of boundaries values explicitly. In both approaches, the obtained reduced order models will contain two parts, the effect of system states and the influence of boundaries control functions. The results obtained from the reduced order model without the control inputs demonstrate a good agreement to the benchmark direct numerical simulations data and prove the high accuracy of the model.

Review History:

Received: 15 July 2017
Revised: 2 November 2018
Accepted: 3 December 2018
Available Online: 19 December 2018

Keywords:

Proper orthogonal decomposition
Galerkin projection
Reduced order model
Boundary control
Viscous Burgers equation

1- Introduction

One of the essential concepts in the field of complex systems is a complete understanding of the turbulent phenomenon. An excellent choice for a better understanding of the disordered flow phenomena which is occurred inevitably at high Reynolds numbers is Direct Numerical Simulations (DNS). However, the present theory of turbulence has still some problems for predicting the important features of some phenomena like turbulent mixing, turbulent convection, and turbulent combustion. It is due to the reason that already the governing equations for the simplest flows such as Newtonian incompressible fluids, have to take into account nonlinear as well as nonlocal properties. Especially, in the linear momentum conservation equation, the nonlinearity arises from the convective acceleration and the pressure gradient terms, whereas nonlocality properties come from the pressure term. Therefore, Burgers equation has three primary terms including the transient, the convective and the diffusion terms. Each of these terms has a specific behavior according to the mathematical specifications of Partial Differential Equation's (PDE). Diffusion term creates the elliptic behavior of the evolution equation such as Burgers equation. Elliptic PDEs have an important property which is influenced from the boundaries values. Also, the viscous property of the flow field is represented by this term. It is clear when the flow is turbulent; the effects of viscosity are amplified in the whole of the flow field. Thus, the control of boundaries' excitations results in the overall control of the flow field's behavior.

Control approaches in engineering and science have attracted considerable interest due to their applications. The essence of control is the ability to utilize real-time sensing, actuation, to manipulate the time-dependent response of a system subject to disturbances and the time variations in the operating regime. Control routine involves controlling a flow field using passive or active devices in order to bring on the desired changes in the behavior of the flow. In 1939, J.M. Burgers [1] proposed a simplified form of the Navier-Stokes (NS) equation, which is called Burgers equation, by just dropping the pressure term. This equation can be investigated as a one-dimensional problem. There is one spatial and one temporal coordinate. Breuer and Petruccione [2] introduced a mesoscopic approach explained by means of the one-dimensional Burgers model of turbulence. They evaluated correlation functions and energy spectra from the appropriate ensemble averages to demonstrate the energy dissipation and energy cascade. The Burgers equation has frequently been used as a model where the dissipation of kinetic energy remains finite in the limit of vanishing viscosity (i.e., the dissipative anomaly). This allows singling out artifacts arising from the manipulation that ignore shock waves [3]. Bec and Konstantin [4] reviewed different methods to study about the solution of turbulence Burgers equation. Bouchaud et al. [5] proposed a simple method to compute the velocity difference in a forced Burgers equation. Honken et al. [6] studied about modeling of turbulence using functional integration of Burgers equation. They investigated the feasibility of modelling turbulence via numeric functional integration by transforming the Burgers equation into a functional integral. Camilo Bayona et al. [7] proposed a numerical approximation for the one-dimensional Burgers equation by means of the Orthogonal Sub-Grid Scales -Variational Multi-Scale (OSGS-VMS) method. Öziş et al. [8] studied about a solution of Burgers equation using the Lie group method. In this method, by similarity transformations, the equation

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is reduced to Ordinary Differential Equations (ODE) whose general solutions are written in terms of the error function. Rudenko et al. [9] presented a one dimensional equation that generalizes the Burgers equation in theory of waves and turbulence models. Burns and Kang [10] studied about control of Burgers equation based on the strategy to minimize a certain weighted energy functional. They used a numerical scheme for computing the feedback gain functional and employed “α-shifted” linear feedback control laws to obtain a desired degree of stability, on a certain energy space, for the closed-loop nonlinear system. Da Prato and Debussche [11] considered a control problem for a stochastic Burgers equation. This problem is motivated by a model from the control of turbulence. Choi et al. [12] developed a procedure to cast the problem of controlling turbulence based on the optimal control theory through the formalism and language of control theory. Baker et al. [13] used a method to synthesize nonlinear finite-dimensional output feedback controllers for the Burgers equation and the two-dimensional channel incompressible flow.

Design of reduced-order controllers is an interesting aspect of the recent development of computational engineering. The reduced order Proper Orthogonal Decomposition (POD) models have prepared new foundations in computational simulations of engineering problems. These models that obtained from the conservation laws cause an increase in the computational speed. Also, the POD based Reduced Order Models (ROMs) have prepared the appropriate base for coupling different dynamical systems and have helped researches and engineers to test and validate their new ideas and theories.

Different efforts have been performed to construct low order control models. Allan [14] used a reduced order model for linearized NS equation for optimal feedback control and applied model on the entire flow field. The balanced POD, which uses Non-linear Galerkin definition, is one of the methods for improving the prediction of a dynamical system. This method is a good choice for reconstruction of reduced order control models due to the ability of the method to control both the inputs and the related outputs [15]. Ravindran [16] used POD low order model for optimal control of fluid flow past a backward step. The ability of the low dimensional dynamical system in the flow control applications have been shown on a recirculation control problem using blowing on the channel boundaries. Atwell [17] proposed POD based reduced order model for control of Burgers equation with periodic boundary conditions and Non-linear observers. This controller is designed using the MinMax approach. A reduced order POD model has been used to control the viscous Burgers equation containing the periodic boundary conditions with non-linear observers. The model is developed based on two parts, the homogeneous model and the non-linear inhomogeneous part due to the effects of boundaries [18].

In the sequel, the principal concepts of the POD are presented. In the next section, the mathematical formulation of the controlled form of Burgers equation is presented. Then, the Galerkin projection of governing equation and the related low order dynamical systems have been presented. Finally, the discretization of the ROM equation, the order reduction manner and the results are discussed in the next sections.

2- Proper Orthogonal Decomposition

The POD Reduced-order modeling begins by finding the empirical eigenfunctions using the Karhunen-Loève decomposition. Then the flow variables are approximated using the expansions of these eigenmodes. The governing equations are projected into the eigenfunctions space to obtain the sets of equations for the coefficients of the expansions that can be solved to predict the behavior of the flow variables in space and time. The POD is remarkable in the selection of bases functions that are optimal and not necessarily appropriate. The POD was introduced to the turbulence community by Lumley in 1967. Before that, it was already known in the statistics as the Karhunen-Loève expansion. Lumley proposed that a coherent structure can be defined with functions of the spatial variables that have maximum energy content. That is, coherent structures are linear combinations of ϕ's, which maximize the following expression [19]:

$$\max \frac{\left\langle (\phi, u)^2 \right\rangle}{\left(\phi, \phi\right)}, \tag{1}$$

where (ϕ , u) is the inner product of the basis vector ϕ with the field, u Note <·> is the time-averaging operation. It can be shown that the POD basis vectors are eigenfunctions of the Kernel K given by [20]:

$$K = \left\langle u, u' \right\rangle, \tag{2}$$

where u' denotes the Hermitian of u. This equation is converted to the Fredholm’s second kind integral equation and its discretization leads to an eigenvalues problem. In this work, the Singular Value Decomposition (SVD) method has been used to solve this eigenvalues problem [21].

3- Governing Equation

Burgers equation is a non-linear PDE which has various applications such as in the fluid mechanics, acoustics and gas dynamics. This equation is a simplified form of the Navier-Stokes equation by just dropping the pressure term and therefore is considered as a convection-diffusion equation. The Burgers equation is nonlinear and one expects to predict phenomena similar to turbulence. However, as it has been shown by Hopf [22] and Cole [23], the homogeneous Burgers equation does not contain the most important property attributed to turbulence. This equation in non-dimensional form is as:

$$\begin{aligned} &\partial_t u + (u.\nabla)u = \frac{1}{Re}\nabla^2 u, \\ &u(x,0) = u_0, \end{aligned} \tag{3}$$

In the domain, Ω with time period [0, T] . The Reynolds number is defined as $Re=(u\times L)/\nu$, and the boundary conditions are as:

$$\begin{aligned} u = u_{\mathrm{L}}, \quad \text{On} \quad \Gamma_L \times \left[0, T\right] \\ u = u_{\mathrm{R}}, \quad \text{On} \quad \Gamma_R \times \left[0, T\right] \end{aligned} \tag{4}$$

Also, the Burgers equation is very similar to the heat equation. The first term is the transient behavior of the desired quantity, the second term is similar to convection heat transfer and the right-hand side term represents the diffusion. Thus, all of these descriptions and properties demonstrate the importance of Burgers equation in the fluid mechanics and the non-linear dynamical systems.

4- Galerkin Projection and POD Surrogate Reduced Order Control Model

Any variable (ensemble members) can be rewritten as the summation of mean and the fluctuation parts, as:

$$u = \overline{u} + u' \tag{5}$$

The first part is a time-averaged value of snapshots while the second part can be written using an expansion of the POD eigenfunctions, as [16]:

$$u' = \sum_{i=1}^{N} a^i(t)\phi^i(\vec{x}) \tag{6}$$

In the above equation, N is the number of modes, $a^i(t)$ and $\phi^i(\vec{x})$ are the modal coefficients (temporal modes) and the related spatial (POD) modes respectively. Now, If the Burgers equation (Eq. (3)) are expanded using Eqs. (5) and (6) and based on the assumption that the time derivative of the mean part is zero, new relations are obtained. Next, these equations are projected along the POD modes and a system of ODE's is obtained, which is called dynamical system.

To construct an appropriate surrogate model for boundaries control and excitations of Burgers equation, two approaches have been performed. In the first, a source term was added to the governing equation of representative dynamical system. This term is contributed as an expansion of POD modes without control inputs or by specified control function. For removing the inhomogeneities on the boundaries, their related values are subtracted from all of the snapshots with a determined control input.

The other approach is based on rewriting the diffusion term (right-hand side of Eq. (3)) as an expanded form which contains the effect of boundaries values explicitly. In both approaches, the reduced order dynamical system contains two parts, the effect of system states and the influence of boundaries control functions.

4- 1- Imposing control function method

The POD eigenfunctions only present the information given by the ensemble of data and thus the snapshots of the uncontrolled system cannot be used for the controlled system. To obtain the snapshots for the controlled system, an ensemble contains members with a specified control input has been taken. The inhomogeneities on the boundary have been removed by subtracting an appropriate function from each snapshot before calculation of the POD basis. A good choice to generate this ensemble is to take the field solution with a fixed value of boundary control function. Then the modified snapshot is now

$$u(x,t^k) = u(x,t^k) - u_{\mathbf{c}},\ k = 1,2,\ldots\ldots,N \tag{7}$$

Let as rewritten the velocity vector as the summation of mean, the control influence, and the fluctuation parts, as:

$$u = \overline{u} + u_{\mathbf{c}} + u'. \tag{8}$$

The first part is a time-averaged value of snapshots ensemble members. The second (control enforces) and the third parts can be written using the expansion of the POD eigenfunctions, as:

$$u'(\vec{x},t) = \sum_{i=1}^{N} a^i(t)\phi^i(\vec{x}), \tag{9}$$

and,

$$u_c(\vec{x},t) = \sum_{i=1}^{N} b^i(t)\phi^i(\vec{x}), \tag{10}$$

In the above equations, $a^i(t)$ and $\phi^i(\vec{x})$ are the modal coefficients and the spatial modes of the considered field without the influence of boundary control. It means that ϕ^i satisfies zero (homogeneous) boundary conditions on each controlled boundaries. In Eq. (10), $b^i(t)$ are the modal coefficients of the boundary influence part of the snapshots. As to be noted before, the boundary control part was represented as a function of spatial modes without control. Therefore, the problem states are only POD eigenfunctions without the boundary control excitation function. If the viscous Burgers equation (Eq. (3)) is expanded using Eqs. (5) and (6) and the outcome equations are projected along the POD modes, the related dynamical system is obtained, as:

$$\frac{da^k}{dt} + A_{kij}a^i a^j + B_{ki}a^i + B'_{ki}b^i + C_k = 0. \tag{11}$$

Or:

$$\dot{a}^k = f(a^k) + \tilde{B}\,\tilde{u}, \tag{12}$$

The first part in the above equation is the effects of problem state without control while the second part is boundary excitation effects. The first item of the right-hand side of Eq. (12) is as:

$$\begin{aligned}
&f(a^k) = -A_{kij}a^i a^j - B_{ki}a^i - C_k\\
&A_{kij} = \left\langle \phi^i \frac{d\phi^j}{dx}, \phi^k \right\rangle.\\
&B_{ki} = \left\langle \overline{u}\frac{d\phi^i}{dx}, \phi^k \right\rangle + \left\langle \phi^i \frac{d\overline{u}}{dx}, \phi^k \right\rangle - \frac{1}{Re}\left\langle \frac{d^2\phi^j}{dx^2}, \phi^k \right\rangle,\\
&\tilde{B} = B'_{ki} = B_{ki}
\end{aligned} \tag{13}$$

Then, Eq. (11) will be solved to compute the time variations of the modal coefficients which are used for reconstruction of the field under controlling the boundaries or neutral conditions.

4- 2- Direct boundary excitation treating method

The second approach to imposing the boundary influence is based on the direct representation of their effects in the governing equation of surrogate reduced order model. This method is based on the decomposition of the viscous term in some parts which contain effects of boundaries and the values of the interior domain. If the viscous Burgers equation (Eq. (3)) is projected along the POD eigendirections space and based on the assumption that time derivative of the mean part is zero, the following relation is obtained:

$$\left(\partial_t u,\phi^k\right)+\left(\left(u.\nabla\right)u,\phi^k\right)=\frac{1}{Re}\left(\nabla^2 u,\phi^k\right), \tag{14}$$

The integral form of one dimensional model along the spatial direction is as:

$$\int_{x_L}^{x_R}\left(\partial_t u+\left(u.\nabla\right)u\right)\phi^k\,dx=\frac{1}{Re}\int_{x_L}^{x_R}\left(\frac{\partial^2 u}{\partial x^2}\right)\phi^k\,dx \tag{15}$$

The diffusion part of the above equation is expanded and the outcome model will be:

$$\begin{aligned}&\int_{x_L}^{x_R}\left(\partial_t u+\left(u.\nabla\right)u\right)\phi^k\,dx=\\&\frac{1}{Re}\left[u(x_L)\frac{\partial\phi^k(x_L)}{\partial x}-u(x_R)\frac{\partial\phi^k(x_R)}{\partial x}+\right.\\&\left.\sum_{j=1}^{N_m}a^j\int_{x_L}^{x_R}\left(\frac{\partial^2\phi^j}{\partial x^2}\right)\phi^k\,dx\right]\end{aligned} \tag{16}$$

The outcome dynamical system is as:

$$\frac{da^k}{dt}+A_{kij}a^ia^j+B_{kj}a^j+\bar{B}\,\tilde{u}+C_k=0.$$

or (17)

$$\dot{a}^k=f(a^k)+\tilde{B}\,\tilde{u},$$

The first part in the above equation is the effects of problem state without control while the second part is boundary excitation effects. The right-hand side of Eq. (17) is defined as:

$$\begin{aligned}&f(a^k)=-A_{kij}a^ia^j-B_{ki}a^i-C_k\\&A_{kij}=\left\langle\phi^i\frac{d\phi^j}{dx},\phi^k\right\rangle.\\&B_{ki}=\left\langle\bar{u}\frac{d\phi^i}{dx},\phi^k\right\rangle+\left\langle\phi^i\frac{d\bar{u}}{dx},\phi^k\right\rangle-\frac{1}{Re}\left\langle\frac{d^2\phi^j}{dx^2},\phi^k\right\rangle,\\&\tilde{B}=\bar{B}\end{aligned} \tag{18}$$

Similar to the previous section, Eq. (17) should be solved to compute the time variations of modal coefficients for reconstruction of the field under the effects of boundary instigation or the neutral conditions.

5- Discretization of ROM Equation and Calculation of Model Coefficients

The derivatives associated with the reduced order model coefficients were computed using a finite difference method with appropriate accuracy. An explicit, fourth-order-accurate, four-stage Runge-Kutta scheme was used for time integration of the ROM equations. The time-step studying shows that if the step size is equal to DNS snapshots increment, the behavior of the surrogate model will be adequately stable.

5- 1- Spatial discretization

The spatial derivatives of the convective (non-linear) term of the dynamical system's equation have been discretized using a first order upwind method, as:

$$u\frac{\partial u}{\partial x}=\begin{cases}u_{i,j}\dfrac{u_{i,j}-u_{i-1,j}}{\Delta x} & if\quad u_{i,j}>0\\[2mm] u_{i,j}\dfrac{u_{i+1,j}-u_{i,j}}{\Delta x} & if\quad u_{i,j}<0\end{cases}, \tag{19}$$

For spatial discretization of the diffusion part of linear coefficients (first order and second order derivatives) in Eqs. (11) and (17) used a second order central differencing method. This formulation for interior nodes of the computational domain is as follows:

$$\frac{\partial u}{\partial x}=\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}, \tag{20}$$

For the points on the boundaries a forward or backward second order differential formula is used as follows:

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{-3u_{i,j}+4u_{i+1,j}-u_{i+2,j}}{2\Delta x}\\\frac{\partial u}{\partial x}&=\frac{3u_{i,j}-4u_{i-1,j}+u_{i-2,j}}{2\Delta x}\end{aligned} \tag{21}$$

6- Model Order Reduction Criterion

Normally, when the number of modes is increased, the reconstruction is performed with better accuracy. It is required to use the optimal number of modes for data reconstruction. This is equivalent to capturing the highest level of energy and the least number of modes for model construction (such as Fig. 3). In this manner, a fraction number is defined for automatic selection of modes as follow:

$$\kappa=\frac{\sum_{i=1}^{N_r}\lambda_i}{\sum_{i=1}^{N_{total}}\lambda_i}, \tag{22}$$

where κ is about 99.9% and N_r is the optimum number of modes for reduced order model construction [20].

7- Direct Numerical Simulation

The snapshots for this test case have been generated using a one dimensional viscous Burgers equation solver based on a high order finite difference method. For clarifying the accuracy of direct simulation code a sample problem has been solved with this solver and obtained results have been compared with reference data. For this purpose consider a stationary sinusoidal wave in the beginning as:

$$u(x,0)=\sin(2\pi x),\qquad x\in(0,1) \tag{23}$$

With periodic boundary conditions:

$$u(0,t)= u(1,t) \qquad t \in (0,\mathrm{T}) \tag{24}$$

The Burgers equation has been numerically solved for Reynolds number of 100 and the obtained results have been stored at times equal to 0.2, 0.3 and 0.4. Fig. 1 shows a comparison between the results of direct simulation code and benchmark data at these times. It is clear that the direct simulation code has good accuracy and agreement with reference data.

8- Results and Discussion

The results obtained for an unsteady incompressible viscous flow governed by Burgers equation at Re = 100, are demonstrated in this section. For validation of low-dimensional POD model, the related results, which are obtained without control enforces, have been compared with the related DNS data.

8- 1- Reduced order model of Burgers equation based on the indirect boundary excitation

In this section direct numerical simulation and the reduced order modeling of Burgers equation under the following conditions is performed:

$$u(x,0) = \sin(\pi x), \tag{25}$$

And the following boundary conditions for controlled boundaries:

$$u(0,t)= 0.0 \quad , \quad u(L,t) =0.3 \times \sin(\pi t), \tag{26}$$

An ensemble with 100 members in different time steps with equal time increments and in a specific time span has been considered as an input data. Fig. 2 shows the distribution of four members of the snapshots ensemble of the field at 1st, 30th, 50th, and 80th time steps while Fig. 3 illustrates the distribution of four members of boundary actuation snapshots ensemble. After the solution of the eigenvalues problem, the POD modes are calculated. In Fig. 4, distribution of the four strongest modes of the field without subtracting the boundary Non-homogeneities is shown.

Fig. 5 shows the eigenvalues distribution versus the mode number and the energy spectrum of the first forty and nineteen POD modes respectively. Distribution of eigenvalues illustrates that their amplitude converges to a very small value after 12th mode. It is to be noted, the criterion for choosing the number of required modes to construct ROM can be verified from these graphs. Based on this criterion, eight modes have been used for the construction of the surrogate reduced order model which can capture 99.9% of the field's kinetic energy.

8- 1- 1- Short time period reduced order modeling without boundary excitation

By the method presented in the previous sections, the surrogate reduced order model is constructed using eight modes. Then, the time marching procedure is performed using an initial condition which is equal to the values of modal coefficients (obtained from the snapshots projection) in the first time step. In Fig. 6, time variations of the first four modal coefficients for uncontrolled reduced order model are shown. To verify the accuracy of the reduced order model, the outcome results have been compared to related DNS data. It is obvious that the low order model predicts relatively accurate results.

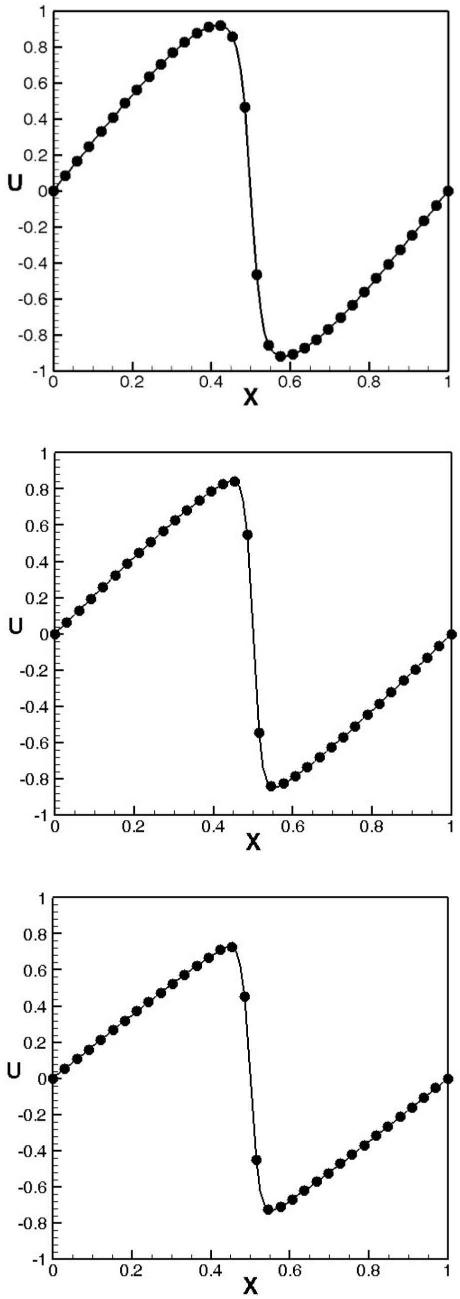

Fig. 1. Comparison between solution of Burgers equation at Reynolds Number of 100 at t =0.2 (Left-Up), t = 0.3 (Right-Up) and t = 0.4 (Bottom), Point (Reference data [24]), Solid Line (Present Computation)

Fig. 7 shows a comparison between the time variations of field variable at x = 0.7, from the projection of an ensemble, which include the excited boundaries and the low order POD model with activating control part. The control part of the reduced order model is same to excitation function of the ensemble with the related boundary and the state part of the model is reconstructed using an ensemble with homogeneous boundary. It is clear that the outcome surrogate reduced order POD control model of viscous Burgers equation predicts results with good agreement.

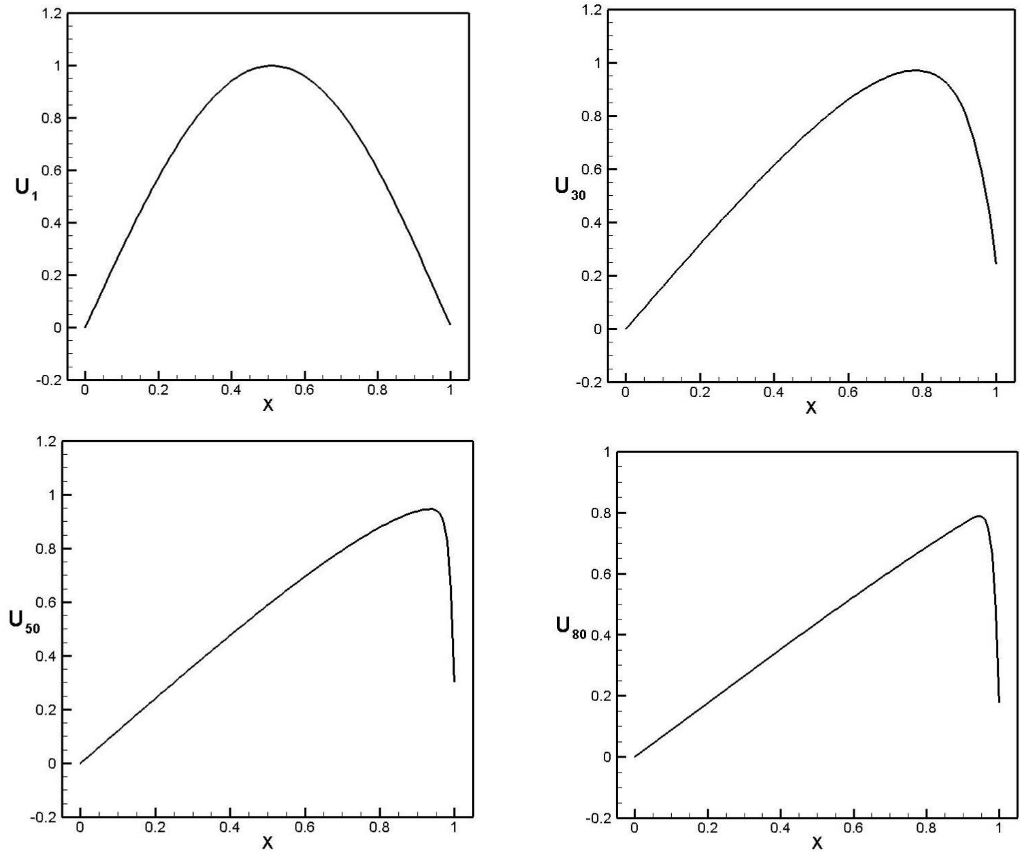

Fig. 2. Distribution of four members of snapshots ensemble at Reynolds Number of 100

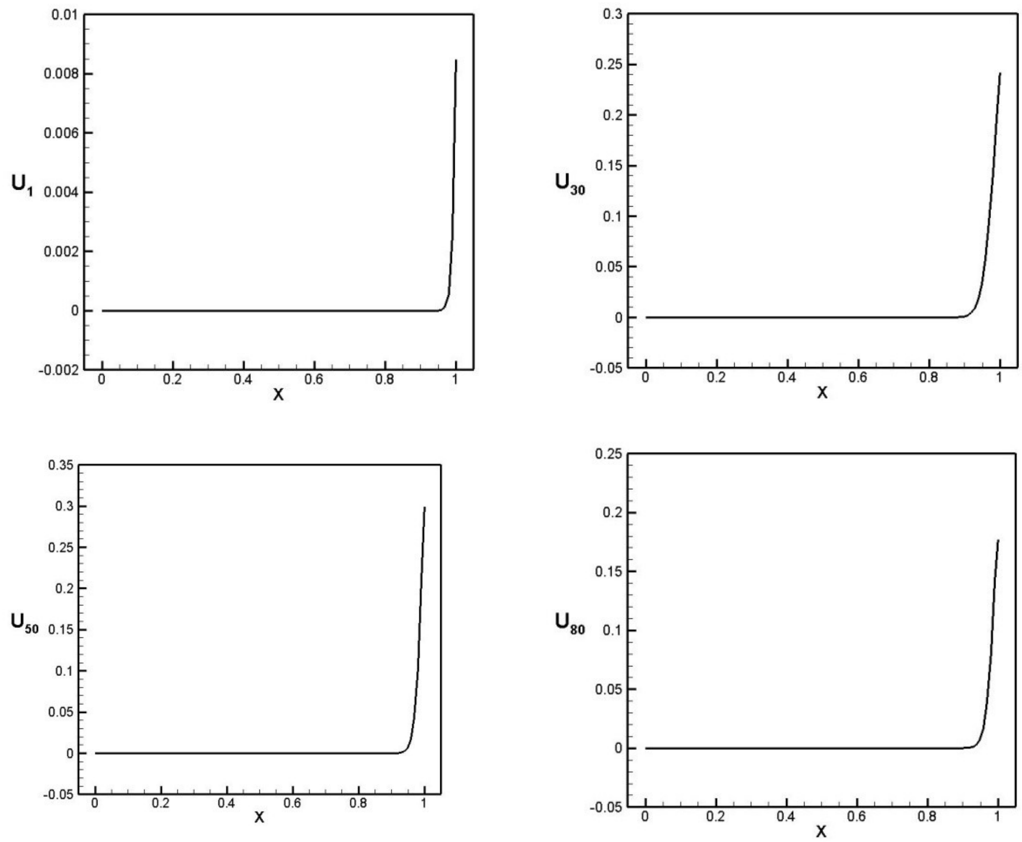

Fig. 3. Distribution of four members of boundary actuation snapshots ensemble at Reynolds Number of 100

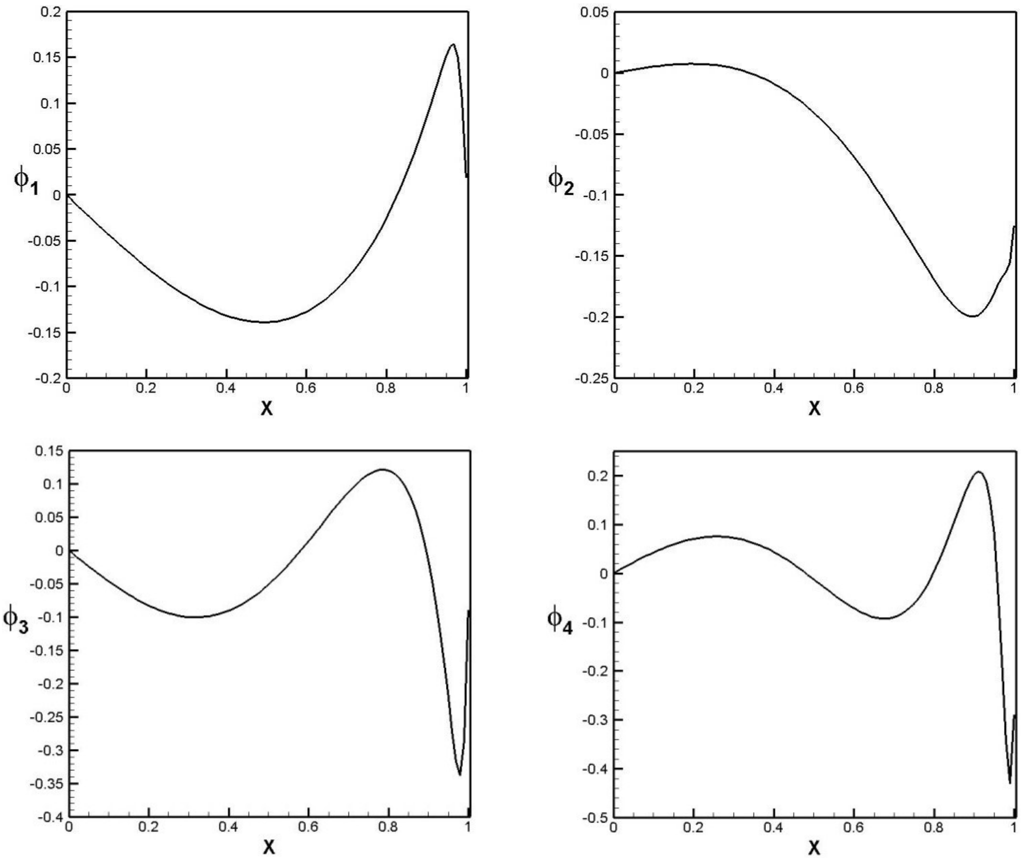

Fig. 4. Distribution of the four strongest modes without subtracting boundary non-homogeneities at *Re*=100

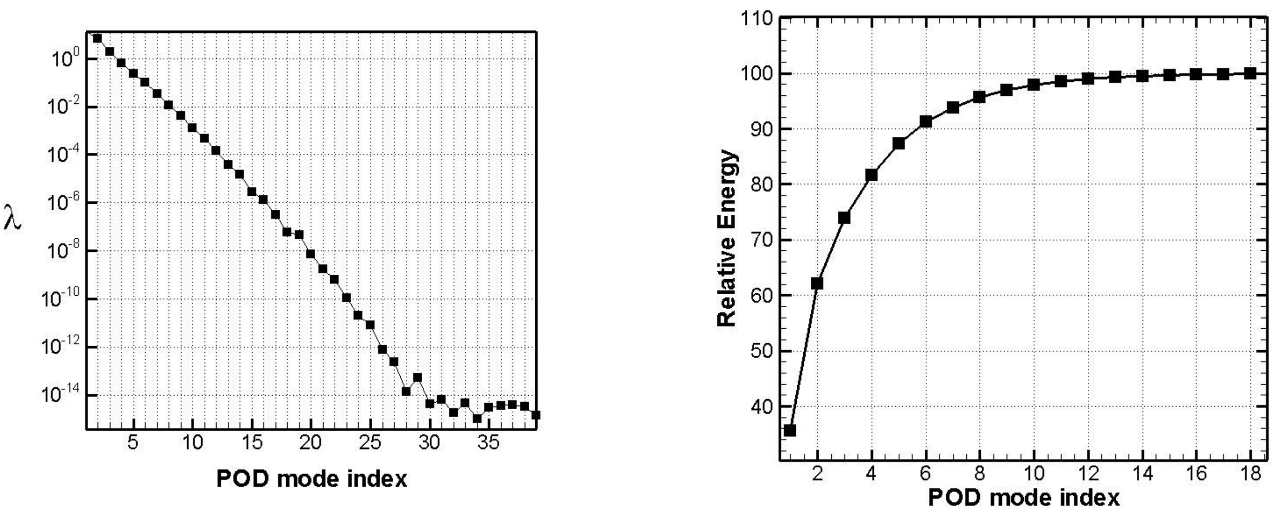

Fig. 5. Eigenvalues distribution and energy spectrum of POD modes in logarithmic scale

8- 1- 2- Reduced order modeling in short time period with the boundary excitation

In this section, the results of reduced order POD control model with boundary excitation which is different from the initial snapshots control function are presented. The test case is a sinusoidal wave in the first step and for the boundaries snapshots ensemble, a sinusoidal variation of boundary values versus time is applied. The ROM of this test case has been constructed using a similar number of modes to previous sections. Therefore, the obtained low-order model behaves as a controlled dynamical system. It was mentioned in section 3, that using the controlled temporal modes extension to ROM, the model contains two parts, the effect of problem state and the boundary control function. Therefore, the low order POD model is integrated in short time period to predict the effects of the influenced control part.

Fig. 8 shows the time variations of the field variable on the left boundary obtained from the low-dimensional POD control model compared to the model without control enforce. This figure affirms that the model deviates right at the beginning

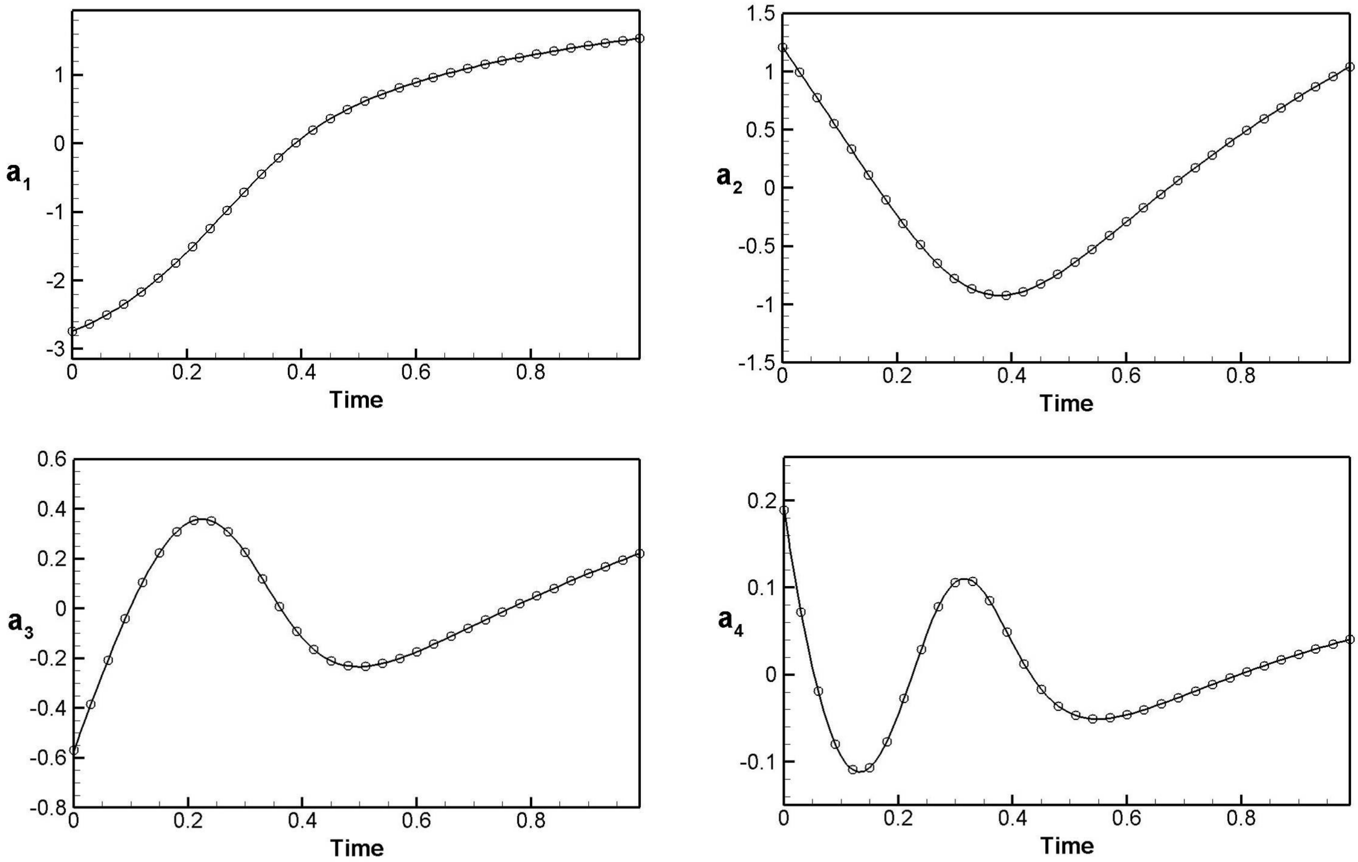

Fig. 6. Temporal variations of first four modal coefficients at Reynolds Number of 100, (○ Snapshots projection), (– – – Reduced order POD model without control).

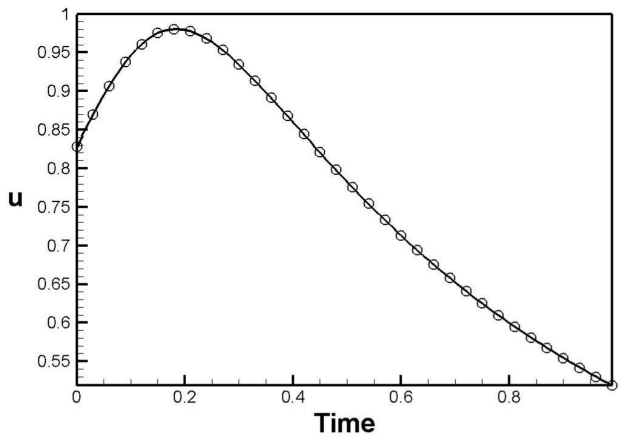

Fig. 7. Comparison between the prediction of reduced order model on $x = 0.7$ at Reynolds number of 100 (○ Snapshots projection with excited boundary) and (– – – Reduced order POD model with Control).

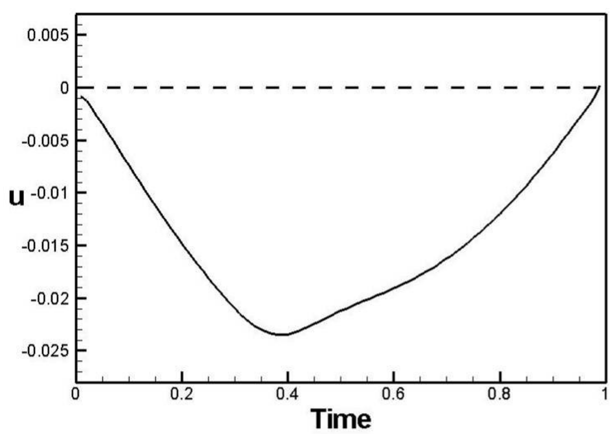

Fig. 8. Comparison between the prediction of ROM on the left boundary at Reynolds number of 100 (– – – Reduced order POD model without control) and (—Reduced order POD model with control).

time step for a model with an enabled control part. In contrast, the reduced order model without control is presented as fixed values. In Fig. 9, time variations of field variable, which are computed using the uncontrolled reduced order POD model and the controlled model, are shown.

It is concluded from the Figs. 8 and 9 that the model behaves appropriately when switching between the boundary control on and off. Fig. 10 shows the comparison between the time variations of field variable at x=0.7, which are obtained from the uncontrolled and controlled low order POD model.

8- 2- Reduced order modeling of Burgers equation based on the direct boundary excitation

The snapshots for this test case also have been generated using a one dimensional viscous Burgers equation solver based on an accurate finite difference approach. The simulation is performed with an initial condition as:

$$u(x,0) = \sin(\pi x), \tag{27}$$

and the following boundary conditions for the controlled

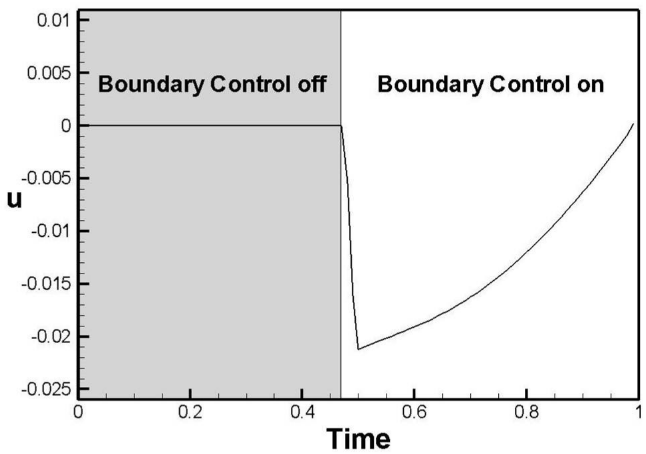

Fig. 9. Comparison between the prediction of reduced order model on the left boundary at Reynolds number of 100 (– – – Reduced order POD model without control) and (— Reduced order POD model with control).

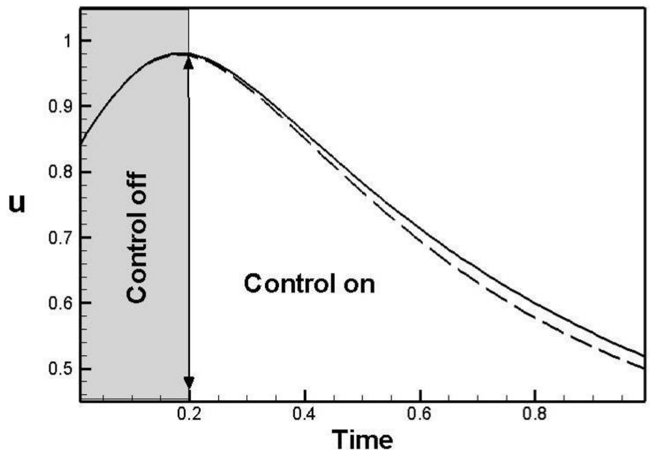

Fig. 10. Comparison between the prediction of reduced order model on x=0.7 at Reynolds number of 100 (— Reduced order POD model without Control) and (– – – Reduced order POD model with Control).

system:

$$u(0,t) = 1.0 \ , \ u(L,t) = 0.5 \tag{28}$$

8- 2- 1- The short time period modeling with the prescribed control function

The results obtained for the unsteady one dimensional Burgers equation with the prescribed control function at *Re* =100 are demonstrated in this section. For validation of the reduced order POD model, the outcome results, which are obtained with specified boundaries function, are compared with the DNS data. An ensemble with 100 members in different time steps with equal time increments and in a specific time span was considered as the input data.

Fig. 11 shows the distribution of four members of snapshots ensemble of filed at 1[st], 30[th], 50[th] and 80[th] time steps. Also, for this problem, an eigenvalues problem has been solved to calculate the POD modes. The distribution of the four strongest modes is shown in Fig. 12.

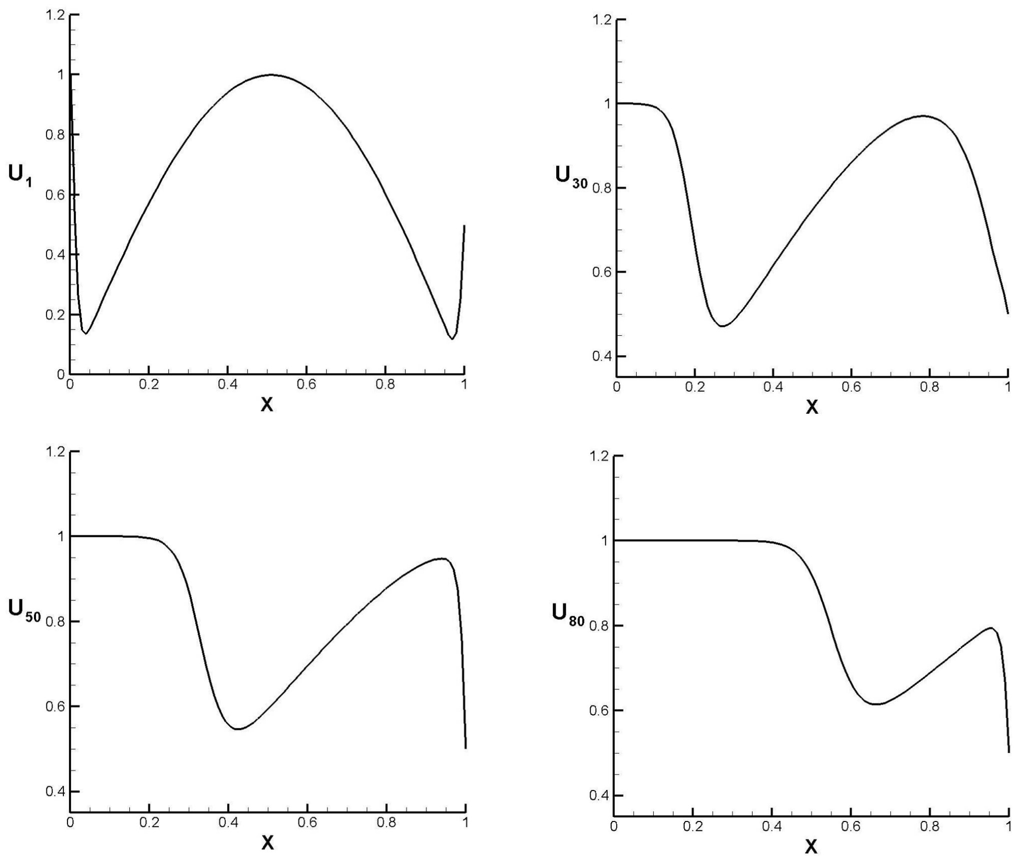

Fig. 11. Distribution of four members of snapshots ensemble with specified boundary function at *Re*=100

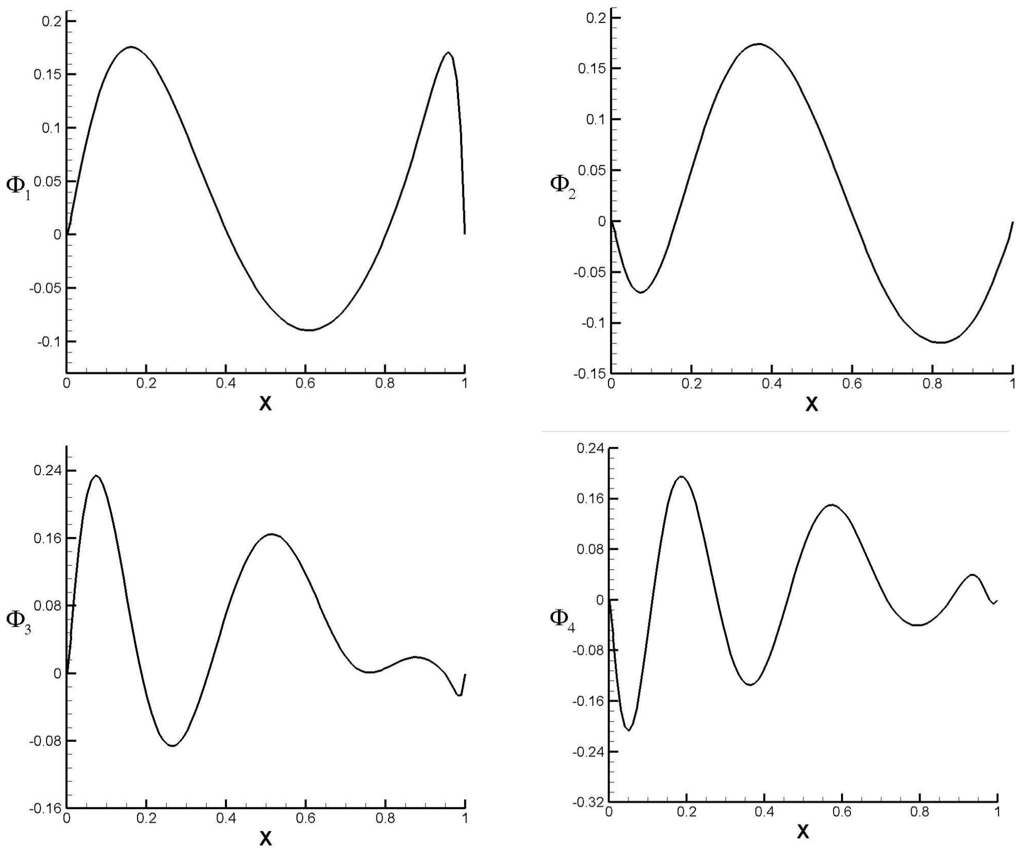

Fig. 12. Distribution of the four strongest modes with the subtracting boundary inhomogeneities at *Re*=100

Fig. 13 shows the distribution of eigenvalues versus modes number and the energy spectrum of the first forty POD modes. The results illustrate that the amplitude of eigenvalues leads to a minimal value after the 12th POD mode. Note, the criteria for choosing the number of modes can be verified from these results. Based on the criteria for the low-order model construction, fourteen modes have been used to construct the reduced order model which can capture 99.9% of the field's energy.

After construction of the ROM, the time marching is performed using an initial condition which is equal to the modal coefficients (obtained from the snapshots projection) in the first time step. In Fig. 14, the time variations of the first four modal coefficients for controlled reduced order model with specified boundaries specification (compared to High-dimensional DNS data) are shown. It is evident from this figure that the model predicts relatively accurate results.

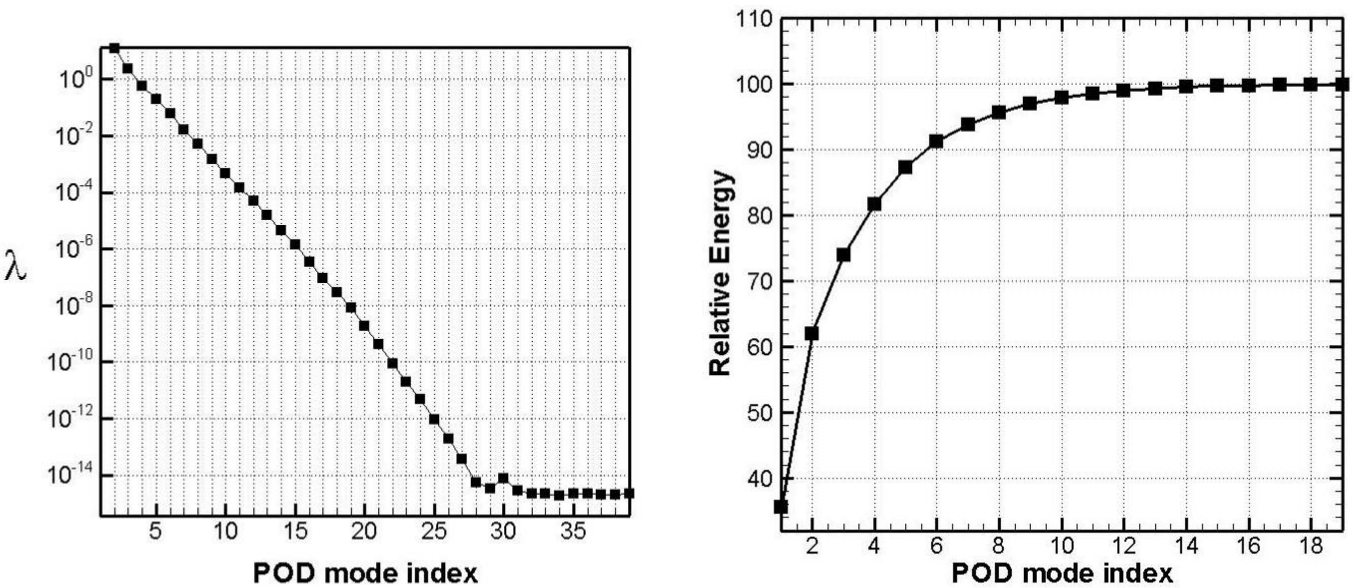

Fig. 13. The Eigenvalues distribution and the energy spectrum of POD modes in logarithmic scale

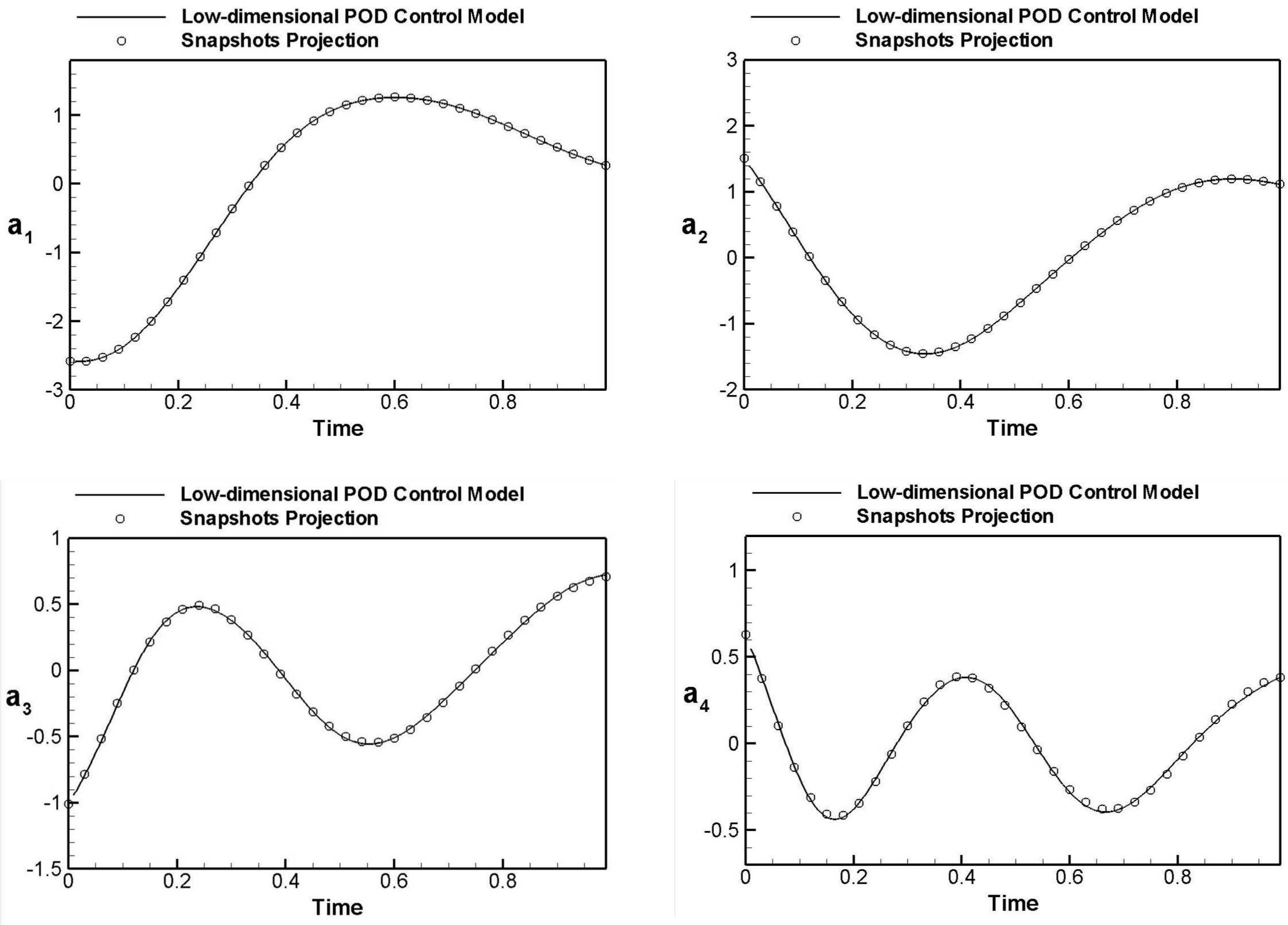

Fig. 14. Comparison between time variations of modal coefficients at *Re* =100, (○ Snapshots Projection), (– – – Reduced Order POD model with specified control function)

8- 2- 2- Reduced order modeling with boundary excitation in a short time period

In this section, the results obtained from the reduced order POD control model are presented. Fig. 15 shows the time variations of the field variable at x = 0.3 obtained from the reduced order POD control model compared to the model without control enforcement. This figure affirms that the model deviates right at the beginning time step for a model with an enabled boundary control part. In Fig. 16, time variations of field variable, which are computed using the uncontrolled reduced order POD model and the controlled model at x=0.3 is shown. Fig. 17 shows the comparison between the time variations of field variable at x = 0.7, which are obtained from the uncontrolled and the controlled reduced order POD model. It is concluded from Figs. 16 and 17 that the model appropriately has good behavior when switching between boundary control turn on or off.

9- Conclusions

In recent years, many efforts have been carried out on advancing the state of the art of proper orthogonal decomposition based low dimensional modeling. The improvement of this method is to apply for the coupled dynamical systems and PDE's control problems is often discussed in the literature. To this effect, this paper contributes a new extension of POD-based reduced order dynamical system and demonstrates its suitability for the control of the unsteady viscous Burgers equation. It is clear that the POD is a robust method for the estimation and the simulation of the steady and unsteady flows, respectively. In this work, a POD snapshots method was used for calculation of the POD modes. By projection of the Burgers equation along the POD modes, the reduced order dynamical system was reconstructed as an initial value problem. For enforcing the effects of boundary control into reduced order model, an ensemble of observations with fixed control function on the boundaries has been used. The control part of the snapshots is rewritten as an expansion of modes which are obtained from the ensemble without control. An order-reduction manner was used to choose the minimum number of modes for reconstruction of the dynamical system and therefore it prepares a low-dimensional model for fast prediction of the flow field. The present method is used to reconstruct a low order POD control model for short time integration. It is clear that the current dynamical system gives suitable results for switching between enabled boundary control and disabled excitation routine.

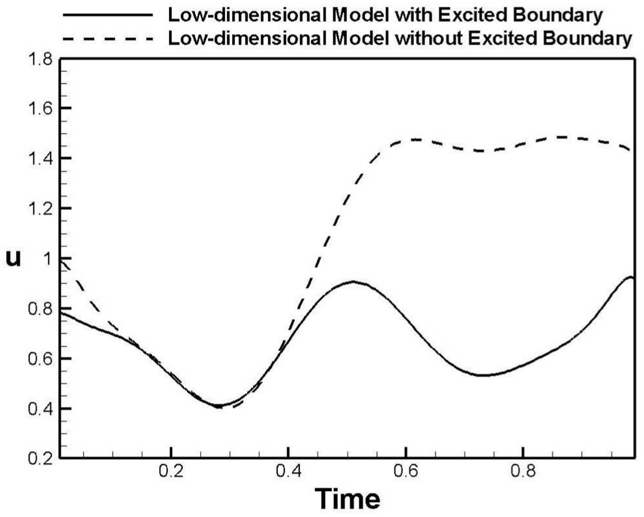

Fig. 15. Comparison between the prediction of reduced order model on *x*=0.3 at *Re*=100 (– – – Reduced order POD model without control) and (—Reduced order POD model with control)

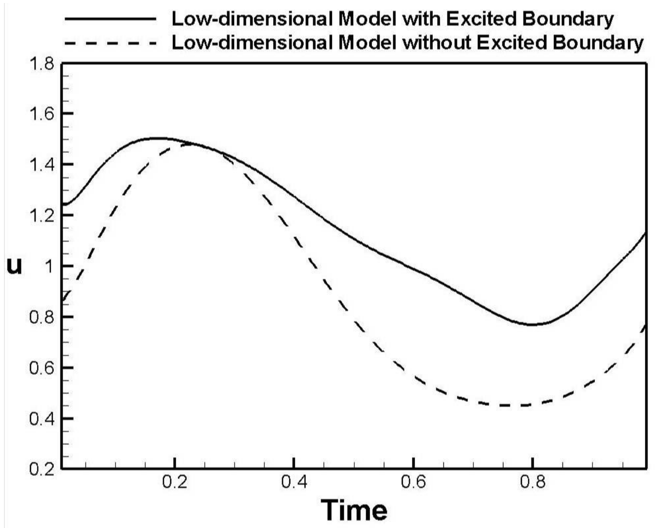

Fig. 16. Comparison between the prediction the value of the desired function on *x* = 0.7 at *Re* =100 (– – – Low-dimensional POD model without control) and (—Low-dimensional POD model with control)

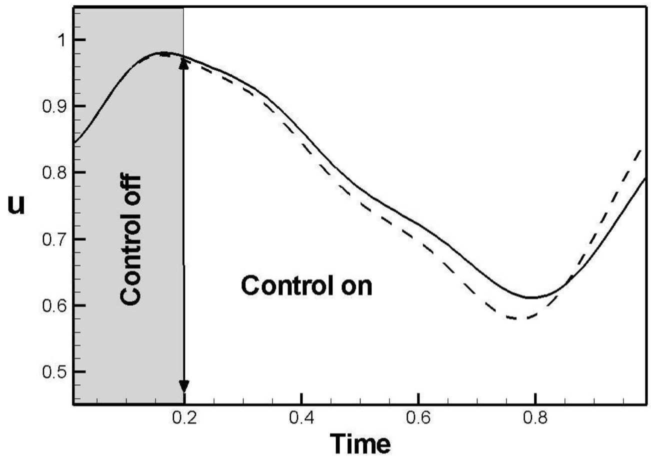

Fig. 17. Comparison between the prediction of the reduced order model on *x* = 0.7 at *Re* =100, (— Reduced order POD model without control) and (– – – Reduced Order POD model with control)

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Please cite this article using:

M. K. Moayyedi, Reduced Order Model for Boundary Instigation of Burgers Equation of Turbulence Using Direct and Indirect Control Approaches, *AUT J. Mech. Eng.*, 3(2) (2019) 173-186.

DOI: 10.22060/ajme.2018.14240.5740