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The Complementary Odd Weibull Power Series Distribution: Properties and Applications

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ABSTRACT: In this paper, a new four-parameters model called the complementary odd Weibull power series (COWPS) distribution is defined and its properties are explored. This new distribution exhibits several new and well-known hazard rate shapes such as increasing, decreasing, bathtub-shaped and J-shape hazard rates. Some of its mathematical properties are obtained including moments, quantiles reliability, and moment generating functions. The maximum likelihood estimation method is used to estimate the vector of parameters. A simulation study is presented to investigate the performance of the estimators. Finally, The usefulness of the model has been demonstrated by applying it to a real-life dataset.

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1. Introduction

In 2006, Cooray introduced a three-parameter generalization of the Weibull distribution called the odd Weibull distribution with cumulative distribution function (cdf) and probability density function (pdf) given by:

$$G(x;\alpha,\beta,\gamma)=\frac{\left(e^{\alpha x^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha x^{\beta}}-1\right)^{\gamma}},\quad x>0, \tag{1.1}$$

and

$$g(x;\alpha,\beta,\gamma)=\frac{\alpha\beta\gamma x^{\beta-1}e^{\alpha x^{\beta}}\left(e^{\alpha x^{\beta}}-1\right)^{\gamma-1}}{\left\{1+\left(e^{\alpha x^{\beta}}-1\right)^{\gamma}\right\}^{2}},\quad x>0, \tag{1.2}$$

where $\alpha>0$ and $\beta\gamma>0$ (Cooray, 2006). Then the odd Weibull distribution has been widely used in reliability and applications to real data. Based on odd Weibull distribution, many authors have extensively defined and studied the generalization of this model. Some well-known generalizations are the beta odd Weibull distribution by Cordeiro et al. (2015), the Kumaraswamy odd Weibull distribution by Alizadeh et al. (2015), The Zografos-Balakrishnan odd Weibull distribution by Cordeiro et al. (2015), and the odd Weibull Poisson distribution by Alizadeh et al. (2017).

In the other hand, during the recent decade, many compound distributions have been presented by complementary risk motivation by compounding a lifetime model and a member of the power series family. These models

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motivated by a system consisting of parallel components with an unknown amount of components. For instance, Cancho et al. (2011) proposed the complementary exponential Poisson distribution. In the same context, Flores et al. (2013) introduced the complementary exponential power series distribution, Munteanu et al. (2014) presented complementary Weibull power series distribution, and Cordeiro and Rodrigo (2014) introduced the complementary extended Weibull power series distribution. In this paper, we introduce the complementary odd Weibull power series distribution (COWPS), which is obtained by compounding odd Weibull and power series distributions in a parallel structure. The compounding procedure follows key ideas of Marshall and Olkin (1997), which builds a wider and more flexible family of continuous lifetime distributions.

The contents of this paper are organized as follows. Section 2 introduces the COWPS distribution; section 3 derives some mathematical properties; section 4 presents five special cases of the COWPS distribution; estimation of the parameters of the COWPS distribution by maximum likelihood is investigated in section 5; simulation studies are given in Section 6; application to a real dataset is illustrated in section 7; the paper is concluded in section 8.

2. The new distribution

Suppose that N is a discrete random variable following a zero-truncated power series distribution with mass function (Noack, 1950) given by

$$P(N=n)=\frac{a_n\theta^n}{C(\theta)},\quad n=1,2,...,$$

where a_n depends only on n, $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ and $\theta>0$ is such that $C(\theta)$ is finite. Table 1 shows useful quantities of some power series distributions (truncated at zero) such as Poisson, geometric, logarithmic, negative binomial and binomial distributions.

Table 1: Members of The power series family.

Distribution	pdf	θ	Extended parameter space after compounding	a_n	$C(\theta)$
Zero truncated Poisson	$e^{-\theta}\theta^n/n!\left(1-e^{-\theta}\right)$	$\theta>0$	$\theta\in(-\infty,+\infty)$	$1/n!$	$e^{\theta}-1$
Geometric	$(1-\theta)\,\theta^{n-1}$	$0<\theta<1$	$\theta\in(-\infty,0)\bigcup(0,1)$	1	$\theta/(1-\theta)$
Logarithmic	$-\theta^n/n\log(1-\theta)$	$0<\theta<1$	$\theta\in(-\infty,0)\bigcup(0,1)$	$1/n$	-log$(1-\theta)$
Negative binomial	$\binom{n+m-1}{n}(1-\theta)^m\theta^n/1-(1-\theta)^m$	$0<\theta<1$	$\theta\in(-\infty,0)\bigcup(0,1)$	$\binom{n+m-1}{n}$	$(1-\theta)^{-m}-1$
Zero truncated binomial	$\binom{m}{n}\theta^n/\left((1+\theta)^m-1\right)$	$0<\theta<\infty$	$\theta\in(-1,0)\bigcup(0,+\infty)$	$\binom{m}{n}$	$(1+\theta)^m-1$

Let's also consider $X_1,X_2,...,X_N$ be a random sample following the odd Weibull distribution, independent of N, with cdf and pdf given by (1.1) and (1.2) respectively. The complementary odd Weibull power series (COWPS) distribution is defined by the marginal cdf $Y=\max\{X_i\}_{i=1}^{N}$:

$$\begin{aligned}F(y;\boldsymbol{\xi}) &= \{C(\theta)\}^{-1}C\left(\frac{\theta\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}\right)\\ &= \{C(\theta)\}^{-1}C\left(\theta\left\{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}^{-1}\right),\quad y>0,\end{aligned}\tag{2.1}$$

where $\boldsymbol{\xi}=(\alpha,\beta,\gamma,\theta)$ is parameters vector of COWPS family of distributions. Furthermore, the random variable Y, following (2.1), extends some distributions, which have been introduced in the literature. The odd Weibull distribution is the particular case for $N=1$. The EPS (Chahkandi and Ganjali, 2008) and WPS (Morais and Barreto-Souza, 2011) distributions are obtained by taking $\gamma=1$ and $\gamma=1,\beta=1$ respectively. Since

$$\lim_{\theta\to0+}F(y;\boldsymbol{\xi})=\lim_{\theta\to0+}\{C(\theta)\}^{-1}C\left(\theta\left\{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}^{-1}\right)=\left\{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}^{-1},$$

the OW distribution is a limiting particular case.

3. Some useful properties

Some basic statistical and mathematical properties of the COWPS distribution are provided in this section.

3.1. Reliability functions

The probability density, survival and hazard rate (hrf) functions are given by

$$f(y;\boldsymbol{\xi}) = \frac{\alpha\beta\gamma\theta y^{\beta-1}e^{\alpha y^{\beta}}\left(e^{\alpha y^{\beta}}-1\right)^{\gamma-1}}{C(\theta)\left\{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}^{2}} C'\left(\frac{\theta\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}\right), \quad y>0, \tag{3.1}$$

$$\bar{F}(y;\xi) = 1-\{C(\theta)\}^{-1}C\left(\frac{\theta\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}\right), \quad y>0,$$

and

$$\begin{aligned} h(y;\boldsymbol{\xi}) &= \frac{\alpha\beta\gamma\theta y^{\beta-1}e^{\alpha y^{\beta}}\left(e^{\alpha y^{\beta}}-1\right)^{\gamma-1}}{\left\{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}^{2}} C'\left(\frac{\theta\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}\right) \\ &\quad \left\{C(\theta)-C\left(\frac{\theta\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}\right)\right\}^{-1}, \quad y>0. \end{aligned}$$

The COWPS is simulated by inverting $F(y;\boldsymbol{\xi}) = U$ in (2.1) as follows: if U has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation

$$Q(U;\boldsymbol{\xi}) = \left\{\frac{1}{\alpha}\log\left(1+\phi(U,\theta)\right)\right\}^{\frac{1}{\beta}}, \tag{3.2}$$

has the pdf (3.1), where

$$\phi(U,\theta) = \frac{C^{-1}\left(UC(\theta)\right)}{\theta - C^{-1}\left(UC(\theta)\right)},$$

and $C^{-1}(.)$ is the inverse of $C(.)$ function.

3.2. Moments, moment generating, and mean residual lifetime functions

Let Y be a COWPS random variable with pdf (3.1). Using the concept of power series and some other mathematical expansions, we derived two linear representations for the cdf and pdf of COWPS distribution. Since

$$\begin{aligned} \left\{\frac{\left(1-e^{-\alpha y^{\beta}}\right)^{\gamma}}{e^{-\alpha\gamma y^{\beta}}+\left(1-e^{-\alpha y^{\beta}}\right)^{\gamma}}\right\}^{n} &= \frac{\sum_{k=0}^{\infty}\lambda_k e^{-\alpha k y^{\beta}}}{\sum_{k=0}^{\infty}\rho_k e^{-\alpha k y^{\beta}}} \\ &= \sum_{r=0}^{\infty} c_r e^{-\alpha r y^{\beta}}, \end{aligned}$$

where ρ_k is defined by Cordeiro et al. (2015),

$$\lambda_k = \lambda_k(n,\gamma) = (-1)^k\binom{\gamma n}{k},$$

$$c_r = c_r(n,\gamma) = \frac{1}{\rho_0}\left(\rho_r - \frac{1}{\rho_0}\sum_{s=1}^{r}\rho_s c_{r-s},\right),$$

and $c_0 = \lambda_0/\rho_0$. Then, equations (3.1) and (2.1) can be expressed as

$$F(y,\boldsymbol{\xi}) = \theta^n\{C(\theta)\}^{-1}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}c_r(n,\gamma)e^{-\alpha r y^{\beta}}, \quad y>0,$$

and

$$f(y,\boldsymbol{\xi}) = \alpha\beta\theta^n\{C(\theta)\}^{-1}y^{\beta-1}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}d_r(n,\gamma)e^{-\alpha r y^{\beta}}, \quad y>0, \tag{3.3}$$

where $d_r(n,\gamma) = -rc_r(n,\gamma)$. Equation (3.3) is the main result of this section. So, several mathematical properties of the proposed family such as moments and moment generating function can be obtained by using this expansion. The formula for the sth moment of Y is obtained from (3.3) as

$$\begin{aligned}
\mu'_s &= \alpha\beta\theta^n[C(\theta)]^{-1}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}d_r(n,\gamma)\int_0^{+\infty}y^{s+\beta-1}e^{-\alpha r y^{\beta}}dy \\
&= \theta^n[C(\theta)]^{-1}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}d_r(n,\gamma)\frac{\Gamma\left(\frac{s}{\beta}+1\right)}{\alpha^{\frac{s}{\beta}}r^{\frac{s}{\beta}+1}} \\
&= \alpha^{-\frac{s}{\beta}}\theta^n[C(\theta)]^{-1}\Gamma\left(\frac{s}{\beta}+1\right)\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}d_r(n,\gamma)r^{-\frac{s}{\beta}-1}.
\end{aligned}$$

The moment generating function of Y is determined from (3.1) by direct integration and is given by

$$\begin{aligned}
M_Y(t) &= E\left[e^{tY}\right] = E\left[\sum_{s=0}^{\infty}\frac{(ty)^s}{s!}\right] = \sum_{s=0}^{\infty}\frac{t^s}{s!}\mu'_s \\
&= \alpha^{-\frac{s}{\beta}}\theta^n[C(\theta)]^{-1}\Gamma\left(\frac{s}{\beta}+1\right)[C(\theta)]^{-1}\sum_{s=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}d_r(n,\gamma)r^{-\frac{s}{\beta}-1}.
\end{aligned}$$

Given survival to time y_0, the residual life is the period from y_0 until the time of failure. The mean residual lifetime of the COWPS distribution is given by

$$\begin{aligned}
m(y_0) &= [S(y_0)]^{-1}\int_{y_0}^{\infty}vf(v)dv - y_0 \\
&= [S(y_0)]^{-1}\alpha\beta\theta^n\{C(\theta)\}^{-1}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}d_r(n,\gamma)\int_{y_0}^{\infty}v^{\beta}e^{-\alpha r v^{\beta}}dv - y_0 \\
&= \frac{\theta^n\Gamma_{y_0}\left(\beta^{-1}+1\right)}{\alpha^{\frac{1}{\beta}}\left[S(y_0)\right]C(\theta)}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}d_r(n,\gamma)r^{-\frac{1}{\beta}-1} - y_0.
\end{aligned}$$

4. Some special cases

Some special cases of the COWPS distribution are introduced in this section. To illustrate the flexibility of the distributions, graphs of the pdf and hazard rate function for some selected values of the parameters are presented.

4.1. Complementary odd Weibull Poisson (COWP)

The complementary odd Weibull Poisson (COWP) distribution is defined by pdf (3.1) with $C(\theta) = \exp(\theta) - 1$, leading to

$$f(y;\boldsymbol{\xi}) = \frac{\alpha\beta\gamma\theta y^{\beta-1}e^{\alpha y^{\beta}}\left(e^{\alpha y^{\beta}}-1\right)^{\gamma-1}}{\left(e^{\theta}-1\right)\left\{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}^{2}}\exp\left(\frac{\theta\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}\right), \quad y>0,$$

where $\alpha >$ and $\beta\gamma > 0$. The θ parameter space in COWP distribution is extended to $\mathbb{R}-\{0\}$ for more flexibility of this distribution. Similar extensions of the parameter space may be done to other COWPS distributions in addition to the COWP distribution, as can be viewed in Table 1 (For more details see Goldoust et al., 2017).

4.2. Complementary odd Weibull geometric (COWG)

The complementary odd Weibull geometric (COWG) distribution is defined by pdf (3.1) with $C(\theta) = \theta/(1-\theta)$, leading to

$$f(y;\boldsymbol{\xi}) = \frac{\alpha\beta\gamma(1-\theta)y^{\beta-1}e^{\alpha y^{\beta}}\left(e^{\alpha y^{\beta}}-1\right)^{\gamma-1}}{\left\{1+(1-\theta)\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}^{2}}, \quad y>0, \tag{4.1}$$

where $\alpha >$, $\beta\gamma > 0$ and $\theta \in (-\infty, 1)$. Figures 1 and 2 show graphs of the probability density and hazard rate functions of the COWG distribution for some selected values of the parameters.

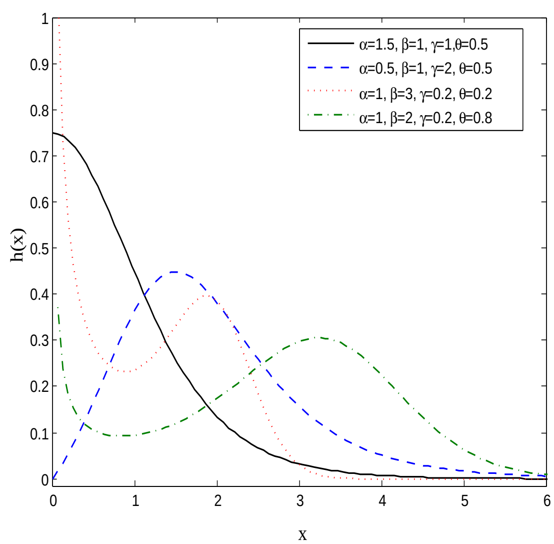

Figure 1: Graphs of the COWG pdf for some values of its parameters.

4.3. Complementary odd Weibull logarithmic (COWL)

The complementary odd Weibull logarithmic (COWL) distribution is defined by pdf (3.1) with $C(\theta) = -\log(1-\theta)$, leading to

$$f(y;\boldsymbol{\xi}) = \frac{\alpha\beta\gamma\theta y^{\beta-1}e^{\alpha y^{\beta}}\left(e^{\alpha y^{\beta}}-1\right)^{\gamma-1}}{-\log\left(1-\theta\right)\left\{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}\left\{1+\left(1-\theta\right)\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}}, \quad y>0,$$

where $\alpha >$, $\beta\gamma > 0$ and $\theta \in (-\infty, 1)$.

4.4. Complementary odd Weibull binomial (COWB)

The complementary odd Weibull binomial (COWB) distribution is defined by pdf (3.1) with $C(\theta) = (1+\theta)^{m}-1$, leading to

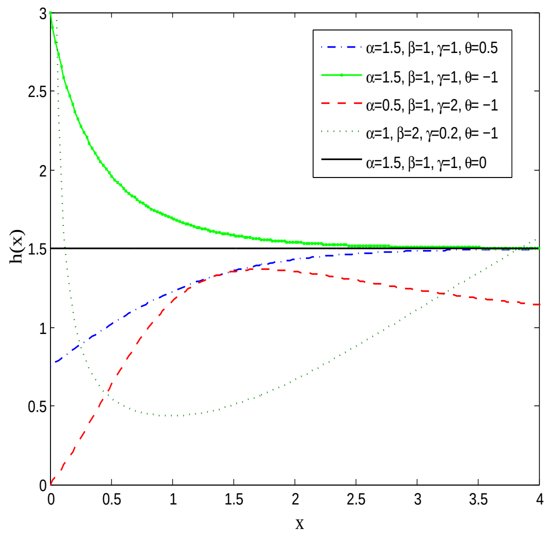

Figure 2: Graphs of the COWG hrf for some values of its parameters.

$$f(y;\boldsymbol{\xi}) = \left(\frac{\alpha\beta\gamma r\theta y^{\beta-1}e^{\alpha y^{\beta}}\left(e^{\alpha y^{\beta}}-1\right)^{\gamma-1}}{\left((1+\theta)^{r}-1\right)\left\{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}\right\}^{2}}\right)\left(\frac{1+(1+\theta)\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha y^{\beta}}-1\right)^{\gamma}}\right)^{r-1}, \quad y>0,$$

where $\alpha >, \beta\gamma > 0$ and $\theta \in (-1, +\infty)$.

5. Maximum likelihood estimation

Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample with observed values $y_1, y_2, \ldots, y_n$ from the COWPS family of distributions with an unknown vector of parameters $\boldsymbol{\xi} = (\alpha, \beta, \gamma, \theta)$. The log-likelihood function of $\boldsymbol{\xi}$ is

$$\begin{aligned}
\ell\left(\boldsymbol{\xi}|\mathbf{y}\right) &= n\log[\alpha] + n\log[\beta] + n\log[\gamma] + n\log[\theta] - n\log[C(\theta)] + (\beta-1)\sum_{i=1}^{n}\log[y_i] \\
&+\alpha\sum_{i=1}^{n}{y_i}^{\beta} + (\gamma-1)\sum_{i=1}^{n}\log\left[e^{\alpha y_i^{\beta}}-1\right] - 2\sum_{i=1}^{n}\log\left[1+\left(e^{\alpha y_i^{\beta}}-1\right)^{\gamma}\right] \\
&+\sum_{i=1}^{n}\log\left[C'\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)\right]. \qquad (5.1)
\end{aligned}$$

By differentiating (5.1) with respect to α, β, γ and θ, we obtain the components of score vector $U_n(\xi) = (\partial\ell/\partial\alpha, \partial\ell/\partial\beta, \partial\ell/\partial\gamma, \partial\ell/\partial\theta)$, where

$$\begin{aligned}
\frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}{y_i}^{\beta} + (\gamma-1)\sum_{i=1}^{n}\frac{y_i^{\beta}e^{\alpha y_i^{\beta}}}{e^{\alpha y_i^{\beta}}-1} - 2\gamma\sum_{i=1}^{n}\frac{y_i^{\beta}\left(e^{\alpha y_i^{\beta}}-1\right)^{\gamma-1}}{1+\left(e^{\alpha y_i^{\beta}}-1\right)^{\gamma}} \\
&+\gamma\theta\sum_{i=1}^{n}\frac{y_i^{\beta}e^{\alpha y_i^{\beta}}\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma-1}C''\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)}{\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{2}C'\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)},
\end{aligned}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta}+\sum_{i=1}^{n}\log\left[y_i\right]+\alpha\left(\gamma-1\right)\sum_{i=1}^{n}\frac{\log[y_i]y_i^{\beta}e^{\alpha y_i^{\beta}}}{e^{\alpha y_i^{\beta}}-1}-2\alpha\gamma\sum_{i=1}^{n}\frac{\log[y_i]y_i^{\beta}\left(e^{\alpha y_i^{\beta}}-1\right)^{\gamma-1}}{1+\left(e^{\alpha y_i^{\beta}}-1\right)^{\gamma}}$$
$$+\alpha\gamma\theta\sum_{i=1}^{n}\frac{\log[y_i]y_i^{\beta}e^{\alpha y_i^{\beta}}\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma-1}C''\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)}{\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{2}C'\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)},$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma}+\sum_{i=1}^{n}\log\left[e^{\alpha y_i^{\beta}}-1\right]-2\sum_{i=1}^{n}\frac{\log\left[e^{\alpha y_i^{\beta}}-1\right]\left(e^{\alpha y_i^{\beta}}-1\right)^{\gamma}}{1+\left(e^{\alpha y_i^{\beta}}-1\right)^{\gamma}}$$
$$+\theta\sum_{i=1}^{n}\frac{\log\left[e^{\alpha y_i^{\beta}}-1\right]\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}C''\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)}{\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{2}C'\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)},$$

and

$$\frac{\partial \ell}{\partial \theta}=\frac{n}{\theta}-\frac{nC'\left(\theta\right)}{C\left(\theta\right)}+\sum_{i=1}^{n}\frac{C''\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)}{\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}C'\left(\theta\left\{\left(e^{\alpha y_i^{\beta}}-1\right)^{-\gamma}+1\right\}^{-1}\right)},$$

where $C'\left(.\right)$ and $C''\left(.\right)$ are the first and second derivative of $C(.)$ respectively. The maximum likelihood estimators are obtained by equating $U_n\left(\xi\right)$ to zero. This non-linear system of equations has not closed form and the estimated values of the parameters $(\widehat{\boldsymbol{\xi}})$ must be found by using iterative methods. For interval estimations and hypothesis tests on the parameters in $\boldsymbol{\xi}$, we require the 4×4 observed information matrix $I_n(\boldsymbol{\xi})$ as follow

$$I_n(\boldsymbol{\xi})=\begin{pmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\gamma} & \frac{\partial^2\ell}{\partial\alpha\partial\theta} \\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\gamma} & \frac{\partial^2\ell}{\partial\beta\partial\theta} \\ \frac{\partial^2\ell}{\partial\gamma\partial\alpha} & \frac{\partial^2\ell}{\partial\gamma\partial\beta} & \frac{\partial^2\ell}{\partial\gamma^2} & \frac{\partial^2\ell}{\partial\gamma\partial\theta} \\ \frac{\partial^2\ell}{\partial\theta\partial\alpha} & \frac{\partial^2\ell}{\partial\theta\partial\beta} & \frac{\partial^2\ell}{\partial\theta\partial\gamma} & \frac{\partial^2\ell}{\partial\theta^2} \end{pmatrix}.$$

Under standard regularity conditions when $n\to\infty$, the distribution of $\left(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\right)$ could be approximated by a multivariate normal $\mathrm{N}_4\left(\mathbf{0}, I_n(\widehat{\boldsymbol{\xi}})^{-1}\right)$ distribution to construct approximate confidence intervals for the parameters and tests of hypotheses (Cox and Hinkley, 1979).

6. Simulation study

In this section, the results of a simulation study are presented. We evaluate the performance of the maximum likelihood estimates of the COWG distribution as the special case of COWPS distribution by using (3.2) with respect to sample size n. We repeated simulation study $k = 1000$ times with sample size $n = 100, 200, 500, 1000$ and parameter values $I : \alpha = 0.5$, $\beta = 1.5$, $\gamma = 0.7$, $\theta = 0.5$ and $II : \alpha = 1$, $\beta = 2$, $\gamma = 1.5$, $\theta = 0.7$ then the parameters are estimated by ML method. The bias and mean squared error (MSE) of the parameters are given respectively by

$$\mathrm{bias}_{\xi}\left(n\right)=\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\xi}_i-\xi\right),$$

Table 2: The mean, bias and MSE of the MLE estimators from 1000 samples.

	I				II			
n	$\boldsymbol{\xi}$	Average	Bias	MSE	$\boldsymbol{\xi}$	Average	Bias	MSE
100	α	0.0601	0.101	0.096	α	1.355	0.355	0.472
	β	1.578	0.078	0.098	β	0.1.731	0.269	0.342
	γ	0.643	-0.057	0.035	γ	1.372	-0.128	0.142
	θ	0.432	-0.068	0.069	θ	0.641	-0.059	0.086
200	α	0.586	0.086	0.072	α	1.203	0.203	0.311
	β	1.541	0.041	0.079	β	1.845	-0.155	0.301
	γ	0.671	-0.029	0.021	γ	1.389	-0.111	0.069
	θ	0.451	-0.049	0.054	θ	0.735	0.035	0.069
500	α	0.551	0.053	0.51	α	1.102	0.102	0.213
	β	1.528	0.028	0.055	β	1.982	-0.018	0.206
	γ	0.689	-0.011	0.005	γ	1.459	-0.041	0.057
	θ	0.467	-0.032	0.033	θ	0.711	0.011	0.052
1000	α	0.512	0.012	0.031	α	1.032	0.032	0.101
	β	1.513	0.013	0.039	β	2.080	0.080	0.095
	γ	0.695	-0.005	0.003	γ	1.485	-0.015	0.027
	θ	0.498	-0.002	0.026	θ	0.696	-0.004	0.034

and

$$\mathrm{MSE}_{\xi}\left(n\right)=\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\xi}_i-\xi\right)^2,$$

for $\xi=\alpha,\beta,\theta,\lambda$ where $\hat{\xi}_i$ is ith MLE of ξ.

The empirical results are presented in Table 2. The results indicate that the maximum likelihood estimators carry out well for estimating the parameters of the COWG model. According to Table 2, it can be concluded that as the sample size n increases, the MSEs decay toward zero. Furthermore, the bias of estimated values of the parameters is greatly reduced as the sample size n is increased.

7. Illustrative real data examples

In this section, we prepare an application to real data to demonstrate the importance of the COWPS through the special model complementary odd Weibull geometric (COWG) with pdf (4.1). The data are times to the death of 26 psychiatric patients. This dataset has been studied by Elbatal et al. (2015). The data are: 1, 1, 2, 22, 30, 28, 32, 11, 14, 36, 1, 33, 33, 37, 35, 25, 31, 22, 26, 24, 35, 34, 30, 35, 40, 39.

This new four-parameters distribution is compared to its sub-models and some well-known lifetime distributions. The following different distributions have been used: exponential distribution (Exp), Weibull distribution (Wei), odd Weibull distribution (OW) (Cooray, 2006), beta Weibull distribution (BW) (Famoye and Olumolade, 2005) and beta generalized exponential distribution (BGE) (Barreto-Souza et al., 2008) to analyze the data. Estimates of the parameters of COWPS distribution and their standard error, -log-likelihood $(-\ell(\widehat{\boldsymbol{\xi}}))$, Kolmogorov-Smirnov statistic (K-S) and its p-value, Akaike information criterion (AIC), corrected Akaike information criterion (AICc) and Bayesian information criterion (BIC) are reported in Table 3.

Table 3 shows that the COWG distribution gives the best-fit respect to all indices. Due to the $\ell(\widehat{\boldsymbol{\xi}})$ value, the largest K-S p-value, the smallest AIC value, the smallest AICc value, and the smallest BIC value are obtained for the COWG distribution. Furthermore, the histogram of the dataset and plots the estimated densities are displayed in Figure 3.

Finally, the total time on test (TTT) transform procedure, proposed by Aarset (1987), is provided. TTT-transform is a tool to identify the hazard behavior of the distributions. Graph of the TTT-transform is displayed in Figure 4 for psychiatric patient's dataset. Aarset (1987) proposed that If the TTT-transform graph is convex and concave, the hrf will have J-shape.

Table 3: Estimates and goodness-of-fit measures for the psychiatric patient's dataset.

Model	$\widehat{\boldsymbol{\xi}}$	$-\ell(\widehat{\boldsymbol{\xi}})$	K-S	p-value	AIC	AICc	BIC
Exp	0.0396	109.969	0.471	0.042	221.938	222.105	223.196
$SE\left(\widehat{\boldsymbol{\xi}}\right)$	0.0078						
Wei	1.6659, 0.0040	106.617	0.395	0.076	217.234	217.756	219.750
$SE\left(\widehat{\boldsymbol{\xi}}\right)$	(0.14207, 0.0019)						
OW	2.3569, 0.0021, 0.2612	102.166	0.267	0.0.185	210.332	213.189	214.106
$SE\left(\widehat{\boldsymbol{\xi}}\right)$	(0.1763, 0.0008, 0.1296)						
BW	2.8460, 0.0021, 0.1813, 0.0285	96.722	0.207	0.431	201.544	204.544	206.476
$SE\left(\widehat{\boldsymbol{\xi}}\right)$	(0.4389, 0.0004, 0.1482, 0.0433)						
BGE	8.4938, 0.0173, 0.1583, 202.2628	101.603	0.39	0.251	211.206	214.206	216.238
$SE\left(\widehat{\boldsymbol{\xi}}\right)$	(0.0097, 4.8377, 0.0867, 245.0212)						
COWG	2.3823, 0.0038, 0.1709, 0.8118	93.371	0.147	0.541	194.742	196.647	199.774
$SE\left(\widehat{\boldsymbol{\xi}}\right)$	(0.1768, 0.0015, 0.0837, 0.1201)						

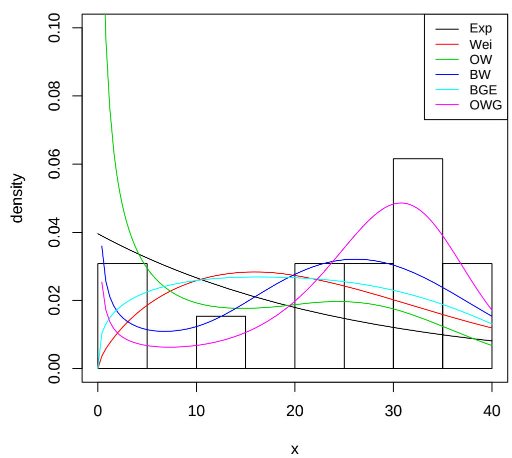

Figure 3: Fitted densities to the psychiatric patient's dataset.

Graph of the estimated hazard rate function is displayed in Figure 4 and it has J-shape. Hence, the COWG distribution could be the appropriate model for the fitting of this dataset.

8. Concluding Remarks

We introduce a new four-parameters distribution, so-called complementary odd Weibull power series (COWPS), which includes several distributions widely used in the lifetime literature. Mathematical properties including the moments and moments generating functions were presented. The maximum likelihood estimation technique is used to estimate the model parameters. A simulation study is presented to investigate the performance of the estimators. Finally, an application of the new distribution has also been demonstrated with real-life data. The results, compared with sub-model and other well-known distributions, revealed that the COWPS provides a better fit for modeling real-life data.

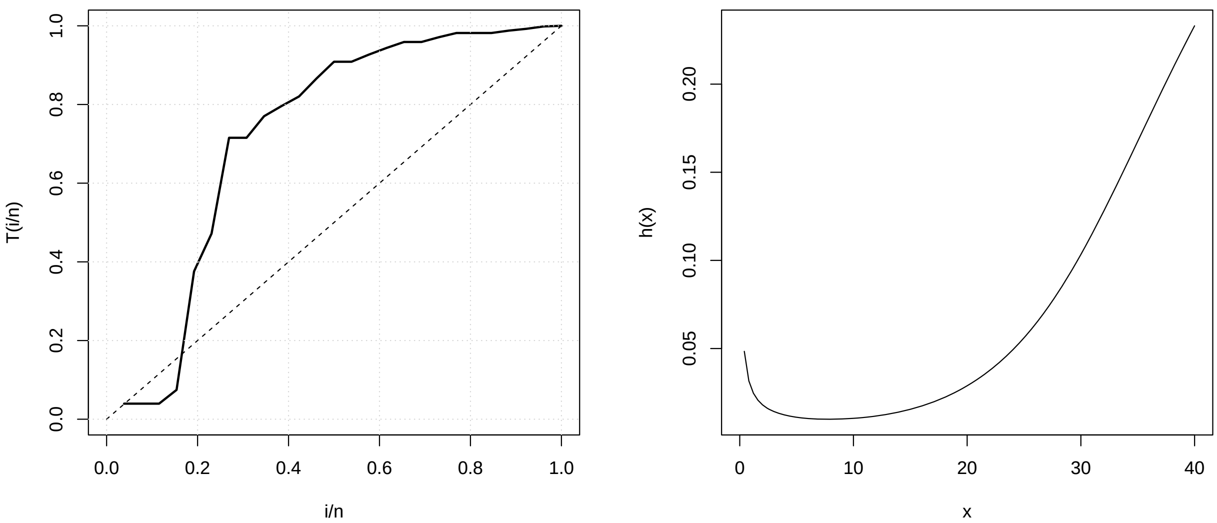

Figure 4: Graphs of the estimated hrf and total test time.

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