AUT Journal of Mathematics and Computing

AUT J. Math. Com., 1(1) (2020) 89-94
DOI: 10.22060/ajmc.2019.16174.1022

The Validity of a Thompson's Problem for PSL(4, 7)

Behrooz Khosravi*[a], Cyrus Kalantarpour[b]

[a] *Department of Mathematics and Computer Science, Amirkabir University of Technology, 424, Hafez Ave., Tehran 15914, Iran*
[b] *Department of Mathematics and Computer Science, Amirkabir University of Technology, 424, Hafez Ave., Tehran 15914, Iran*

ABSTRACT: Let $\pi_e(G)$ be the set of elements orders of G. Also let s_n be the number of elements of order n in G and $\mathrm{nse}(G) = \{s_n \mid n \in \pi_e(G)\}$. In this paper, we prove that if G is a group such that $\mathrm{nse}(G) = \mathrm{nse}(\mathrm{PSL}(4, 7))$, $19 \,|\, |G|$ and $19^2 \nmid |G|$, then $G \cong \mathrm{PSL}(4, 7)$. As a consequence of this result, it follows that Thompson's problem is satisfied for the simple group PSL(4, 7).

Review History:

Received:21 April 2019
Revised:11 June 2019
Accepted:12 June 2019
Available Online:1 February 2020

Keywords:

Thompson's problem
Characterization
Number of elements of the same order
Projective special linear group
Hall subgroup
NSE
Sporadic groups
Python

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. The set of element orders of G is denoted by $\pi_e(G)$. Let $k \in \pi_e(G)$. We denote the number of elements of order k in G by $s_k(G)$ or simply by s_k. Let $\mathrm{nse}(G) = \{s_k | k \in \pi_e(G)\}$. For a finite group G and positive integer n, let $L_n(G) = \{g \in G | g^n = 1\}$. The groups G_1 and G_2 are called of same order type if and only if $|L_n(G_1)| = |L_n(G_2)|$, for each $n \in \mathbb{N}$. In 1987, J. G. Thompson posed a question as follows:

Thompson's Problem. Suppose G_1 and G_2 are the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

It is well known that if G_1 and G_2 are of the same order type, then $|G_1| = |G_2|$ and $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$. It is interesting to investigate Thompson's problem by $|G|$ and $\mathrm{nse}(G)$. The following example shows that there are finite groups which are not characterizable by $\mathrm{nse}(G)$ and $|G|$. If $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = \mathrm{PSL}(3, 4) \rtimes \mathrm{C}_2$, then $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$ and $|G_1| = |G_2|$, but $G_1 \ncong G_2$.

Let G be a group and H be one of the following groups. Then $|G| = |H|$ and $\mathrm{nse}(G) = \mathrm{nse}(H)$ if and only if $G \cong H$:

- All sporadic simple groups. [1]
- Simple K_i-groups, where $i \in \{3, 4\}$. ([11],[13])

*Corresponding author.
E-mail addresses: khosravibbb@yahoo.com, siruscag@aut.ac.ir

In [8], it is proved that $A_4 \cong \mathrm{PSL}(2,3)$, $A_5 \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ and $A_6 \cong \mathrm{PSL}(2,9)$ are uniquely determined by $\mathrm{nse}(G)$. Recently, it is shown that $\mathrm{PSL}(2,81)$ (see [10]), $\mathrm{PSL}(3,4)$ (see [9]), $\mathrm{PSU}(3,4)$ (see [3]), are characterizable by nse. So far it is shown that $\mathrm{PSL}(2,q)$ (see [2]) and $\mathrm{PSL}(n,2)$ (see [17]), with some conditions on q and the cardinal of the group are characterizable by their nse. In this paper we prove that if G is a group such that $\mathrm{nse}(G) = \mathrm{nse}(\mathrm{PSL}(4,7))$, $19 \big| |G|$ and $19^2 \nmid |G|$, then $G \cong \mathrm{PSL}(4,7)$. Since $|\mathrm{PSL}(4,7)| = 2317591180800$, we use GAP [16], Python and Microsoft Excel to handle extreme computations for proving $\pi(G) = \{2,3,5,7,19\}$, the computation of $s_2, s_3, s_5, s_7, s_{19}, \ldots$ and the number of Sylow subgroups. Let p be a prime number and $S = {}^dX_m(q)$ belong to a family of finite simple groups of Lie type with rank m on a field of characteristic p. Let $Y = \{p_1, p_2, \ldots, p_n\}$ be a set of primes. We write $ord_p(S,Y) = \alpha$ if $GF(p^\alpha)$ is the smallest field of characteristic p such that $Y \subseteq \pi(S)$.

If p is a prime number, then we write: $p^\alpha \| |G|$, if $p^\alpha \big| |G|$ and $p^{\alpha+1} \nmid |G|$. We denote the set of all Sylow p-subgroups in G by $\mathrm{Syl}_p(G)$.

2. Preliminary Lemmas

Lemma 2.1. *[5] Let G be a finite group and n be a positive integer such that $n \big| |G|$. If $L_n(G) = \{g \in G | g^n = 1\}$, then $n \big| |L_n(G)|$.*

Lemma 2.2. *[14] Let G be a group containing more than two elements. If $s = sup\{s_n | n \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2-1)$.*

Lemma 2.3. *[11] Let G be a simple K_3-group. Then G is isomorphic to one of the following groups: A_5, A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(4,2)$, $\mathrm{PSU}(3,3)$.*

Lemma 2.4. *[13] Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(i) $A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2$.

(ii) $\mathrm{PSL}(2,\mathrm{r})$, *where r is a prime number and $r^2-1 = 2^a3^bt^c$, where $a \geq 1, b \geq 1, c \geq 1$ and t is a prime greater than 3.*

(iii) $\mathrm{PSL}(2,2^\mathrm{m})$, *where $2^m - 1 = u$, $2^m + 1 = 3t^b$, where $m \geq 2$, u,t are primes and $t \geq 3$ and $b \geq 1$.*

(iv) $\mathrm{PSL}(2,3^\mathrm{m})$, *where $3^m + 1 = 4t$, and $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$ with $m \geq 2$, u,t are odd primes, $b \geq 1, c \geq 1$.*

(v) one of the following 28 simple groups:
$\mathrm{PSL}(2,16), \mathrm{PSL}(2,25), \mathrm{PSL}(2,49), \mathrm{PSL}(2,81), \mathrm{PSL}(3,4), \mathrm{PSL}(3,5), \mathrm{PSL}(3,7), \mathrm{PSL}(3,8)$,
$\mathrm{PSL}(3,17), \mathrm{PSL}(4,3), \mathrm{S}_4(4), \mathrm{S}_4(5), \mathrm{S}_4(7), \mathrm{S}_4(9), \mathrm{S}_6(2), \mathrm{O}_8^+(2), \mathrm{G}_2(3), \mathrm{PSU}(3,4), \mathrm{PSU}(3,5)$,
$\mathrm{PSU}(3,7), \mathrm{PSU}(3,8), \mathrm{PSU}(3,9), \mathrm{PSU}(4,3), \mathrm{PSU}(5,2), \mathrm{Sz}(8), \mathrm{Sz}(32), {}^3\mathrm{D}_4(2), {}^2\mathrm{F}_4(2)'$.

Lemma 2.5. *[7] Let G be a simple K_5-group, Then G is isomorphic to one of the following groups.*

(i) $\mathrm{PSL}(2,\mathrm{q})$, *where $|\pi(q^2-1)| = 4$.*

(ii) $\mathrm{PSU}(3,\mathrm{q})$, *where $|\pi((q^2-1)(q^3-1))| = 4$.*

(iv) $O_5(q)$, *where $|\pi(q^4-1)| = 4$.*

(v) $Sz(2^{2m+1})$, *where $|\pi((2^{2m+1}-1)(2^{4m+2}+1))| = 4$*

(vi) $R(q)$, *where q is an odd power of 3, $|\pi(q^2-1)| = 3$ and $|\pi(q^2-q+1)| = 1$*

(vii) $A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, \mathrm{PSL}(4,4), \mathrm{PSL}(4,5), \mathrm{PSL}(4,7)$

(viii) $\mathrm{PSL}(5,2), \mathrm{PSL}(5,3), \mathrm{PSL}(6,2), \mathrm{O}_7(3), \mathrm{O}_9(2), \mathrm{PS}_{\mathrm{p}_6}(3), \mathrm{PS}_{\mathrm{p}_8}(2), \mathrm{PSU}_4(4)$

(ix) $\mathrm{PSU}(4,5), \mathrm{PSU}(4,7), \mathrm{PSU}(4,9), \mathrm{PSU}(5,3), \mathrm{PSU}(6,2), \mathrm{O_8}^+(3)$

(x) $O_8{}^-(2), {}^3D_4(3), G_2(4), G_2(5), G_2(7), G_2(9)$.

Lemma 2.6. *[6] Let G be a finite solvable group and $|G| = mn$, where $m = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_r}^{\alpha_r}$ and $(m,n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and h_m be the number of Hall $\pi-$subgroups of G. Then $h_m = {q_1}^{\beta_1}{q_2}^{\beta_2}\ldots{q_s}^{\beta_s}$ satisfies the following conditions, for all $i \in \{1, \ldots, s\}$:*

(i) ${q_i}^{\beta_i} \equiv 1 \pmod{p_j}$ *for some p_j.*

(ii) the order of some chief factor of G is divided by ${q_i}^{\beta_i}$.

Lemma 2.7. *[12] Let G be a group and P be a cyclic subgroup of G of order p^α. If there is a prime r such that $p^\alpha r \in \pi_e(G)$, then $s_{p^\alpha r}(G) = s_r(C_G(P))s_{p^\alpha}(G)$. In particular, $\phi(r)s_{p^\alpha} \big| s_{p^\alpha r}$, where $\phi(r)$ is the Euler function.*

3. Main Results

Throughout this section always we suppose that $q = p^\alpha$, where p is a prime.

Lemma 3.1. *Let S be a simple group and $\pi(S) = \{2, 3, 5, 7, 19\}$ and $|S| \leq |\mathrm{PSL}(4, 7)|$. Then $S \cong \mathrm{PSL}(4, 7)$ or $\mathrm{PSU}(3, 19)$.*

Proof. Let $A = \{2, 3, 5, 7, 19\}$. Since S is a simple K_5-group, using Lemma 2.5, we consider each possibility:
First we note that $G \ncong Sz(2^{2m+1})$, since $3 \nmid |Sz|$.

Case (a). Let S be isomorphic to $\mathrm{PSL}(2,\mathrm{q})$, where q satisfies $|\pi(q^2-1)| = 4$. Since $|\mathrm{PSL}(2,\mathrm{q})| = \mathrm{q}(\mathrm{q}^2-1)/(2,\mathrm{q}-1)$ and $\pi(S) = A$, if $q = 2^\alpha$ or $q = 3^\alpha$, then $\alpha = 18$, since $ord_2(S, \pi(A)) = ord_3(S, \pi(A)) = 18$. Similarly if $q = 5^\alpha$, then $\alpha = 9$, if $q = 7^\alpha$, then $\alpha = 6$, if $q = 19^\alpha$, then $\alpha = 3$. But in all cases we get a contradiction by $|\pi(q^2-1)| = 4$. Similarly $S \ncong O_5(q)$

Case (b). If $S \cong \mathrm{PSU}(3,\mathrm{q})$ and $q = 2^\alpha$, then $\alpha = 12$, since $ord_2(S, \pi(A)) = 12$. Similarly if $q = 3^\alpha$, then $\alpha = 6$, if $q = 5^\alpha$, then $\alpha = 10$, if $q = 7^\alpha$, then $\alpha = 6$ and this is contrary to $|\pi((q^2-1)(q^3-1))| = 4$. If $q = 19^\alpha$, then $S \cong \mathrm{PSU}(3, 19)$.

Case (c). If $S \cong R(q)$, since $|R(q)| = q^3(q^{3n+1}(q-1))$, where $q = 3^{2n+1}$, then $5 \nmid |R(q)|$, since $3^{6n+3} = 27^{3n+1} \overset{10}{\equiv} 3$ or 7 and $3^{6n+3} + 1 \overset{10}{\equiv} 4$ or 8. Also $3^{2n+1} - 1 \overset{10}{\equiv} 6$ or 2. So this is a contradiction.

Case (d). If S is isomorphic to one of the 30 other groups that are mentioned in Lemma 2.5, then by [4], we can see $\pi(S) \neq \{2, 3, 5, 7, 19\}$ so this is a contradiction, except for $S = \mathrm{PSL}(4, 7)$.

□

Remark. Using a simple program in GAP, we determine $\mathrm{nse}(\mathrm{PSL}(4, 7)) = \{\lambda_1, ..., \lambda_{27}\}$, which are presented in the appendix of this paper.

Theorem 3.2. *If G is a group such that $\mathrm{nse}(G) = \mathrm{nse}(\mathrm{PSL}(4, 7)) = \{\lambda_1, ..., \lambda_{27}\}$ and $19 \| |G|$, then $G \cong \mathrm{PSL}(4, 7)$.*

Proof. By Lemma 2.2, G is a finite group. We know that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \pi_e(G)$, by Lemma 2.1 and the above discussion we have:

$$\begin{cases} \phi(m)|s_m, \\ m|\sum_{d|m} s_d. \end{cases} \tag{1}$$

Using (1) we can see that if $\{2, 3, 5, 7, 19, 25\} \subseteq \pi_e(G)$, then $s_3 = 2^1 5^2 7^3 11^1 20691$, $s_5 \in \{2^4 \times 3^2 \times 7^4 \times 19 \times 43 \times 1731,\ 2^6 \times 3^4 \times 7^6 \times 19\}$, $s_7 = 2^5 \times 3^2 \times 5^2 \times 13 \times 19 \times 43 \times 1811$, $s_{19} = 2^1 0 \times 3^3 \times 5^2 \times 7^6$, $s_{25} = 2^6 \times 3^4 \times 5 \times 7^6 \times 19$. We first prove that $\pi(G) \subseteq \{2, 3, 5, 7, 19\}$. We know that $s_2 = \lambda_2$, since λ_2 is the only odd number in $\mathrm{nse}(G)$, so $2\big||G|$. Using relations in (1) we see that $\pi(G) \subseteq \cup_{\lambda_i \in \mathrm{nse}(G)} \pi(1 + \lambda_i)$ and so $\pi(G) \subseteq B = \{2, 3, 5, 7, 19, b_6, ..., b_{60}\}$[1]. If $b_6 = 1471213 \in \pi(G)$, then $\phi(b_6) \nmid \lambda_i$, for every $i \in \{1, 2, ..., 27\}$, which is a contradiction by (1). Therefore $b_6 \notin \pi(G)$. Similar to the above discussion we get that $b_i \notin \pi(G)$, for each $b_i \in B_\delta$, where B_δ is defined in Appendix. For $b_{21} = 257$, using (1), if $257 \in \pi(G)$, then $s_{257} \in \{\lambda_4, \lambda_{16}, \lambda_{18}, \lambda_{22}, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{27}\}$. But we have no elements of order 2×257, since $2 \times 257 \nmid (s_1 + s_2 + s_{257} + t)$, for every $t \in \mathrm{nse}(G)$. So $P_{257} \in \mathrm{Syl}_{257}(G)$ acts fixed point freely on the set of elements of order 2. Therefore $|P_{257}|\big|s_2$ and this is a contradiction. Now if $b_i \in B \setminus (\{2, 3, 5, 7, 19\} \cup B_\delta)$, then similarly to the above $2b_i \notin \pi_e(G)$. Using the fact that P_{b_i} acts on the set of elements of order 19, fixed point freely, we get a contradiction. Therefore $\{2, 19\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 19\}$. Since $19 \| |G|$, so P_{19} is cyclic. Now if there exists an element of order 2×19, by (1) we have $2 \times 19\big|(1 + s_2 + s_{19} + s_{2\times19})$ and this is a contradiction, since by Lemma 2.7, $\phi(19) \times s_{19}|s_{2\times19}$. So there is no element of order 2×19. Therefore P_2 acts fixed point freely on the set of elements of order 19 and $|P_2|\big|s_{19}$ therefore $|P_2|\big|2^{10}$. Since P_{19} is cyclic and $exp(p_{19}) = 19$, the number of Sylow 19-subgroups of G equals $s_{19}/\phi(19) = 2^9 \times 3 \times 5^2 \times 7^6\big||G|$ so $\{3, 5, 7\} \subseteq \pi(G)$. Therefore $\pi(G) = \{2, 3, 5, 7, 19\}$. If there exists an element of order 5×19, by Lemma 2.7, $4 \times s_{19}\big|s_{5\times19}$ and we get a contradiction in the similar way to the above. So $|P_5|\big|s_{19}$ and therefore $|P_5|\big|5^2$. Similarly, if $7 \in \pi(G)$ by Lemma 2.7, we get that $s_{7\times19} = \lambda_{25}$, which is contrary to (1). So $|P_7|\big|s_{19}$ so $|P_7|\big|7^6$. Now we show that P_5 is cyclic. If $exp(P_5) = 5$ and $|P_5| = 25$, then by Lemma 2.1, $25\big|(1 + s_5)$, but this is a contradiction since $1 + s_5 \in \{5 \times 29 \times 1801 \times 187129, 5 \times 17 \times 251 \times 543143\}$. Therefore P_5 is cyclic. Similarly to the above we get

[1] see Appendix 4.1

that there is no element of order 25×3. Therefore $|p_3|\big|s_{25}$ so $|p_3|\big|3^4$. Also we know that the summation of all members of $nse(G)$ is less than $|G|$, so $2^7 \times 3^4 \times 7^6 \times 19 \times 91 = 1 + \sum_{i=1}^{27} \lambda_i \leq |G|$ and by the above discussion $|G| \leq 2^{10} \times 3^4 \times 5^2 \times 7^6 \times 19$. So $|G| = |\mathrm{PSL}(4,7)| = 2^9 \times 3^4 \times 5^2 \times 7^6 \times 19$ or $|G| = 2|\mathrm{PSL}(4,7)|$.

We claim that $|G| = |\mathrm{PSL}(4,7)|$. Otherwise suppose $|G| = 2|\mathrm{PSL}(4,7)|$. Since $exp(P_{19}) = 19$, it follows that $n_{19} = \frac{s_{19}}{\phi(19)} = 2^9 \times 3 \times 5^2 \times 7^6$ and by Lemma 2.6, we get that G is insolvable. So there is a subnormal series $1 \trianglelefteq K \trianglelefteq L \trianglelefteq G$ such that L/K is a simple K_i-group where, $i \in \{3,4,5\}$. Now using Lemmas 2.3, 2.4 and 2.5 we consider each possibility for L/K, separately.

If L/K is isomorphic to a simple K_3-group, by Lemma 2.1, it is isomorphic to one of the following groups: $A_5, A_6, \mathrm{PSL}(2,7), \mathrm{PSL}(2,8), \mathrm{PSU}(4,2), \mathrm{PSU}(3,3)$. Let $L/K \cong A_5$. Then $|G/L|\big|2^8 \times 3^3 \times 7^6 \times 5 \times 19$. Let $A/K := C_{G/K}(L/K)$. Then $A/K \cap L/K = 1$. Also $(G/K)/(A/K) \hookrightarrow Aut(L/K) = S_5$ and so $G/A \hookrightarrow S_5$. Since $A/K, L/K \lhd G/K$, $A/K \times L/K \leq G/K$. Therefore $|L/K|\big||G/A|$ and so $G/A \cong A_5$ or S_5. So $|A| = 2^8 \times 3^2 \times 5 \times 7^6 \times 19$ or $2^7 \times 3^2 \times 5 \times 7^6 \times 19$. By Sylow's theorem and using easy GAP computations, $n_{19}(A) \in \{1, 20, 96, 210, 343, 1008, ..., 2^7 \times 3 \times 5 \times 7^6\}$. Since $A \lhd G$, we have $n_{19}(A) = n_{19}(G)$, and so $s_{19}(G) \in \{18, 360, 1728, 3780, 6174, ..., 2^8 \times 3^3 \times 5 \times 7^6\}$, which is a contradiction. Now let $L/K \cong \mathrm{PSL}(2,8)$, we know that $|Aut(L/K)| = 1512$ and it has just one non-trivial normal subgroup N such that $|N| = 2^3 \times 3^2 \times 7$. So $|A| \in \{2^7 \times 3^2 \times 5^2 \times 7^5 \times 19,\ 2^7 \times 3 \times 5^2 \times 7^5 \times 19\}$. Now by Sylow's theorem $n_{19}(A) \in \{1,\ 20,\ 96,\ 210,\ 343,\ 400,\ 1008,\ 1920,\ 2205,\ 3136,\ 4200,\ 6860,\ 20160,\ 32928, \ldots,\ 6914880,\ 15126300\}$, which is a contradiction. Similarly we can rule out the other cases and get that L/K is not isomorphic to other K_3-simple groups or K_4-simple groups (for the computations we use the program in Appendix 4.2.1). Therefore L/K is a simple K_5-group and $\pi(L/K) = \{2,3,5,7,19\}$. So by Lemma 3.1, $L/K \cong \mathrm{PSL}(4,7)$ or $\mathrm{PSU}(3,19)$. If $L/K \cong \mathrm{PSU}(3,19)$, exactly with the same manner we get a contradiction using $s_{19}(G)$ (see 4.2.1). Similarly, if $L/K \cong \mathrm{PSL}(4,7)$, then $|K||2$. If $|K| = 2$, then $K \subseteq Z(G)$ so $2\big||Z(G)|$. Therefore $2 \times 19 \in \pi_e(G)$ and this is a contradiction. Therefore $K = 1$, $L = \mathrm{PSL}(4,7)$ and $|G/L| = 2$. So G acts on L. Let κ be the kernel of this action, If $\kappa = C_G(L) \neq 1$, then $G \cong L \times C_G(L)$. Therefore $2 \times 19 \in \pi_e(G)$ and this is a contradiction. If $|\kappa| = 1$, then $G \leq Aut(L)$ so $G/L \leq Out(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore $G \cong L : T$, where $|T| = 2$ and $s_2(G) > s_2(L) = s_2(\mathrm{PSL}(4,7))$, which is a contradiction. Therefore $G \cong \mathrm{PSL}(4,7)$ and we get the result. □

Acknowledgements

The authors are very thankful to the referee for valuable comments and suggestions which improved the manuscript.

4. Appendices

4.1. Numbers

$\{\lambda_1, ..., \lambda_{27}\} = \{2^5 \times 3^2 \times 5^2 \times 13 \times 19 \times 43 \times 181,\ 3 \times 7^4 \times 19 \times 43, 2 \times 5^2 \times 7^3 \times 11 \times 2069,\ 2^{10} \times 3^3 \times 5^2 \times 7^6,\ 2^5 \times 3^2 \times 5^2 \times 7^4 \times 19 \times 43,\ 2^2 \times 3^2 \times 7^4 \times 19 \times 3853,\ 2^4 \times 3^2 \times 7^4 \times 19 \times 43 \times 173,\ 2^6 \times 3^3 \times 5^2 \times 7^4 \times 13 \times 19,\ 2^7 \times 3^3 \times 5^2 \times 7^4 \times 13 \times 19,\ 2^1 \times 3 \times 5^3 \times 7^4 \times 19 \times 1151,\ 2^6 \times 3^2 \times 5^3 \times 7^3 \times 11 \times 13 \times 19,\ 2^6 \times 3^2 \times 5^2 \times 7^4 \times 19 \times 109,\ 2^6 \times 3^4 \times 7^6 \times 19,\ 2^6 \times 3^4 \times 5 \times 7^6 \times 19,\ 2^7 \times 3^4 \times 7^6 \times 19,\ 2^8 \times 3^4 \times 7^6 \times 19,\ 2^7 \times 3^4 \times 5 \times 7^6 \times 19, 2^8 \times 3^4 \times 5 \times 7^6 \times 19, 2^3 \times 3^2 \times 5^2 \times 7^5 \times 19127,\ 2^7 \times 3^3 \times 5^2 \times 7^5 \times 19, 2^5 \times 3^2 \times 5^3 \times 7^5 \times 19^2, 2^8 \times 3^3 \times 5^2 \times 7^5 \times 19, 2^{11} \times 3^3 \times 5^2 \times 7^6, 2^{10} \times 3^2 \times 5^2 \times 7^6, 2^{11} \times 3^4 \times 5^2 \times 7^6, 2^6 \times 3^2 \times 5^2 \times 7^6 \times 19,\ 2^9 \times 3^2 \times 5^2 \times 7^6 \times 19\}$

The b_i values:

$B = \{b_1, b_2, ..., b_{60}\} = \{2, 3, 5, 7, 19, 1471213, 2299, 14461, 4279946779, 11, 1283967491, 4889, 1294277, 29, 1801, 187129, 61, 79, 5316379, 51239260801, 9257, 22273, 211, 318004891, 13, 1181, 4663817, 17, 251, 543143, 191, 23334587, 23175911809, 23, 4909, 37321, 12379, 9360979, 231759118081, 17623, 4142287, 113, 653, 373909, 218423772001, 97, 631, 883, 1021, 29191, 506501, 101, 421, 49037, 257, 36809, 51577, 139, 21052169, 257510131201\}$.

$B_\delta = \{b_6, b_7, b_{11}, b_{12}, b_{13}, b_{16}, b_{21}, b_{22}, b_{24}, b_{26}, b_{27}, b_{30}, b_{32}, b_{35}, b_{36}, b_{37}, b_{38}, b_{40}, b_{44}, b_{49}, b_{50}, b_{50}, b_{51}, b_{54}, b_{56}, b_{57}, b_{59}\}$.

4.2. Python Codes

Using Version:3.7.0a4

4.2.1. finding number of Sylow-subgroups of A where $A/K = C_{G/K}(L/K)$

.

```
alpha=2**9*3**4*5**2*7**6*19 #|G|=alpha or beta
beta=2**10*3**4*5**2*7**6*19
p=[]
q=[]
def factors(n):
```

```
    global leng
    leng=0
    g=int(n)
    for i in range(1,g+1):
        if n%i==0:
            leng=leng+1
            p.append(i)

A={'z_1':2**3*3*7,'z_2':2**3*3**2*7,'z_3':2**6*3**2*5*7,'z_4':2**5*3**2*7**3*19,'z_5':2**6*3**4*5,'z_6':2**5*3**3*7,'z_7':2**4*3**2*5**3*7,'z_8':2**9*3**4*7*19,'z_9':2**7*3**6*5*7,'z_10':2**9*3**4*5*7,'z_11':2**2*3*5,'z_12':2**3*3**2*5,'z_13':2**3*3**2*5*7,'z_14':2**6*3**2*5*7,'z_15':2**6*3**4*5*7,'z_16':2**7*3**4*5**2*7} #z_i is the size of simple K_j-Groups candidates for L/K, determinated by Lemma[2.3,2.4,2.5].

B={'l_1':2**4*3*7,'l_2':2**3*3**3*7,'l_3':2**8*3**3*5*7,'l_4':2**6*3**3*7**3*19,'l_5':2**7*3**4*5,'l_6':2**6*3**3*7,'l_7':2**5*3**3*5**3*7,'l_8':2**10*3**6*7*19,'l_9':2**10*3**6*5*7,'l_10':2**9*3**4*5*7,'l_11':2**3*3**2*5,'l_12':2**5*3**2*5,'l_13':2**4*3**2*5*7,'l_14':2**7*3**2*5*7,'l_15':2**7*3**4*5*7,'l_16':2**8*3**4*5**2*7} #l_i's are the size Aut(L\K), where |L/K|=z_i.
NSE=[13841287200,5884851,390316850,81318988800,14123642400,6327720252,
48867802704,25619630400,51239260800,39380601750,67099032000,71603582400,
11587955904,57939779520,23175911808,46351823616,115879559040,231759118080,
72999523800,27590371200,218423772000,55180742400,162637977600,27106329600,
487913932800,32188766400,257510131200]
c0=c1=c2=c3=c4=c5=c6=c7=c8=c9=c10=c11=c12=c13=c14=c15=c16=0
h=0
for i in range(1,17):
    ci=B['l_{}'.format(i)]/A['z_{}'.format(i)]
    factors(ci)
    print("list of c{}".format(i)," divisors:",p)
    for j in range(0,len(p)):
        hj=beta/(p[j]*B['l_{}'.format(i)])
        q.append(hj)
    print("{}".format(i),q) #Computing the size of A.
    for k in range(0,len(q)):#Finding the number of Sylow 19-subgroups of A.
        if int(q[k])==q[k]:
            m=q[k]/19
        for t in range(1,int(m+1)):
            if t % m==0 and (t-1)%19==0:
                if t*18 in NSE:# checking that computed values for s_19 belongs to NSE or not.
                    print("computed values for s_19:",t)
    q=[]
    p=[]
```

4.3. Implementation of (1) relations

.

```
from math import gcd as gcd
def phi(n):
    global c
    c=0
    for i in range(1,n):
        if gcd(i,n)==1:
            c=c+1
    return(c)
########################################
NSE=[13841287200,5884851,390316850,81318988800,14123642400,6327720252,
48867802704,25619630400,51239260800,39380601750,67099032000,71603582400,
11587955904,57939779520,23175911808,46351823616,115879559040,231759118080,
72999523800,27590371200,218423772000,55180742400,162637977600,27106329600,
487913932800,32188766400,257510131200]
########################################
def fc(n,o):
    global a
    a=[]
    for i in NSE:
        if i%phi(n)==0:
            a.append(i)
    if o=="print":
        print("s_{} candidates determinated by first condition:".format(n),a)
    elif o=="return":
        return(a)
```

```
##########################################
def sc(p):
    for i in fc(p,"return"):
        if (1+i)%p==0:
            print(i)
```

References

[1] A. K. Asboei, S. S. S. Amiri, A. Iranmanesh, A. Tehranian, A characterization of sporadic simple groups by nse and order, J. Algebra Appl, 12(2) (2013) 1250158(3 pages).

[2] A. K. Asboei, S. S. S. Amiri, A. Iranmanesh, A new characterization of $\mathrm{PSL}(2,\mathrm{q})$ for some q, Ukrainian Mathematical Journal, 67 (9) (2016) 1297–1305.

[3] Chen .Deqin, A characterization of $\mathrm{PSU}(3,4)$ by nse, International Journal of Algebra and Statistics, 2(1) (2013) 51–56.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of finite groups, Maximal subgroups and ordinary characters for simple groups, Clarendon Press New York, 1985.

[5] J. D. Frobenius, Verallgemeinerung des Sylowschen Satse, Berliner Sitz, (1895) 981–993.

[6] M. Hall, The Theory of Groups, Macmillan, New York 1959.

[7] A. Jafarzadeh, A. Iranmanesh, On simple K_n-groups for $n = 5, 6$, London Mathematical Society Lecture Note Series, 340(2) (2005) 517-526.

[8] M. Khatami, B. Khosravi, Z. Akhlaghi, A new characterization for some linear groups, Monatsh. Math, 163 (2009) 39-50.

[9] S. Liu, A characterization of $\mathrm{PSL}(3,4)$, Science Asia, 39 (2013) 436–439.

[10] Leila. Mousavi, Bijan. Taeri, A characterization of $L_2(81)$ by nse, International Journal of Group Theory, 1(5) (2016) 29-35.

[11] C. G. Shao, W. J. Shi, Q. H. Jiang, A new characterization of simple K_3-groups, Adv. Math, 38 (2009) 327-330.

[12] C. G. Shao, Q. Jiang, New characterization of simple linear groups by nse, J. Algebra Appl, 13, 1350094 (2014).

[13] C. Shao, W. Shi, Q. Jiang, A new characterization of simple K_4-groups, Front. Math. China, 3 (2008) 355-370.

[14] R. Shen, C. Shao, Q. Jiang, W. Shi, V. Mazurov, A new characterization of A_5, Monatsh. Math, 160 (2010) 337-341.

[15] W.J. Shi, On simple K_4-groups, Chine.Sci. Bull, 36 (1991) 1281-1283.

[16] The GAP Group, GAP-Groups, Algorithms and Programming, Version 4.8.3 (2016) `http://www.gap-system.org`

[17] Yong. Yang, Shitian. Liu, A characterization of some linear groups by nse, An. Stiint. Univ. Al. I. Cuza Iasi Mat. (N.S.), vol. 2 (2016).

Please cite this article using:

Behrooz Khosravi, Cyrus Kalantarpour, The Validity of a Thompson's Problem for $\mathrm{PSL}(4,7)$, AUT J. Math. Com., 1(1) (2020) 89-94
DOI: 10.22060/ajmc.2019.16174.1022