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The regularity of binomial edge ideals of graphs

Sara Saeedi Madani*[a], Dariush Kiani[a]

[a]*Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran*

ABSTRACT: In this paper, we study the Castelnuovo-Mumford regularity and the graded Betti numbers of the binomial edge ideals of some classes of graphs. Our special attention is devoted to a conjecture which asserts that the number of maximal cliques of a graph provides an upper bound for the regularity of its binomial edge ideal.

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1. Introduction

The binomial edge ideal of a graph was introduced in [1] by Herzog et al., and in [2] by Ohtani at about the same time. Let G be a finite simple graph with the vertex set $[n]$ and the edge set $E(G)$. Also, let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring over a field K. Then the *binomial edge ideal* of G, denoted by J_G, is the ideal of S, generated by binomials $f_{ij} = x_i y_j - x_j y_i$, where $i < j$ and $\{i, j\} \in E(G)$. Namely,

$$J_G = (f_{ij} : i < j, \{i, j\} \in E(G)).$$

One could also see this ideal as an ideal generated by a collection of 2-minors of a $(2 \times n)$-matrix whose entries are all indeterminates. Many of the homological and algebraic properties and invariants of such ideals were studied in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Among those invariants are the graded Betti numbers and the Castelnuovo-Mumford regularity of J_G. In [14], the authors posed a conjecture about the Castelnuovo-Mumford regularity of the binomial edge ideal of graphs, which asserts that $\mathrm{reg}(J_G) \leq c(G) + 1$, where $c(G)$ is the number of maximal cliques of G. They could verify it for closed graphs in [10]. Later, in [8], the authors gained an upper bound for the Castelnuovo-Mumford regularity of the binomial edge ideal of a graph G on n vertices. Indeed, they showed that $\mathrm{reg}(J_G) \leq n$. In the same paper, a conjecture was also posed which asserts that $\mathrm{reg}(J_G) \leq n - 1$, whenever G is not P_n, the path over n vertices. In [7], this conjecture was proved.

In this paper, we study some of the graded Betti numbers and the Castelnuovo-Mumford regularity of the binomial edge ideals of graphs. Our special attention goes to the proposed upper bound $c(G)+1$ for the Castelnuovo-Mumford regularity of J_G in certain cases. The organization of this paper is as follows. In Section 2, we provide

**Corresponding author.*
E-mail addresses: sarasaeedi@aut.ac.ir, dkiani@aut.ac.ir

some of the definitions, facts and known results which are needed and used throughout the paper. In Section 3, we mainly study the Castelnuovo-Mumford regularity of J_G. Indeed, we reduce this study to a certain case in which there is no "free cut edges" in the graph. For this purpose, we first obtain some results on the graded Betti numbers of J_G. As consequences of our result, we provide some non-chordal graphs for which $c(G)+1$ is indeed an upper bound for the Castelnuovo-Mumford regularity of J_G.

2. Preliminaries

In this section, we review some notions and facts around graphs and binomial edge ideals associated to graphs, which we need throughout.

Let G be a graph and $e=\{v,w\}$ be an edge of G. Then, as usual, the vertices v and w are called the endpoints of e. If $\{e_1,\dots,e_t\}$ is a set of edges of G, then by $G\setminus\{e_1,\dots,e_t\}$ we mean the graph on the same vertex set as G in which the edges $e_1,\dots,e_t$ are omitted. Here, we simply write $G\setminus e$, instead of $G\setminus\{e\}$. An edge e of G whose deletion from the graph, yields a graph with more connected components than G, is called a *cut edge* of G.

Next, we recall a notation from [9]. If v and w are two vertices of a graph $G=(V,E)$ and $e=\{v,w\}$ is not an edge of G, then G_e is defined to be the graph on the vertex set V, and the edge set $E\cup\{\{x,y\} \ : \ x,y\in N_G(v) \text{ or } x,y\in N_G(w)\}$. By $N_G(v)$, we mean the set of all neighbors, (i.e. adjacent vertices) of the vertex v in G.

Recall that $\Delta(G)$ is the *clique complex* of G, the simplicial complex whose facets are the vertex sets of the maximal cliques of G.

Note that we also call a vertex of G a *free vertex* if it is a free vertex of the simplicial complex $\Delta(G)$.

3. Betti numbers and the regularity

In this section, we study some invariants of binomial edge ideals which arise from their minimal graded free resolutions. Indeed, we investigate about the graded Betti numbers and the Castelnuovo-Mumford regularity of the binomial edge ideal of a graph. In particular, we deal with a conjecture posed in [14] on the Castelnuovo-Mumford regularity of binomial edge ideals. Recall that the *Castelnuovo-Mumford regularity* (or simply *regularity*) of J_G is defined as

$$\operatorname{reg}(J_G)=\max\{j-i \ : \ \beta_{i,j}(J_G)\neq 0\}.$$

The following conjecture was posed in [14] and in some cases it has been proved, (see for example [5] and [10]). Denoted by $c(G)$, we mean the number of maximal cliques of G.

Conjecture 3.1. *Let G be a graph. Then* $\operatorname{reg}(J_G)\leq c(G)+1$.

Before stating our contribution to the above conjecture, we need to define a graph obtained from the original graph G by certain type of deletions in the following. Let G be a graph and let e be a cut edge of G such that its endpoints are free in $G\setminus e$. Then we call e, a *free cut edge* of G. Suppose that $\{e_1,\dots,e_t\}$ is the set of all free cut edges of G. Then, we call the graph $G\setminus\{e_1,\dots,e_t\}$, the *reduced graph* of G, and denote it by $\mathcal{R}(G)$. If G does not have any free cut edges, then we set $\mathcal{R}(G)=G$.

The next theorem is indeed the main theorem of this section, and enables us to reduce the study of Conjecture 3.1 to a certain situation.

Theorem 3.2. *Let G be a connected graph on n vertices and let $H_1,\dots,H_q$ be the connected components of $\mathcal{R}(G)$. If Conjecture 3.1 is valid for all $H_1,\dots,H_q$, then it is also valid for G.*

To prove the above theorem, we need some ingredients which we provide in the following.

Proposition 3.3. *Let G be a graph and let e be a cut edge of G. Then*

(a) $\beta_{i,j}(J_G)\leq\beta_{i,j}(J_{G\setminus e})+\beta_{i-1,j-2}(J_{(G\setminus e)_e})$, *for all* $i,j\geq 1$;
(b) $\operatorname{pd}(J_G)\leq\max\{\operatorname{pd}(J_{G\setminus e}),\operatorname{pd}(J_{(G\setminus e)_e})+1\}$;
(c) $\operatorname{reg}(J_G)\leq\max\{\operatorname{reg}(J_{G\setminus e}),\operatorname{reg}(J_{(G\setminus e)_e})+1\}$.

Proof. It is enough to consider the short exact sequence

$$0\longrightarrow S/J_{G\setminus e}:f_e(-2)\xrightarrow{f_e}S/J_{G\setminus e}\longrightarrow S/J_G\to 0.$$

Then, the statements follow by the mapping cone, and the fact that we have $J_{G\setminus e}:f_e=J_{(G\setminus e)_e}$ by [9, Theorem 3.4].

□

Proposition 3.4. *Let G be a graph and let e be a free cut edge of G. Then*

(a) $\beta_{i,j}(J_G) = \beta_{i,j}(J_{G\setminus e}) + \beta_{i-1,j-2}(J_{G\setminus e})$, *for all* $i, j \geq 1$*;*
(b) $\mathrm{pd}(J_G) = \mathrm{pd}(J_{G\setminus e}) + 1$*;*
(c) $\mathrm{reg}(J_G) = \mathrm{reg}(J_{G\setminus e}) + 1$.

Proof. We have $J_{(G\setminus e)_e} = J_{G\setminus e}$, since e is a free cut edge of G. So, one may consider the short exact sequence

$$0 \longrightarrow S/J_{G\setminus e}(-2) \xrightarrow{f_e} S/J_{G\setminus e} \longrightarrow S/J_G \to 0$$

instead of the one mentioned in the proof of Proposition 3.3. Let $\mathcal{E}$ be the minimal graded free resolution of $S/J_{G\setminus e}$. Now, consider the homomorphism of complexes $\Phi : \mathcal{E}(-2) \longrightarrow \mathcal{E}$, as the multiplication by f_e. In fact, it is a lift of the map $S/J_{G\setminus e}(-2) \xrightarrow{f_e} S/J_{G\setminus e}$. Obviously, the mapping cone over Φ resolves S/J_G. In addition, it is also minimal, because $\mathcal{E}$ is minimal and all the maps in the complex homomorphism Φ are of positive degrees.

□

Corollary 3.5. *Let G be a connected graph and let $H_1, \dots, H_q$ be the connected components of $\mathcal{R}(G)$. Then* $\mathrm{reg}(J_G) = \sum_{i=1}^{q} \mathrm{reg}(J_{H_i})$.

Proof. Since $\mathcal{R}(G)$ has q connected components, it follows that G has exactly $q-1$ free cut edges. So, by using Proposition 3.4 repeatedly, we have that $\mathrm{reg}(J_G) = \mathrm{reg}(J_{\mathcal{R}(G)}) + q - 1$. On the other hand, $\mathrm{reg}(J_{\mathcal{R}(G)}) = \sum_{i=1}^{q} \mathrm{reg}(J_{H_i}) - q + 1$, since J_{H_i}'s are generated in disjoint sets of variables. Hence, we have $\mathrm{reg}(J_G) = \sum_{i=1}^{q} \mathrm{reg}(J_{H_i})$.

□

Now we are ready to prove Theorem 3.2:

Proof of Theorem 3.2. By Corollary 3.5, we have $\mathrm{reg}(J_G) = \sum_{i=1}^{q} \mathrm{reg}(J_{H_i})$. Since by our assumption Conjecture 3.1 is true for each H_i, it follows that $\mathrm{reg}(J_{H_i}) \leq c(H_i)+1$ for all $i = 1, \dots, q$. Thus, $\mathrm{reg}(J_G) \leq \sum_{i=1}^{q} c(H_i)+q$. On the other hand, we have

$$c(G) = \sum_{i=1}^{q} c(H_i) + q - 1,$$

because $\mathcal{R}(G)$ has q connected components, and hence G has $q-1$ free cut edges. Therefore, $\mathrm{reg}(J_G) \leq c(G) + 1$, which completes the proof.

It was shown in [15] that Conjecture 3.1 is true for chordal graphs. Next, using Theorem 3.2, we give a family of non-chordal graphs for which Conjecture 3.1 is valid. Let G be the graph depicted in Figure 1. By [7, Theorem 3.2], we have that $\mathrm{reg}(J_G) \leq 5$, since G has 6 vertices. Thus, Conjecture 3.1 is true for G, since we clearly have $c(G) = 4$.

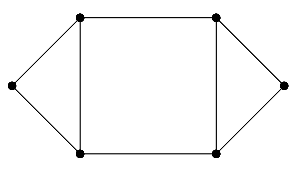

Figure 1: A non-chordal graph G with $\mathrm{reg}(J_G) \leq 5$

Now, taking finitely many copies of the above graph G and joining a free vertex of each of them to the other one's by a cut edge, we obtain a new graph H for which $\mathcal{R}(H)$ is just the disjoint union of some copies of G. Then, Theorem 3.2 implies that $\mathrm{reg}(J_H) \leq c(H) + 1$ satisfying Conjecture 3.1. For example, Figure 2 shows such a graph built up with three copies of G.

We end this section with some further consequences of our results concerning closed graphs and forests. First, combining Proposition 3.4 and [5, Theorem 2.2], we get the following consequence:

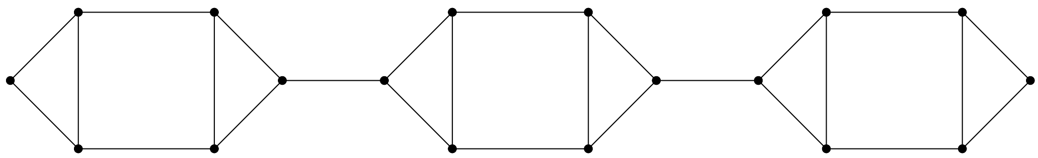

Figure 2: A non-chordal graph H with $\mathrm{reg}(J_H) \leq c(H) + 1$

Corollary 3.6. *Let G be a connected graph and let $H_1, \ldots, H_q$ be the connected components of $\mathcal{R}(G)$. If $\mathcal{R}(G)$ is closed, then*

$$\mathrm{reg}(J_G) = \sum_{i=1}^{q} \ell_i + \sharp\{\text{free cut edges of } G\} + 1,$$

where ℓ_i is the length of a longest induced path in H_i for each i.

Finally, note that it is known that $\beta_{1,j}(J_G) = 0$ for all $j > 2n$ (see [10, Theorem 2.2]). Now, using Proposition 3.3, we strength this in the case of forests.

Corollary 3.7. *Let G be a forest on $n \geq 4$ vertices. Then $\beta_{1,j}(J_G) = 0$, for all $j > n$.*

Proof. We use induction on the number of edges of a forest. If G has no edges, then $J_G = (0)$, and hence the result is obvious. Now, let G be a forest on $n \geq 4$ vertices and with at least one edge. Let e be an edge of G, which is indeed a cut edge of G. Then by Proposition 3.3, we have $\beta_{1,j}(J_G) \leq \beta_{1,j}(J_{G\setminus e}) + \beta_{0,j-2}(J_{(G\setminus e)_e})$, for all $j > n$. We have $\beta_{0,j-2}(J_{(G\setminus e)_e}) = 0$, since $j \geq 5$. Thus, we have $\beta_{1,j}(J_G) \leq \beta_{1,j}(J_{G\setminus e})$, for all $j > n$. By induction hypothesis, we have $\beta_{1,j}(J_{G\setminus e}) = 0$, for all $j > n$. This implies that $\beta_{1,j}(J_G) = 0$ for all $j > n$, as desired.

□

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