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Design of a Robust H∞ Controller for Affine Nonlinear Singular Systems with Norm-bounded Time-varying Uncertainties

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ABSTRACT: In this paper, the problem of robust H∞ control for continuous-time affine nonlinear singular systems with norm-bounded time-varying uncertainties is addressed. The problem is solved locally by employing two different approaches. The first approach is an extended version of the guaranteed cost control method, adapted in order to conform to affine nonlinear singular systems. The second approach is an indirect method in which a known auxiliary system is presented and it is shown that any control law which solves the H∞ control problem for this auxiliary system, also solves the robust H∞ control problem for the original unknown system. In both approaches, sufficient conditions for the solvability of the considered robust H∞ control problem are provided in terms of a generalized Hamilton-Jacobi-Isaacs inequality. In order to show the consistency of our results, solving the robust H∞ control problem for linear uncertain singular systems is considered, and, it is shown that the corresponding results in linear domain are the special cases of our results. Finally, a numerical example is given to illustrate the applicability of the presented approaches.

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1- INTRODUCTION

State-space models are the dominant tool for dynamical systems modeling and they are extensively employed for analyzing and control design purposes. However, considering the potentials of singular models this might not be the case in the near future. Singular models could be thought of as a broader, more capable class of models, comprising the state-space models as a special case. In the last 30 years, singular models have become a widely accepted tool for modeling and simulation of dynamical systems in numerous applications such as electrical circuits and power systems [1], robotics and mechanical systems [2-4], modern control theories [5], chemical processes [6], biological systems and many other areas [7-9]. In control theory, systems described by singular models are usually called *singular* or *descriptor* systems; However, it should be mentioned that the terminology differs within different fields of study and these systems are also known as *differential-algebraic, generalized state-space, semi-state* and *implicit systems*.

Recently, many control theory issues have been extended from state-space domain to singular systems, among which, H_∞ control problem is considered in this paper. This problem has been studied thoroughly in the context of linear singular systems (see, e.g. [10-17] and references therein). Taking the parameter uncertainties into account, linear robust H_∞ control problem is also considered in [12] and [18-21]. Robust H_∞ controllers are designed to satisfy a pre-specified H_∞ performance level and simultaneously assuring the admissibility of the closed-loop system for all acceptable uncertainties.

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As compared to linear singular systems, nonlinear singular systems (NSSs) are intrinsically more challenging, and consequently, fewer results have been reported on the subject. The results reported in [22] and [23] were the first attempts to solve the H_∞ control problem in NSSs. In [23], a self-scheduling H_∞ control of parameter-varying systems was generalized to polytopic NSSs. A more general class of NSSs was considered in [24] in which, based on the dissipativity theory, the problem has been solved with both state and output feedback controllers. In [25-27], affine NSSs with a Hamiltonian realization were studied and the Hamiltonian approach was employed to tackle the problem. In [26], variations in the system parameters were also considered and an adaptive H_∞ controller was obtained. Taking the advantages of *port-controlled Hamiltonian systems*, in [28] the problems of finite-time stabilization and finite-time H_∞ control for a class of *Hamiltonian* NSSs was solved and the results were used to solve the corresponding problem in affine NSSs.

Although the H_∞ control topic for definite NSSs is widely studied, the robust H_∞ control topic, i.e. the H_∞ controller design for indefinite models, is studied partially for these systems. In fact, to the best of the authors' knowledge, all contributions within the robust H_∞ control topic for NSSs were just dedicated to systems that can be modelled as a linear uncertain system affected by bounded nonlinear disturbances [29-35]. Accordingly, the problem has been solved mainly by using linear theories which confines the applicability of the obtained results in some cases. One way to overcome this restriction is to solve the problem for broader classes of nonlinear systems and employing nonlinear theories in the controller design problem.

In this paper, we consider the robust H_∞ control problem in a broader class of NSSs, i.e. input affine systems, and, the problem is solved entirely based on nonlinear control theories. For this purpose, the nominal model is considered at first, and a modified solution to the corresponding H_∞ control problem is presented. This solution has the specific property that, contrary to the existing results, it can be conveniently employed to solve the robust problem. Based on this solution, the robust H_∞ control problem is solved employing two different approaches. The first approach is founded on an extended version of the *guaranteed cost control* (GCC) method and directly solves the robust problem. In this regard, the guaranteed cost control (GCC) definition is modified to comply with the NSSs and the resulted concept is used to solve the problem. The second approach is rather an indirect method in which the model uncertainty is replaced with a weighted disturbance input. Accordingly, a known auxiliary system is presented and it is shown that any control law that solves the H_∞ control problem for the auxiliary system also solves the robust H_∞ control problem for the main unknown system. Consequently, the robust H_∞ control problem turns into an H_∞ control problem for the auxiliary system that can be solved using the provided H_∞ control theorem.

The rest of the paper is organized as follows: In Section 2, the preliminaries and the required definitions are presented. Specifically, this section includes the modified solution of the H_∞ control problem for NSSs. Based on the results of Section 2, our main results are presented in Section 3. We solve the robust H_∞ control problem for NSSs in this section and provide the sufficient condition of the problem solvability in terms of a generalized Hamilton-Jacobi-Isaacs inequality. Section 4 is devoted to a numerical example and Section 5 concludes the paper.

Notation: The standard notation is employed. R is the set of real numbers and $R^+=[0,\infty)\subset R$. R^n is the n-dimensional real Euclidean space and $R^{n\times m}$ is the set of all real $n\times m$ matrices. $\|.\|$ denotes the Euclidean vector norm on R^n and $\|a\|_Q^2=a^TQa, \forall a\in R^n, Q\in R^{n\times n}$. The space $\mathcal{L}_2([t_0,\infty],R^n)$ is the space of square-integrable vector-valued functions. $P>0(P\geq 0)$ for some $P\in R^{n\times n}$, means that the matrix P is positive (semi-positive) definite and the C^k-functions are k-times continuously differentiable ones.

2- PRELIMINARIES AND PROBLEM STATEMENT

2-1- Problem Statement

Consider the class of uncertain nonlinear systems described by the following differential algebraic model:

$$(\Sigma_\Delta)\begin{cases} E\dot{x}(t)=f_\Delta(x,\theta,t)+g_1(x)w(t) \\ \quad +g_{2\Delta}(x,\theta,t)u(t), \quad x(t_0)=x_0 \\ z(t)=h(x)+k_1(x)w(t)+k_2(x)u(t) \end{cases} \tag{1}$$

where $x(t)\in R^n$ is the descriptor state, $w(t)\in\mathcal{L}_2([t_0,\infty],R^s)$ is the disturbance signal, $\boldsymbol{u}\in R^m$ is the control input, and $\boldsymbol{z}\in R^p$ is the *to-be-controlled* output (tracking errors, cost variables, *etc*.). $\boldsymbol{E}$ is a singular matrix with $rank(\boldsymbol{E})=r<n$ and, without any loss of generality, it is assumed to be in the form $\boldsymbol{E}=\begin{bmatrix}\boldsymbol{I}_r & 0\\ 0 & 0\end{bmatrix}$, in which $\boldsymbol{I}_r$ is an $r\times r$ identity matrix.

The mappings $f_\Delta(x)$ and $g_{2\Delta}(x)$ are supposed to be in the form of $f_\Delta(x)=f(x)+\Delta f(x,\theta,t)$, $\Delta f(0,\theta,t)=0$ and $g_{2\Delta}(x)=g_2(x)+\Delta g_2(x,\theta,t)$ with $g_{2\Delta}(0,\theta,t)=0$, where $f:R^n\to R^n$ and $g_2:R^n\to R^{n\times m}$, as well as $g_1:R^n\to R^{n\times s}$, $h_1:R^n\to R^p$, $k_1:R^n\to R^{p\times s}$ and $k_2:R^n\to R^{p\times m}$ are known, real C^∞-functions of x. The unknown parts of model, i.e. $\Delta f:R^n\times R^\theta\times R\to R^n$ and $\Delta g_2:R^n\times R^\theta\times R\to R^{n\times m}$, belong to an admissible uncertainty set Ψ_Δ defined as

$$\Psi_\Delta \overset{\Delta}{=} \left\{\begin{array}{l} \Delta f,\Delta g_2 \mid \Delta f(x,\theta,t)= \\ F(x)\Delta(x,\theta,t)N_1(x),\Delta g_2(x,\theta,t) \\ =G_2(x)\Delta(x,\theta,t)N_2(x), \\ \qquad \Delta^T(x,\theta,t)\Delta(x,\theta,t)\leq I,\forall x \\ \in\Omega_1\subseteq R^n,\forall\theta\in\Psi_\theta,t\in R \end{array}\right\}$$

where Ψ_θ is the set of all permissible parameters and Δ, $\boldsymbol{F}$, $\boldsymbol{G_2}$, $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ are C^∞-functions with appropriate dimensions. $\boldsymbol{F}$, $\boldsymbol{G_2}$, $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ are also known matrix functions and $\Delta(x,\theta,t)$, representing the uncertainty, is a norm-bounded time-varying parameter-dependent function of the state. Furthermore, it is assumed that the system admits a unique solution and $x=0$ is the isolated equilibrium point of the system when $w(t)\equiv 0$.

Remark 1: Considering the above mappings as C^∞-functions is rather conservative and can be relaxed to "*smooth functions*" assumption. By smooth functions we mean the mappings are C^k-functions with 'k' big enough such that the functions can be differentiated to the order needed by the problem in hand.

Singular models such as (1) are more intricate than state-space models and they are capable of modelling more complicated phenomena in real world such as the impulsive response of dynamic systems [36]. Existence of impulse terms in system response may cause saturation of control actuators and may even damage the system. Therefore, in addition to unstable trajectories, impulsive responses are also undesired for singular systems. This fact gives rise to the notions of impulse-freeness and admissibility, defined as follows:

Definition 1 [2]: If the state response of a singular system, starting from an arbitrary initial condition, does not contain impulse terms, then the system is called impulse-free.

Definition 2 [2]: A singular system with a unique solution is called admissible if it is stable and impulse-free.

Since the impulsive behavior of singular systems is undesired, a proper controller is the one that makes the resulted closed-loop system both stable and impulse-free. One way to guarantee the impulse-freeness of a singular system is ensuring that the system is index one. Therefore a class of proper controllers are those which derive the closed-loop system index one. Here, by index one systems we mean singular systems possessing *differentiation index* one. According to [37], for every singular system in the form of

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}=\begin{bmatrix} F_1\left(x_1,x_2,w,u\right) \\ F_2\left(x_1,x_2,w,u\right) \end{bmatrix} \tag{2}$$

the differentiation index of the system is defined as the minimum number of times that $F_2\left(x_1,x_2,w,u\right)$ must be differentiated with respect to time in order to determine $\dot{x}_2$ as a continuous function of (t,x) [38]. This definition of index one systems is locally consistent with the definition of [24], employed in the sequel.

The necessity of impulse-freeness can be considered as the main difference in the definition of H_∞ control problem between singular systems and conventional state-space systems. In other words, the internal stability requirement is replaced with the admissibility requirement in the case of singular systems. Accordingly, the robust H_∞ control problem for model is defined as follows:

Definition 3: The Robust State-Feedback H_∞ (suboptimal) Control Problem (RSFHICP) for the nonlinear singular system (Σ_Δ) is the problem of finding a state-feedback strategy like $\tilde{u}(t)=\tilde{\alpha}(x,t)$, $\tilde{\alpha}:\Omega\times R\rightarrow R^m,\Omega\subseteq R^n$, such that the indefinite closed-loop system

$$\Psi_\Delta\overset{\Delta}{=}\left\{\begin{array}{l} \Delta f,\Delta g_2 \mid \Delta f\left(x,\theta,t\right)= \\ F\left(x\right)\Delta\left(x,\theta,t\right)N_1\left(x\right),\Delta g_2\left(x,\theta,t\right) \\ =G_2\left(x\right)\Delta\left(x,\theta,t\right)N_2\left(x\right), \\ \qquad\qquad \Delta^T\left(x,\theta,t\right)\Delta\left(x,\theta,t\right)\le I,\forall x \\ \in\Omega_1\subseteq R^n,\forall\theta\in\Psi_\theta,t\in R \end{array}\right\} \tag{4}$$

has a local ℓ_2-gain less than or equal to a prescribed "γ" for all $x\left(t_0\right)\in\Omega$, $\forall\Delta f,\Delta g_2\in\Psi_\Delta$, $\forall w(t)\in\mathcal{L}_2\left(\left[t_0,\infty\right],R^s\right)$, and it is admissible when $w\equiv 0$.

Notational remark: Without any ambiguity, the abbreviation RSFHICP (or SFHICP) and the term Robust State-Feedback H_∞ Control Problem (or State-Feedback H_∞ Control Problem) will be used interchangeably, in the sequel.

2-2- H_∞ control problem in nonlinear singular systems

Our method for solving the RSFHICP is founded both conceptually and practically, on the solution of H_∞ control problem. Thus, a restatement of the H_∞ control problem in NSSs is presented which makes the paper self-contained and helps to explain the underlying ideas. This restatement is based on the two-player zero-sum differential game (TPZSDG) theory and can be considered as a slight modification of the corresponding results from [24]. Compared to the results of [24], our solution has the property that it can be conveniently employed to solve the robust problem.

Consider the nominal (known) part of differential algebraic model as follows:

$$(\Sigma)\left\{\begin{array}{l} \qquad E\dot{x}\left(t\right)=f\left(x\right)+ \\ \qquad g_1\left(x\right)w\left(t\right)+g_2\left(x\right)u\left(t\right),\quad x\left(t_0\right)=x_0 \\ z\left(t\right)=h\left(x\right)+k_1\left(x\right)w\left(t\right)+k_2\left(x\right)u\left(t\right) \end{array}\right. \tag{5}$$

In what follows, the following assumption is used which is similar to the corresponding assumption in conventional non-singular case, c.f. ([40]-Ch. 10) and ([39]-Ch. 5) for instance.

Assumption 1: Matrix functions of the system satisfy the following assumptions:

A1-1: $Q\triangleq k_1^Tk_1-\gamma^2I_s<0$

A1-2: $\begin{bmatrix} h & k_2\end{bmatrix}^T\begin{bmatrix} k_1 & k_2\end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & R(x)\end{bmatrix},$

$R(x)>0,\quad h(0)=0,\quad k_2(0)=0.$

These assumptions are the nonlinear versions of the standing assumptions which are commonly used in the literature; e.g. [43]. Relaxing these assumptions is indeed possible, but more complicated formulae must be worked out.

Definition 4: The nonlinear singular system (Σ) with $w(t)\equiv 0$, is ***locally zero-state detectable*** if there exists a neighborhood $\Omega\subseteq R^n$ of the origin such that $\forall x\left(t_0\right)\in\Omega$, $z(t)\equiv 0$ and $u(t)\equiv 0$ implies that $\lim_{t\to\infty}x(t)=0$. The system is ***zero-state detectable*** if $\Omega=R^n$.

Our main theorem in this section, stated below, is essentially based on Theorem 2.15 of [24] and it is similar to Lemma 4.1 of this book or Lemma 5 of [44]; However there are a couple of differences between our method and these lemmas. The validity domain of our results might be generally wider, the related GHJI inequality is derived and, contrary to aforementioned lemmas, the presented theorem can be conveniently employed to solve the robust problem. These characteristics make our results different from Lemma 4.1. of [24] or Lemma 5 of [44].

Theorem 1: Consider a given $\gamma > 0$ and the nonlinear singular system for which Assumption 1 is satisfied. If there exists a positive definite C^3-function $V(Ex), V:\Omega \to R$ and a C^2-function $W(x), W:\Omega \to R^n$ such that:

$$I.\quad (\partial/\partial x)V = EW(x), W(0) = 0 \tag{6}$$

$$II.\quad W^T(x)f(x) - \frac{1}{2}W^T(x)\left[g_1(x)Q^{-1}g_1^T(x) + g_2(x)R^{-1}g_2^T(x)\right]W(x) + \frac{1}{2}h^T(x)h(x) < 0 \tag{7}$$

$$III.\quad E^T W_x(x) = W_x^T(x)E \geq 0, \tag{8}$$

then the control law $u^*(t) = -R^{-1}(x)g_2^T(x)W(x)$ solves the SFHICP for this system, locally in a neighborhood of the origin.

Proof: The proof has two parts; in the first part it will be shown that by employing $u^*(t)$ the closed-loop system has ℓ_2-gain less than or equal to γ; and the admissibility of the closed-loop system will be proved in the second part. Consider the following Hamiltonian:

$$H(x,W,w,u) \triangleq W^T(x)\left[f(x) + g_1(x)w(t) + g_2(x)u(t)\right] + \frac{1}{2}\|z(t)\|^2 - \frac{1}{2}\gamma^2\|w(t)\|^2 \tag{9}$$

Substituting *z(t)* in , it can be seen that this Hamiltonian is a quadratic function of *u(t)* and *w(t)*. Therefore the *completion of squares* technique is employed to obtain the following relation:

$$H(x,W,w,u) = W^T(x)f(x) - \frac{1}{2}W^T(x)\left[g_1(x)Q^{-1}g_1^T(x) + g_2(x)R^{-1}g_2^T(x)\right]W(x) + \frac{1}{2}h^T(x)h(x) + \frac{1}{2}\left\|w(t) - w^*(t)\right\|_Q^2 + \frac{1}{2}\left\|u(t) - u^*(t)\right\|^2 \tag{10}$$

where $u^*(t) = -R^{-1}(x)g_2^T(x)W(x)$ and $w^*(t) = -Q^{-1}(x)g_1^T(x)W(x)$. Equation shows that the couple $(w^*(t), u^*(t))$ satisfies the following inequality:

$$H(u^*(t), w(t)) \leq H(u^*(t), w^*(t)) < 0, \quad \forall w \in \mathcal{W} \tag{11}$$

where $H(u^*(t), w^*(t)) < 0$ is the result of condition . Therefore, if is satisfied and $u^*(t)$ asymptotically stabilizes the closed-loop system, by integrating $H(u^*(t), w^*(t)) < 0$ from $t_0 = 0$ to $t_f \to \infty$ the following inequality is acquired

$$-V(Ex(t_0)) + \frac{1}{2}\int_{t_0}^{\infty}\left[\|z^*(t)\|^2 - \gamma^2\|w^*(t)\|^2\right]dt < 0$$

where $z^*(t)$ is the controlled output of the system when $u^*(t)$ is employed. Defining $\beta(x_0)$ such that it satisfies $\beta(x_0) \leq V(Ex(t_0))$, equation is equivalent to $J(u^*(t), w^*(t)) < 0$, where the cost function(al) $J(u,w)$ is defined as follows:

$$J(u,w) \triangleq \frac{1}{2}\int_0^{\infty}\|z(t)\|^2 dt - \frac{1}{2}\gamma^2\int_0^{\infty}\|w(t)\|^2 dt - \beta(x_0) \tag{13}$$

Considering this inequality and integrating from $t_0 = 0$ to $t_f \to \infty$ yields $J(u^*(t), w(t)) < J(u^*(t), w^*(t)) < 0, \quad \forall w \in \mathcal{W}$ which shows that the closed-loop system has ℓ_2 gain less than or equal to *γ*.

In order to show the admissibility (internal stability and index one property) of the closed-loop system when $w(t) \equiv 0$, Theorem 2.15 of [24] is employed. Condition and are similar to the first and the third properties (Property i. and iii.) of this theorem, respectively. For $w(t) \equiv 0$, and in view of inequality , it can be seen that $H\left(u^*(t), 0\right) \le H\left(u^*(t), w^*(t)\right) < 0$, $\forall w \in \mathcal{W}$ which results in

$$W^T(x^*)\bar{F}(x^*) < 0 \tag{14}$$

where $\bar{F}(x^*) \triangleq f(x^*) + g_2(x^*)u^*(t)$ and $x^*(t)$ is the state of the system when $u^*(t)$ is employed. Inequality shows that the second condition (Property ii.) of this theorem is also satisfied. Consequently, based on the same lines as in the proof of Theorem 2.15 of [24] the closed-loop system is proved to be index one in a neighborhood of the origin such as $\breve{\Omega}$. In order to show the internal stability, it should be noted that every positive definite function *V(Ex)* satisfying - can be considered as an implicit positive definite function of x. This is due to the fact that the system is index one and consequently *Ex=0* implies that *x=0*. On the other hand, in view of , along any trajectory of the closed-loop system the following inequality is satisfied:

$$\dot{V}\left(Ex^*(t)\right)|_{w\equiv 0} = W^T\left(x^*\right)\begin{bmatrix} f\left(x^*\right)+ \\ g_2\left(x^*\right) \\ u^*\left(t\right) \end{bmatrix} < 0 \tag{15}$$

This inequality shows that *V(Ex)* is a Lyapunov function for this system and the closed-loop system is locally asymptotically stable in $\breve{\Omega}$.

Remark 2: Setting *E=I* in equality yields $W(x)=V_x(x)$. Substituting this relation in returns an inequality which is similar to the HJI inequality obtained in the solution of the corresponding problem in nonsingular systems, cf. Theorem 10.1.1 of [40] or Theorem 5.1.2 of [39]. As a result, inequality - is dubbed '*The Generalized Hamilton-Jacobi-Isaacs (GHJI) inequality*' in this paper.

Remark 3: Unlike the conventional HJI inequalities which are non-strict, the GHJI inequality - is a strict one; this strictness guarantees the required impulse-free property of the closed-loop system response. It should be noted that the strictness can be relaxed, as it has been done in Lemma 4.1 of [24], at the expense of putting more restrictions on the considered systems and results. In fact, as it can be inferred from Lemma 4.1 of [24], the considered system need to be *zero-state detectable* and its linearized model must be *impulse-observable* in this case. Note that satisfying these condition get harder for robust problem, i.e. when the model uncertainties are also considered. Furthermore, the validity region of the results is confined to the neighborhood in which the linearized model is a valid approximation for the system.

Theorem 1 completes the groundwork for solving the robust H_∞ control problem and we are ready to present the main results in next section.

3- Robust H_∞ control in nonlinear singular systems

In this section a solution for the robust H_∞ control problem in NSSs is provided by employing two different approaches. The first approach is based on the GCC method in which an uncertain Hamiltonian is used to derive the solution of the RSFHICP. In the second approach an equivalent known model is presented and it is shown that by employing this model and using the results of SFHICP, the robust problem in uncertain model can be solved. We show that in spite of the differences in the idea of the adopted approaches, they are rather equivalent and yield similar results.

3-1- First approach: guaranteed cost control

Guaranteed cost control is an extension of quadratic-stabilization method [39] used to solve the robust H_∞ control problem in nonlinear systems. We extend this approach to NSSs and employ it to solve the RSFHICP in these systems. For this purpose, we first extend the definition of GCC to NSSs as follows:

Definition 5: The function $\tilde{\alpha}(x): \tilde{\Omega} \to R^m$ is said to be an "*Extended Guaranteed Cost Control"* (EGCC) for singular system (Σ_Δ) with cost function if there exists a positive-definite C^3-function $\tilde{V}(Ex), \tilde{V}: \tilde{\Omega} \to R^+$ and a C^2-function $\tilde{W}(x), \tilde{W}: \tilde{\Omega} \to R^n$ such that:

$$i. \quad \left(\partial/\partial x\right)\tilde{V} = E\tilde{W}(x), \tilde{W}(0) = 0 \tag{16}$$

$$ii. \quad \tilde{W}^T\left(x\right)\begin{Bmatrix} f\left(x\right)+\Delta f(x,\theta,t) \\ +g_1\left(x\right)w(t)+\begin{bmatrix} g_2\left(x\right)+ \\ \Delta g_2 \\ (x,\theta,t) \end{bmatrix}\tilde{\alpha}(x) \end{Bmatrix} + \frac{1}{2}\begin{bmatrix} \left\| z\left(t\right)\right\|^2 \\ -\gamma^2\left\| w\left(t\right)\right\|^2 \end{bmatrix} < 0, \quad \forall x \in \tilde{\Omega}, \forall w \in \mathcal{W}, \forall \Delta f, \Delta g_2 \in \Psi_\Delta \tag{17}$$

The above-defined EGCC notion paves the way for finding a robust H_∞ controller in NSSs as it is stated in the following lemma:

Lemma 1: Consider the uncertain singular system (Σ_Δ) and suppose that there exists an extended guaranteed cost control such as $\tilde{\alpha}(x)$ for this system in a neighborhood of the origin like $\tilde{\Omega} \subseteq R^n$. If the function $\tilde{W}(x)$ also satisfies the additional condition $E^T \tilde{W}_x(x) = \tilde{W}_x^T(x) E \geq 0$, then the control law $u(t) = \tilde{\alpha}(x)$ solves the robust H_∞ control problem for (Σ_Δ), locally in a neighborhood of the origin.

Proof: Using the closed-loop system equation

$$E\dot{x}(t) = f\left(x\right) + \Delta f\left(x,\theta,t\right) + g_1\left(x\right)w(t) + \begin{bmatrix} g_2\left(x\right) + \\ \Delta g_2 \\ (x,\theta,t) \end{bmatrix} \tilde{\alpha}(x) \tag{18}$$

and in view of relations -, existence of an EGCC for (Σ_Δ) yields

$$d\tilde{V}(Ex)/dt + \frac{1}{2}\| z\left(t\right)\|^2 - \frac{1}{2}\gamma^2 \| w\left(t\right)\|^2 < 0, \quad \forall x \in \tilde{\Omega}, \forall w \in \mathcal{W}, \forall \Delta f, \Delta g_2 \in \Psi_\Delta \tag{19}$$

Integrating both sides of the above relation from $t_0 = 0$ to $t_f \to \infty$ results in:

$$\frac{1}{2}\int_0^\infty \begin{bmatrix} \| z^*\left(t\right)\|^2 \\ -\gamma^2 \\ \| w^*\left(t\right)\|^2 \end{bmatrix} dt \leq \tilde{V}(Ex_0) \leq \tilde{\beta}(x_0), \quad \forall w \in \mathcal{W}, \forall \Delta f, \Delta g_2 \in \Psi_\Delta \tag{20}$$

which shows that the existence of an EGCC guarantees that the closed-loop system has an ℓ_2-gain less than or equal to γ, locally in $\tilde{\Omega}$. Furthermore, setting $w=0$ and defining $\tilde{F}(x^*) = f(x^*) + g_2(x^*)u^*(t)$, inequality implies that $\tilde{W}^T(x^*)\tilde{F}(x^*) < 0$ for all $\forall \Delta f, \Delta g_2 \in \Psi_\Delta$. This fact together with conditions and the relation $E^T \tilde{W}_x(x) = \tilde{W}_x^T(x) E \geq 0$ show that the closed-loop system satisfies the three conditions of Theorem 2.15 of [24]. Therefore, based on this theorem the closed-loop system has index one for all $\forall \Delta f, \Delta g_2 \in \Psi_\Delta$, locally in a neighborhood of the origin. In addition, the index one property of the closed-loop system makes $\tilde{V}(Ex)$ an explicit positive definite function of x, and consequently an appropriate Lyapunov candidate for analyzing the stability. Setting $w=0$ in inequality shows that $\dot{\tilde{V}}(Ex) < 0$ for all $\forall \Delta f, \Delta g_2 \in \Psi_\Delta$, which means that the closed-loop system is locally asymptotically stable in $\tilde{\Omega}$. To sum up, every extended guaranteed cost control such as $\tilde{\alpha}(x)$ which satisfies the additional condition $E^T \tilde{W}_x(x) = \tilde{W}_x^T(x) E \geq 0$, solves the RSFHICP.

In view of Lemma 1, one way to solve the robust H_∞ control Problem in NSSs is to find an EGCC in these systems. Based on this fact, sufficient conditions for the solvability of the robust H_∞ control Problem in NSSs are presented in the following theorem:

Theorem 2: Consider a given $\gamma > 0$ and the uncertain singular system (Σ_Δ). Supposes that Assumption 1 is satisfied and there exist a positive-definite C^3-function $\tilde{V}(Ex), \tilde{V}: \tilde{\Omega} \to R^+$ and a C^2-function $\tilde{W}(x), \tilde{W}: \tilde{\Omega} \to R^n$ such that:

$$I. \quad \left(\partial/\partial x\right)\tilde{V}(Ex) = E\tilde{W}(x), \tilde{W}(0) = 0 \tag{21}$$

$$II. \quad \tilde{W}^T\left(x\right) f\left(x\right) + \frac{1}{2}\tilde{W}^T\left(x\right) \begin{bmatrix} \frac{1}{\varepsilon_1^2} F(x)F^T(x) \\ + \frac{1}{\varepsilon_2^2} G_2(x)G_2^T(x) \\ -g_1\left(x\right)Q^{-1}(x)g_1^T(x) \\ -g_2\left(x\right)\tilde{R}^{-1} (x)g_2^T(x) \end{bmatrix} \tilde{W}(x) + \frac{1}{2}\varepsilon_1^2 N_1^T(x)N_1(x) + \frac{1}{2}h^T(x)h(x) < 0 \tag{22}$$

$$III. \quad E^T \tilde{W}_x(x) = \tilde{W}_x^T(x) E \geq 0 \tag{23}$$

where ε_1 and ε_2 are some nonzero scalars. Then the control law $\tilde{u}^*(t) = -\tilde{R}^{-1}(x)g_2^T(x)\tilde{W}(x)$, where $\tilde{R}(x) = R(x) + \varepsilon_2^2 N_2^T(x)N_2(x)$, solves the RSFHICP for these systems, locally in $\tilde{\Omega} \subseteq R^n$.

Proof: Considering Lemma 1, it suffices to show that if conditions - are satisfied, then $\tilde{u}^*(t)$ is an EGCC for (Σ_Δ). In this regard, the uncertain Hamiltonian $H_\Delta(x, \tilde{W}, w, u)$ is defined as follows:

$$H_\Delta\left(x, \tilde{W}, w, u\right) \triangleq \tilde{W}^T\left(x\right) \left\{ \begin{matrix} f\left(x\right) + \Delta f \\ \left(x,\theta,t\right) \end{matrix} + g_1\left(x\right) w\left(t\right) + \begin{bmatrix} g_2\left(x\right) \\ +\Delta g_2 \\ \left(x,\theta,t\right) \end{bmatrix} u\left(t\right) \right\} + \frac{1}{2}\left[\left\|z\left(t\right)\right\|^2 - \gamma^2 \left\|w\left(t\right)\right\|^2\right] \tag{24}$$

Using the following facts:

$$\tilde{W}^T(x)F(x)\Delta(x,\theta,t)N_1(x)\leq\frac{1}{2\varepsilon_1^2}\tilde{W}^T(x)F(x)F^T(x)\tilde{W}(x)+\frac{1}{2}\varepsilon_1^2N_1^T(x)N_1(x) \tag{25}$$

$$\tilde{W}^T(x)G_2(x)\Delta(x,\theta,t)N_2(x)u(t)\leq\frac{1}{2\varepsilon_2^2}\tilde{W}^T(x)G_2(x)G_2^T(x)\tilde{W}(x)+\frac{1}{2}\varepsilon_2^2u^T(t)N_2^T(x)N_2(x)u(t) \tag{26}$$

results in

$$\begin{aligned}H_\Delta(x,\tilde{W},w,\tilde{u}^*)\leq{}&\tilde{W}^T(x)f(x)+\frac{1}{2}\tilde{W}^T(x)\Big[-g_1(x)Q^{-1}(x)g_1^T(x)+\frac{1}{\varepsilon_1^2}F(x)F^T(x)\\&+\frac{1}{\varepsilon_2^2}G_2(x)G_2^T(x)-g_2(x)\tilde{R}^{-1}(x)g_2^T(x)\Big]\tilde{W}(x)+\frac{1}{2}\varepsilon_1^2N_1^T(x)N_1(x)\\&+\frac{1}{2}h^T(x)h(x)+\frac{1}{2}\left\|\tilde{u}(t)+\tilde{R}^{-1}(x)g_2^T(x)\tilde{W}(x)\right\|_{\tilde{R}}^2\\&+\frac{1}{2}\left\|w(t)+Q^{-1}(x)g_1^T(x)\tilde{W}(x)\right\|_Q^2,\ \forall w\in\mathcal{W}\end{aligned} \tag{27}$$

If condition (II), i.e. the generalized HJI inequality is satisfied, then

$$H_\Delta(x,\tilde{W},w,\tilde{u}^*)\leq 0,\quad\forall w\in\mathcal{W} \tag{28}$$

Therefore, according to definition 5, if conditions (I)-(III) are satisfied, $\tilde{u}^*(t)$ is an EGCC for (Σ_Δ) and solves the robust H_∞ control problem in the considered NSSs.

To this point, guaranteed cost control approach has been extended to nonlinear singular systems and it has been shown that the resulted EGCC solves the robust H_∞ control problem in NSSs, provided that the additional condition $E^T\tilde{W}_x(x)=\tilde{W}_x^T(x)E\geq 0$ is satisfied. Accordingly, sufficient conditions for solvability of this problem in NSSs and the corresponding control law is provided in Theorem 2. In the next part, this problem is solved using a different approach which is based on offering an equivalent known model for (Σ_Δ).

3-2- Second approach: Employing the auxiliary system

Our approach in this section is based on an idea which is employed previously for nonsingular systems. In this approach, the RSFHICP is turned into a SFHICP and can be solved using the SFHICP theories such as Theorem 1. Accordingly, we offer an auxiliary known system such that any control law which solves the SFHICP for this system, also solves the robust SFHICP for the original uncertain system. In this regard, the model uncertainties in (Σ_Δ) should be replaced by an equivalent disturbance input which is scaled to become bounded energy. Hence, the following assumption is required:

Assumption 2: There exist nonzero scalars α, β and smooth mappings like $\hat{F}$, $\hat{G}_2$, $\hat{N}_1$ and $\hat{N}_2$ such that the equalities $\hat{F}(\xi)\Delta\hat{N}_1(\xi)=F(\xi)\Delta N_1(\xi)$ and $\hat{G}_2(\xi)\Delta\hat{N}_2(\xi)=G_2(\xi)\Delta N_2(\xi)$ are satisfied, and the signals $\delta_1(t)=(\alpha/\gamma)\Delta\hat{N}_1(\xi)$ and $\delta_2(t)=(\beta/\gamma)\Delta\hat{N}_2(\xi)u(t)$ belong to $\mathcal{L}_2(t_0,\infty]$.

Regarding Assumption 2, we define the following auxiliary model:

$$(\Sigma_a)\begin{cases}E\dot{\xi}(t)=f(\xi)+\hat{g}_1(\xi)w_a(t)+g_2(\xi)u(t),\quad\xi(t_0)=\xi_0\\ z_a(t)=\hat{h}(\xi)+\hat{k}_1(\xi)w_a(t)+\hat{k}_2(\xi)u(t)\end{cases} \tag{29}$$

where $\xi(t)\in R^n$ is the state, $w_a(t)\in\mathcal{L}_2(t_0,\infty]$ is the equivalent disturbance input, $u(t)\in R^m$ is the control input and $z_a(t)$ is the to-be-controlled output of the auxiliary system (Σ_a). The mappings f, g_1, g_2, h, k_1, k_2, F, G_2, N_1, N_2 and $\Delta(x,\theta,t)$ are borrowed from (Σ_Δ) and they were defined in . $\hat{g}_1(\xi)=\begin{bmatrix}g_1(\xi) & (\gamma/\alpha)\hat{F}(\xi) & (\gamma/\beta)\hat{G}_2(\xi)\end{bmatrix}$ and

$$w_a(t)=\begin{bmatrix}w^T(t) & \delta_1^T(t) & \delta_2^T(t)\end{bmatrix}^T \tag{30}$$

The mappings defining $z_a(t)$ are as follows:

$$\hat{h}(\xi)=\begin{bmatrix} h^T(\xi) & \alpha\hat{N}_1^T(\xi) & 0_{1\times k}\end{bmatrix},\ \hat{h}(0)=0,$$
$$\hat{k}_1(\xi)=\begin{bmatrix} k_1(\xi) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}_{(p+2k)\times(s+2l)}, \tag{31}$$
$$\hat{k}_2(\xi)=\begin{bmatrix} k_2^T(\xi) & 0 & \beta\hat{N}_2^T(\xi)\end{bmatrix}^T_{m\times(p+2k)},$$
$$\hat{k}_2(0)=0.$$

The cost function for the auxiliary system (Σ_a) is defined as follows:

$$J_a(w_a,u)=\frac{1}{2}\int_0^\infty \| z_a(t)\|^2 dt -\frac{1}{2}\gamma^2\int_0^\infty \| w_a(t)\|^2 dt-\beta(\xi_0) \tag{32}$$

The following Lemma states the relation between the solution of the SFHICP in (Σ_a) and the RSFHICP solution for the original system (Σ_Δ):

Lemma 2: Consider the uncertain singular system (Σ_Δ) and the corresponding auxiliary system (Σ_a) with their respected cost functions $J(w,u)$ and $J_a(w_a,u)$. Let $\hat{\alpha}(.)$ be a state-feedback control law which solves the H$_\infty$ control problem for the auxiliary system (Σ_a). Then it follows that $\hat{\alpha}(.)$ also solves the robust H$_\infty$ control problem for the main system (Σ_Δ).

Proof: Comparing (Σ_Δ) and the auxiliary system (Σ_a) shows that the state evolution equations of these systems are identical. This fact implies that for any common control input the state trajectories of the two systems are the same. Consequently, any control law which solves the SFHICP for the auxiliary system can also makes the state trajectory of the uncertain closed-loop system (Σ_Δ^{cl}) impulse-free and asymptotically stable. In addition, for the given $\gamma>0$ the resulted auxiliary closed-loop system has an ℓ_2-gain less than or equal to γ. This in turn yields

$$J_a(w_a,\hat{\alpha}(\xi))=\frac{1}{2}\| \hat{z}_a \|_2^2 -\frac{1}{2}\gamma^2\| w_a \|_2^2-\beta(x_0)\le 0. \tag{33}$$

where $\hat{z}_a(t)$ is the to-be-controlled output of the closed-loop auxiliary system (Σ_a^{cl}) when $u(t)=\hat{\alpha}(\xi)$ is used. Expanding the above inequality and using the fact that $x(t)=\xi(t), \forall t\ge t_0$ together with equations and results in

$$J_a(w_a,\hat{\alpha}(\xi))= J(w,\hat{\alpha}(x))+\frac{1}{2}\alpha^2\left[\|\hat{N}_1\|_2^2-\|\Delta\hat{N}_1\|_2^2\right]+\frac{1}{2}\beta^2\left[\|\hat{N}_2\hat{\alpha}(\xi)\|_2^2-\|\Delta\hat{N}_2\hat{\alpha}(\xi)\|_2^2\right] \tag{34}$$

where $J(w,\hat{\alpha}(x))$ is the cost function of the uncertain closed-loop system (Σ_a^{cl}) when $u(t)=\hat{\alpha}(x)$ is used. Since Δ satisfies $\Delta^T\Delta\le I$, equality along with inequality show that $J(w,\hat{\alpha}(x))\le 0$ which in turn means that the uncertain closed-loop system (Σ_Δ^{cl}) has an ℓ_2-gain less than or equal to γ.

The above lemma is used in the following theorem which presents some sufficient conditions for the solvability of RSFHICP in NSS modelled by and the corresponding robust H$_\infty$ controller:

Theorem 3: Consider a given $\gamma>0$ and the uncertain nonlinear singular system (Σ_Δ) for which Assumptions 1 and 2 are satisfied. If there exists a positive definite C^3-function $\hat{V}(Ex), \hat{V}:\hat{\Omega}\to R^+$ and a C^2-function $\hat{W}(x), \hat{W}:\hat{\Omega}\to R^n$ such that:

$$I.\quad (\partial/\partial x)\hat{V}(Ex)=E\hat{W}(x), \hat{W}(0)=0 \tag{35}$$

$$II.\quad \hat{W}^T(x)f(x)+\frac{1}{2}\hat{W}^T(x)\left[\frac{1}{\alpha^2}\hat{F}(x)\hat{F}^T(x)+\frac{1}{\beta^2}\hat{G}_2(x)\hat{G}_2^T(x)-g_1(x)Q^{-1}(x)g_1^T(x)-g_2(x)\hat{R}^{-1}(x)g_2^T(x)\right]\hat{W}(x)+\frac{1}{2}\alpha^2\hat{N}_1^T(x)\hat{N}_1(x)+\frac{1}{2}h^T(x)h(x)<0 \tag{36}$$

$$III.\quad E^T\hat{W}_x(x)=\hat{W}_x^T(x)E\ge 0 \tag{37}$$

where $\hat{R}(x)=R(x)+\beta^2\hat{N}_2^T(x)\hat{N}_2(x)$, then the control law $u^*(t)=-\hat{R}^{-1}(x)g_2^T(x)\hat{W}(x)$ solves the RSFHICP locally in a neighborhood of the origin.

Proof: In view of Lemma 2, it suffices to find an H$_\infty$ control law for the corresponding auxiliary system (Σ_a) and use it as a robust H$_\infty$ controller for the original system. In this regard, and based on the fact that (Σ_Δ) satisfies Assumption 1, it can be easily shown that (Σ_a) satisfies the corresponding assumption as well, i.e.,

$$\hat{Q} \triangleq \hat{k}_1^T \hat{k}_1 - \gamma^2 I_{(s+2l)} < 0$$

$$\begin{bmatrix} \hat{h} & \hat{k}_2 \end{bmatrix}^T \begin{bmatrix} \hat{k}_1 & \hat{k}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{R}(x) \end{bmatrix}, \quad R(x) = R(x) + \beta^2 \hat{N}_2^T(x)$$

$$\hat{N}_2(x) > 0, \quad \hat{h}(0) = 0, \quad \hat{k}_2(0) = 0.$$

Observe that condition of Theorem 1 for (Σ_a) is obtained as follows:

$$\hat{W}^T(\xi) f(\xi) - \frac{1}{2}\hat{W}^T(\xi) \begin{bmatrix} \hat{g}_1(\xi) Q^{-1} + \hat{g}_2(\xi) R^{-1} \\ (\xi)\hat{g}_1^T(\xi)(\xi)\hat{g}_2^T(\xi) \end{bmatrix} \hat{W}(\xi) + \frac{1}{2}\hat{h}^T(\xi)\hat{h}(\xi) < 0 \tag{38}$$

Substituting $\hat{g}_1, \hat{Q}, \hat{g}_2, \hat{R}$ and $\hat{h}$ in yields:

$$\hat{W}^T(\xi) f(\xi) + \frac{1}{2}\hat{W}^T(\xi) \begin{bmatrix} \frac{1}{\alpha^2}\hat{F}(\xi)\hat{F}^T(\xi) - g_1(\xi) Q^{-1} \\ (\xi) g_1^T(\xi) - g_2(\xi)\hat{R}^{-1}(\xi) g_2^T(\xi) \\ + \frac{1}{\beta^2}\hat{G}_2(\xi)\hat{G}_2^T(\xi) \end{bmatrix} \hat{W}(\xi) + \frac{1}{2}\alpha^2 \hat{N}_1^T (\xi)\hat{N}_1(\xi) + \frac{1}{2} h^T(\xi) h(\xi) < 0 \tag{39}$$

which is the same as condition . Since the conditions and are also similar to the corresponding ones in Theorem 1, the fulfilment of conditions - implies that (Σ_a) satisfies the requirements of Theorem 1. Hence, based on this theorem, the control law $u^*(t) = -\hat{R}^{-1}(\xi) g_2^T(\xi)\hat{W}(\xi)$ solves the SFHICP in (Σ_a), which in turn means that $u^*(t) = -\hat{R}^{-1}(x) g_2^T(x)\hat{W}(x)$ solves the RSFHICP in (Σ_Δ).

Remark 4: By comparing Theorem 2 and Theorem 3, we can see that although the equivalent system approach and the EGCC method are based on different ideas, they yield the same results. Particularly, if ε_1 and ε_2 in - are set equal to α and β in Assumption 2 respectively, and the auxiliary system parameters can be chosen as $\hat{F} = F$, $\hat{G}_2 = G_2$, $\hat{N}_1 = N_1$ and $\hat{N}_2 = N_2$, then the two theorems amounts exactly to each other. This fact also shows the consistency of the derived results.

Employing the nonlinear domain results to solve the analogous problem in linear case and comparing the outcomes with the existing linear literature is a convenient verification method and a beneficial practice in nonlinear system theories. Thus, in the next part our results are used to solve the robust H_∞ control problem in linear time-invariant (LTI) singular systems.

3-3- Robust H_∞ control of linear time-invariant singular systems

Linear singular systems can be considered as a special case of the affine NSSs considered above. For instance, consider the LTI form of (Σ_Δ) as follows:

$$(\Sigma_l) \begin{cases} E\dot{x}(t) = (A + \Delta A)x(t) + B_1 w(t) + \\ (B_2 + \Delta B_2)u(t),\ x(t_0) = x_0 \\ z(t) = Cx(t) + D_{12}u(t) \end{cases} \tag{40}$$

ΔA and *ΔB*$_2$ are unknown matrices representing norm-bounded uncertainties, which are assumed to be in the following forms:

$$\Delta A = M\Delta(t) N_1, \qquad \Delta B_2 = M\Delta(t) N_2 \tag{41}$$

where M, N_1 and N_2 are known real constant matrices with appropriate dimensions and the uncertain matrix $\Delta(t)$ satisfies $\Delta^T(t)\Delta(t) \le I$.

It is assumed that the nominal system is regular and C-Observable [2]. It is also assumed that the system coefficients are such that the relation $D_{12}^T[C \quad D_{12}] = [0_{m\times n} \quad I_m]$ is satisfied. Employing the state feedback structure suggests the following relations for $V(x)$ and $W(x)$:

$$V(x) = \frac{1}{2}x^T E^T X x, \qquad W(x) = Xx \tag{42}$$

where $X \in R^{n\times n}$ is a constant matrix with the property $E^T X = X^T E \ge 0$. Putting the systems coefficients and the above candidates for $V(x)$ and $W(x)$ in the generalized HJI inequality yields the following inequality:

$$A^T X + X^T A + X^T \Big(\frac{1}{\gamma^2} B_1 B_1^T + \frac{1}{2\varepsilon} M M^T - B_2 \left(D_{12}^T D_{12} + 2\varepsilon N_2^T N_2 \right)^{-1} B_2^T \Big) X + 2\varepsilon N_1^T N_1 + C^T C < 0 \tag{43}$$

where $\varepsilon = \varepsilon_1^2/2 = \varepsilon_2^2/2,\ \varepsilon \neq 0$. Therefore, Theorem 2 implies

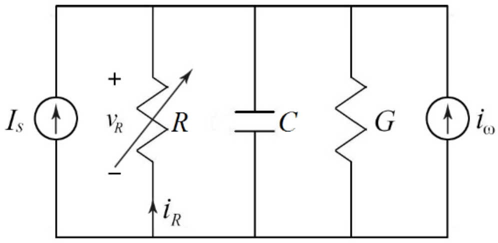

Fig. 1. The electrical circuit schematic

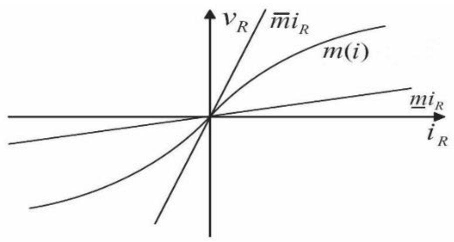

Fig. 2. Nonlinear function v = m(i)

that if there exists a constant real matrix $X \in R^{n\times n}$, satisfying the above generalized Riccati equation, then the control input $u^*(t) = -\left(D_{12}^T D_{12} + 2\varepsilon N_2^T N\right)^{-1} B_2^T Xx$ solves the robust H$_\infty$ control problem for (Σ_1). Using the Schur complement inequality is rewritten as follows:

$$\begin{bmatrix} A_{cl}^T X + X^T A_{cl} + C_{cl}^T C_{cl} + 2\varepsilon \left(N_1 + N_2 K\right)^T \left(N_1 + N_2 K\right) & X^T B_1 & X^T M \\ B_1^T X & -\gamma^2 I & 0 \\ M^T X & 0 & -\varepsilon I \end{bmatrix} < 0 \quad (44)$$

where $A_{cl} \triangleq A + B_2 K, C_{cl} \triangleq C + D_{12} K$ --and $K \triangleq -\left(D_{12}^T D_{12} + 2\varepsilon N_2^T N_2\right)^{-1} B_2^T X$. Based on the fact that $\|G\|_\infty = \|G^T\|_\infty$, where *G(s)* denote the closed-loop system transfer function, the dual condition for inequality is obtained. defining $Y \triangleq K\Omega$ and $\Omega(X,Q) \triangleq XE + SQ$ as it is stated in [[12]- Ch. 5], the obtained dual inequalities turn into the following inequalities:

$$E^T X = X^T E \geq 0 \quad (45)$$

$$\begin{bmatrix} A_{cl}\Omega + \Omega^T A_{cl}^T + B_1 B_1^T + 2\varepsilon M M^T & \Omega^T C_{cl}^T & \Omega^T N_1^T + Y^T N_2^T \\ C_{cl}\Omega & -\gamma^2 I & 0 \\ N_1 \Omega + N_2 Y & 0 & -\varepsilon I \end{bmatrix} < 0 \quad (46)$$

which is similar to the condition presented in Theorem 5.8. of [[12]- Ch. 5]. It should be noted that the term $B_1 B_1^T$ and the coefficient of $\varepsilon M M^T$ are missed in this theorem which is possibly due to a typographical error, c.f. Theorem 5.3. of the same book.

4- ILLUSTRATIVE EXAMPLE

In order to illustrate the performance of presented approach, example 2.3 of [45] is considered where two current sources are added to the circuit as shown in Fig. 1. In this figure *G* represents a conductance with the *i-v* relation $i = G(v) = av - bv^3,\ \ a,b > 0$. *C* is a linear capacitance and *R* is assumed to be a nonlinear current-controlled resistor with uncertain *v-i* relation, which is confined to the area between $\underline{m}i$ and $\overline{m}i$, $0 < \underline{m} < 1 < \overline{m}$, as shown in Fig. 2.

The dynamic behavior of the above electric circuit can be modeled as:

$$\begin{cases} \dot{x}_1 = -\dfrac{a}{C}x_1 + \dfrac{b}{C}x_1^3 + \dfrac{1}{C}x_2 + \dfrac{1}{C}u + \dfrac{1}{C}w \\ 0 = -x_1 + m(x_2) \\ z = \begin{bmatrix} x_1 & x_2 & u \end{bmatrix}^T \end{cases} \quad (47)$$

where $x(t) = \begin{bmatrix} v_R(t) & i_R(t) \end{bmatrix}^T$, $w(t)=i_w$ and $u=I_s$. The uncertain *v-i* relation of *R* could be modeled as $m(i) = (m_0 + \delta(i)\tilde{m})i$, in which $m_0 = (\overline{m} + \underline{m})/2$, $\tilde{m} = (\overline{m} - \underline{m})/2$ and $\delta(i) \in R$ is uncertain with the property $|\delta(i)| < 1$. Consequently, representing

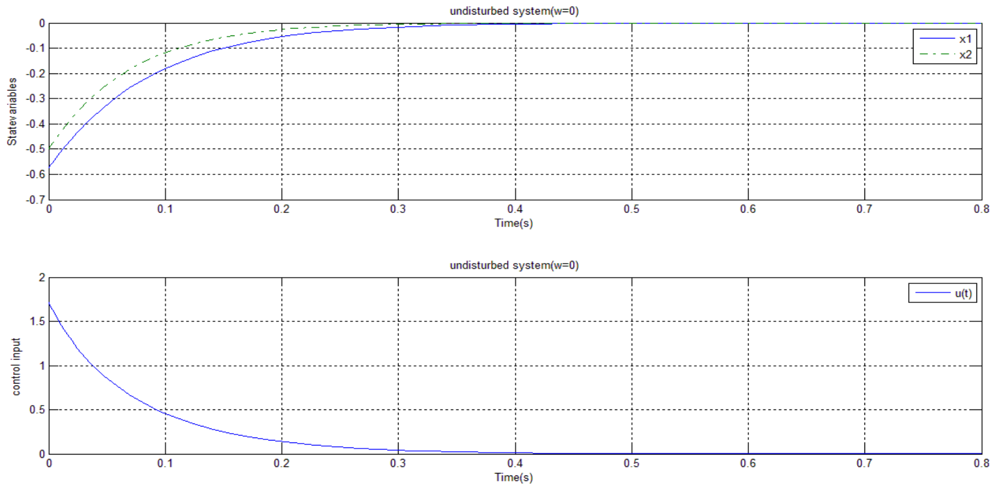

Fig. 3. State trajectory and control signal of the disturbance-free closed-loop system

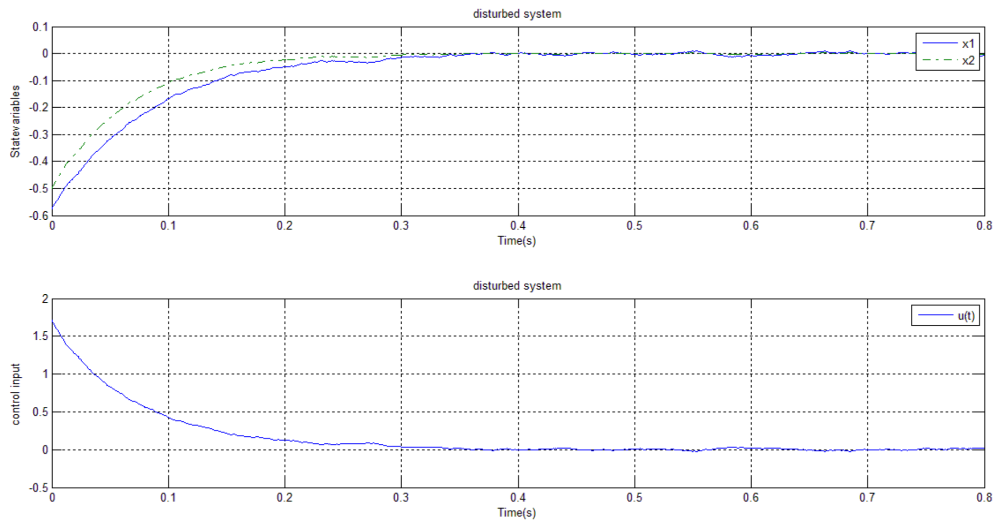

Fig. 4. State trajectory and control signal of the disturbed closed-loop system

in the form of model shows that $x=0$ is the isolated equilibrium point and the assumptions of Theorem 2 are satisfied. Employing this theorem for $\gamma = \sqrt{2}$, the following HJI inequality is obtained:

$$\tilde{W}^T(x)\begin{bmatrix} -\dfrac{a}{C}x_1 + \dfrac{b}{C}x_1^3 + \dfrac{1}{C}x_2 \\ -x_1 + m_0 x_2 \end{bmatrix} + \frac{1}{2}\tilde{W}^T(x)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\tilde{W} (x) + \frac{1}{2}x_1^2 + \frac{\tilde{m}^2+1}{2}x_1^2 < 0 \tag{48}$$

One solution to is $\tilde{W}^T(x) = \left[\bar{k}x_1 / \left(\bar{a} - bx_1^2\right) \quad 0\right]$, in which $\bar{k} \geq C\left(1+\tilde{m}^2+\underline{m}^2\right)/2\underline{m}^2$, $a > 1/\underline{m}$, $\bar{a} = a - 1/\underline{m}$ and $\Omega = \left\{x \mid |x_1| < \sqrt{\bar{a}/b}\right\}$. Therefore, the corresponding solution to generalized HJI inequality - is the positive-definite function $\tilde{V}(x) = -\frac{\bar{k}}{2b}\ln(1-\frac{bx_1^2}{\bar{a}})$, which leads to robust H_∞ controller $\tilde{u}^*(t) = -\frac{\bar{k}}{C}\frac{x_1}{\bar{a}-bx_1^2}$.

Fig. 3 and Fig. 4 show the results of a numerical simulation, carried out with Ex_0=[-0.5743   0] and the following numerical values:

$$\bar{m} = 2,\ \ \underline{m} = 0.2,\ \ a = 10,\ \ b = 2.5,\ \ C = 1_F$$

It can be observed from Fig. 3 ($w \equiv 0$) that the closed-loop system is internally stable. Furthermore, cost function (3) for the disturbed closed-loop system ($w \neq 0$) turns to be negative (equals to -3.1725 in our simulation) which means that the corresponding ℓ_2-gain is less than or equal to $\gamma = \sqrt{2}$. Simulation results confirm that in spite of the disturbance and the model uncertainty, the obtained control input is effective.

5- CONCLUSION

The problem of robust H_∞ controller design for continuous-time affine nonlinear singular systems has been investigated in this paper. The underlying idea was the differential games theory and a modified solution to the H_∞ control problem has been presented which has provided a convenient basis for solving the robust problem. The robust H_∞ control problem has been tackled using two different approaches. It has been observed that in spite of the differences between the adopted approaches, they yield in similar results and they could be evenly employed. Both approaches have yielded a sufficient condition for the solvability of the robust H_∞ control problem in terms of an extended version of the celebrated Hamilton-Jacobi-Isaacs inequality. It has also been shown that the existing results for linear singular systems are special cases of the presented results. A numerical example has been employed which has demonstrated the effectiveness of the presented results.

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