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On GDW-Randers metrics on tangent Lie groups

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ABSTRACT: Let G be a Lie group equipped with a left-invariant Randers metric F. Suppose that F^v and F^c denote the vertical and complete lift of F on TG, respectively. We give the necessary and sufficient conditions under which F^v and F^c are generalized Douglas-Weyl metrics. Then, we characterize all 2-step nilpotent Lie groups G such that their tangent Lie groups (TG, F^c) are generalized Douglas-Weyl Randers metrics.

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1. Introduction

A Randers metric $F(x,y) = \alpha(x,y) + \beta(x,y)$ is a Finsler metric which is defined as the sum of a Riemannian metric $\alpha(x,y) := \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta(x,y) := b_i(x)y^i$ such that the Riemannian metric controls the related form by $||\beta||_\alpha < 1$. The history of Randers metrics goes back to G. Randers's research on general relativity of 4-dimensional Riemannian manifolds. He regarded these metrics not as Finsler metrics, but as affinely connected Riemannian metrics [14]. This non-Riemannian metric was first recognized as a kind of Finsler metric by Ingarden, who first named it Randers metric [8]. Since then it has been widely applied in many areas, including electron optics and biology. In Finsler geometry, the class of Randers metrics is computable and this may lead to a better understanding of non-Riemannian curvature properties of Finsler metrics.

The study of the Riemannian geometry of tangent bundles started with Sasaki's paper [16]. He showed that any Riemannian metric g on the base manifold M induces a Riemannian metric on TM by using vertical and horizontal lifts. If we replace the horizontal lift with a complete lift then we have another way for constructing Riemannian metrics on TM. In a series of papers, Yano-Kobayashi used this way and studied many geometric properties of such lifted metrics (see [9], [19] and [20]). It is well known that the tangent bundle of every Lie group has a natural Lie group structure [6]. The interpolation between algebraic and geometric properties of Lie groups leads us to

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important results in this field. Therefore, it is natural to study the left invariant Riemannian-Finsler structures on the tangent bundle of Lie groups.

Let G be a Lie group equipped with a left invariant Riemannian metric α. In [1], by using complete and vertical lifts of left invariant vector fields, Asgari-Salimi Moghdam introduced a left invariant Riemannian metric $\tilde{\alpha}$ on the tangent Lie group TG. They find the Levi-Civita connection and sectional curvature of $(TG,\tilde{\alpha})$ in terms of the Levi-Civita connection and sectional curvature of (G,α). Also, they presented the Levi-Civita connection, sectional curvature and Ricci tensor formulas of $(TG,\tilde{\alpha})$ in terms of structure constants of the Lie algebra of G. In [2, 13], they studied the Riemannian geometry of tangent bundle of two families of Lie groups.

Using the left-invariant Riemannian metric $\tilde{\alpha}$ on TG, we define the complete and vertical lifts of a left invariant Randers metric $F = F(x,y) = \sqrt{\alpha_x(y,y)} + \alpha_x(U,y)$ on G to TG by F^c and F^v, respectively, and define them by following

$$F^c\big((x,y),\tilde{z}\big) := \sqrt{\tilde{\alpha}(\tilde{z},\tilde{z})} + \tilde{\alpha}\big(U^c(x,y),\tilde{z}\big), \quad F^v\big((x,y),\tilde{z}\big) := \sqrt{\tilde{\alpha}(\tilde{z},\tilde{z})} + \tilde{\alpha}\big(U^v(x,y),\tilde{z}\big), \tag{1}$$

where $x \in G$, $y \in T_xG$, $\tilde{z} \in T_{(x,y)}TG$ and $U = \beta^\sharp$. Since $\|U^c\|_{\tilde{\alpha}} = \|U^v\|_{\tilde{\alpha}} = \|U\|_\alpha < 1$, then F^c and F^v are left-invariant Randers metrics on TG. There is still another kind of lifting of vector fields so-called the horizontal lift. It is remarkable that, the horizontal lift of a vector field on a Lie group G is not necessary a vector field on TG and it needs to furnish the Lie group G with an extra structure, i.e., connection. Thus, we do not use horizontal lifts for our purpose.

In Finsler geometry, there are several well-known projective invariants such as Douglas curvature and Weyl curvature (see [10] and [11]). Douglas curvature is a non-Riemannian projective invariant constructed from the Berwald curvature. The notion of Douglas metric was proposed by Bácsó-Matsumoto as a generalization of Berwald metric [4]. Finsler metrics with $\mathbf{D} = 0$ are called Douglas metrics. Other than Douglas metrics, there is another projective invariant in Finsler geometry, namely

$$D^i{}_{jkl|m}y^m = T_{jkl}y^i$$

that is hold for some tensor T_{jkl}, where $D^i{}_{jkl|m}$ denotes the horizontal covariant derivatives of Douglas curvature $D^i{}_{jkl}$ with respect to the Berwald connection of F. This equation is equivalent to that for any linearly parallel vector fields $u = u(t)$, $v = v(t)$ and $w = w(t)$ along a geodesic $c(t)$, there is a function $T = T(t)$ such that $\frac{d}{dt}\big[D_{\dot{c}}(u,v,w)\big] = T\dot{c}$. The geometric meaning of this identity is that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic [10]. For a manifold M, let $\mathcal{G}DW(M)$ denotes the class of all Finsler metrics satisfying the above relation for some tensor T_{jkl}. In [5], Bácsó-Papp showed that $\mathcal{G}DW(M)$ is closed under projective changes. All generalized Douglas-Weyl Randers metrics are characterized in [10]. Recently, the study on $\mathcal{G}DW(M)$ has attracted many geometricians, see [17] and [18].

In this paper, we study a Lie group G equipped with a left-invariant Randers metric F and find the relation between the projective geometry of (TG,F^c) and (TG,F^v) and the projective geometry of (G,F), where F^c and F^v are complete and vertical lift of F, respectively. For this aim, let $\{X_i\}_{i=1}^n$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$ with respect to α. Then, for every vector $Z \in \mathfrak{g} := T_eG$, we put

$$\mathcal{B}(Z) := \sum_{i=1}^{n}[2\nabla_{X_i}X_i, Z] + [2X_i, \nabla_{X_i}Z] + \frac{1}{2}\left[X_i, ad_Z^*X_i + ad_ZX_i\right], \tag{2}$$

where ad_Z^* is the adjoint of ad_Z with respect to α. Also, for any vectors $X, Y, Z \in \mathfrak{g}$ let us define

$$\mathcal{C}(X,Y,Z) := \Big\langle U,\ [\nabla_XY, Z] + [Y, \nabla_XZ]\Big\rangle, \tag{3}$$

where $<,>$ denotes the inner product on $\mathfrak{g}$ induced by the Riemannian metric α. Then, we prove the following.

Theorem 1.1. *Let G be an n-dimensional Lie group equipped with a left-invariant Randers metric $F(x,y) = \sqrt{\alpha_x(y,y)} + \alpha_x(U,y)$ defined by the underlying left-invariant Riemannian metric α and the left-invariant vector field U such that $\|U\|_\alpha < 1$. Then the followings hold*

(i) F^c is a generalized Douglas-Weyl metric on TG if and only if the following hold

$$\Big\langle U,\ [\nabla_XY - \frac{1}{2}[X,Y], Z]\Big\rangle = \frac{1}{2n-1}\langle X,Y\rangle \sum_{i=1}^{n}\Big\langle U,\ 2[\nabla_{X_i}X_i, Z] + [X_i, \nabla_{X_i}Z]\Big\rangle,$$
$$\Big\langle U, [[X,Z],Y]\Big\rangle + \Big\langle U, [[X,Y],Z]\Big\rangle = 0,$$

where X, Y and Z are arbitrary left-invariant vector fields, and $\{X_i\}_{i=1}^n$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$ with respect to α.

(ii) F^v is a generalized Douglas-Weyl metric on TG if and only if the followings hold

$$\mathcal{C}(X,Y,Z) - \frac{1}{2}\Big\langle U,\ [[X,Y],Z] + \big[Y,[X,Z]\big]\Big\rangle = \frac{1}{2n-1}\Big\{\langle X,Y\rangle\big\langle U,\mathcal{B}(Z)\big\rangle - \langle X,Z\rangle\big\langle U,\mathcal{B}(Y)\big\rangle\Big\},$$

$$\mathcal{C}(X,Y,Z) + \frac{1}{2}\big\langle U,[Y,ad^*_Z X]\big\rangle = \frac{1}{2n-1}\langle X,Y\rangle\big\langle U,\mathcal{B}(Z)\big\rangle,$$

$$\mathcal{C}(X,Y,Z) + \frac{1}{2}\big\langle U,\ [ad^*_X Y,Z] + [Y,ad^*_X Z]\big\rangle = 0,$$

where $X,Y,Z \in \mathfrak{g}$.

The class of two-step nilpotent Lie groups equipped with left-invariant Randers metrics plays an important role in mathematical physics and geometrical analysis [12][15]. As an application of Theorem 1.1, we characterize 5-dimensional two-step nilpotent Lie groups such that (TG, F^c) is a generalized Douglas-Weyl metric (see Theorem 3.5).

2. Preliminaries

Let M be an n-dimensional smooth manifold and TM be its tangent bundle. A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties:
(i) F is smooth on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ on T_xM is positive definite

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y+su+tv)\Big]_{s,t=0}, \quad u,v \in T_xM.$$

For every $x \in M$, we denote the Minkowski norm on T_xM induced by F with $F_x := F|_{T_xM}$. The Cartan torsion $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ describes the non-Euclidean feature of F_x, which defined by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0}, \quad u,v,w \in T_xM.$$

The Cartan torsion is the family of $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$. The significant of Cartan torsion is $\mathbf{C}$=0 if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Diecke's Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

Given a Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on slit tangent bundle TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x,y)$ are called the spray coefficients and given by following

$$G^i(x,y) := \frac{1}{4}g^{il}\Big\{\frac{\partial^2 F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big\}.$$

The vector field $\mathbf{G}$ is called the associated spray to (M,F). In local coordinates, a curve $c = c(t)$ is a geodesic of F if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(c,\dot{c}) = 0$.

For a non-zero tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j\partial y^k\partial y^l}.$$

The tensor $\mathbf{B}$ is called the Berwald curvature of Finsler metric F. Then F is called a Berwald metric if $\mathbf{B} = 0$.

For a non-zero vector $y \in T_xM_0$, define $\mathbf{D}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y)u^iv^jw^k \frac{\partial}{\partial x^i}|_x$, where

$$D^i{}_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left[G^i - \frac{2}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right]. \tag{4}$$

$\mathbf{D}$ is called the Douglas curvature of F and F is called a Douglas metric if satisfies $\mathbf{D} = 0$.

Let M be an n-dimensional smooth manifold and TM be its tangent bundle. Suppose that X is an arbitrary vector field on M. Then X defines two types of (local) one-parameter group of diffeomorphisms on TM as follows

$$\Phi_t(y) := (T_x\phi_t)(y), \qquad \forall x \in M, \ \forall y \in T_xM, \tag{5}$$
$$\Psi_t(y) := y + tX(x), \qquad \forall x \in M, \ \forall y \in T_xM, \tag{6}$$

where ϕ_t is the one-parameter group generated by the vector field X on M and $T_x\varphi_t$ denotes the derivation of φ_t at point x. The infinitesimal generator of one parameter groups of diffeomorphisms Φ_t and Ψ_t are called the complete lift (denoted by X^c) and vertical lift (denoted byX^v) of X, respectively.

Let $(x^i)(i = 1, \cdots, n)$ be a local coordinate system in an open subset $\mathcal{U}$ of M. Then we denote the induced local coordinate system on $\pi^{-1}(\mathcal{U})$ by $(x^i, y^i)(i = 1, \cdots, n)$, where $\pi : TM \longrightarrow M$ is the canonical projection map. Let X be a vector field on M with local representation $X|_{\mathcal{U}} = \sum_{i=1}^m \xi^i \partial/\partial x^i$. Then the local representation of its vertical and complete lifts on TM are as follows:

$$(X\,|_{\mathcal{U}})^v = \sum_{i=1}^m \xi^i \frac{\partial}{\partial y^i}, \tag{7}$$
$$(X\,|_{\mathcal{U}})^c = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^m y^j \frac{\partial \xi^i}{\partial x^j}\frac{\partial}{\partial y^i}. \tag{8}$$

The Lie brackets of vertical and complete lifts of vector fields satisfy the following relations [2]

$$[X^v, Y^v] = 0, \quad [X^c, Y^c] = [X,Y]^c, \quad [X^v, Y^c] = [X,Y]^v. \tag{9}$$

Let G be a real n-dimensional connected Lie group with the multiplication denoted by

$$\begin{cases} \mu : & G \times G \longrightarrow G \\ & (x,y) \longmapsto xy. \end{cases}$$

Here the identity element is denoted by e. The left and right translations along $y \in G$ are defined by the following

$$\begin{cases} L_y : & G \longrightarrow G \\ & x \longmapsto yx, \end{cases}, \qquad \begin{cases} R_y : & G \longrightarrow G \\ & x \longmapsto xy. \end{cases}$$

Then, TG is also a Lie group with the following multiplication

$$\big(T_{(x,y)}\mu\big)(v,w) = (T_yL_x)(w) + (T_xR_y)(v), \quad v \in T_xG, \quad w \in T_yG, \quad x, y \in G. \tag{10}$$

A vector field X on a Lie group G is said to be left-invariant if it is invariant under every left translation of G. In [2], it is shown that if X is a left-invariant vector field on G, then X^c and X^v are left-invariant vector fields on TG. This result together with the local representation of vertical and complete lifts of vector fields show that: if $\{X_1, \cdots, X_n\}$ is a basis for the Lie algebra $\mathfrak{g}$ of G, then $\{X_1^c, \cdots, X_n^c, X_1^v, \cdots, X_n^v\}$ is a basis for the Lie algebra $\tilde{\mathfrak{g}}$ of TG.

Let G be a smooth n-dimensional connected Lie group endowed with a Riemannian metric $\alpha = a_{ij}dx^i \otimes dx^j$. We denote the inverse of (a_{ij}) by (a^{ij}). We know that α induces the musical bijection between 1-forms and vector fields on G, which is denoted by $\flat : T_xG \longrightarrow T_x^*G$ and given by $y \mapsto \alpha_x(y, -)$. The inverse of $\flat$ is denoted by $\sharp : T_x^*G \longrightarrow T_xG$.

Suppose that $\beta = b_i(x)dx^i$ is a 1-form on G, in which we have used Einstein's convention for summation. Then $\beta^\sharp = b^i \frac{\partial}{\partial x^i}$, where $b^i := a^{ij}b_j$. Consider β such that $\|\beta\|_\alpha := \sqrt{a_{ij}b^ib^j} < 1$. A Randers metric F on G is defined by $F(x,y) = \alpha(x,y) + \beta(x,y)$, $\forall x \in M, \ \ \forall y \in T_xM$, where

$$\alpha(x,y) = \sqrt{a_{ij}y^iy^j} = \sqrt{\alpha_x(y,y)}, \qquad \beta(x,y) = (\beta^\sharp)^\flat(y) = \alpha_x(\beta^\sharp, y).$$

Now, we put

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}) \ , \ \ s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$
$$r_i := b^m r_{im}, \ s_i := b^m s_{im}, \ r_{i0} := r_{ij}y^j \ , \ \ s_{i0} := s_{ij}y^j, \ r_0 := r_j y^j, \ s_0 := s_j y^j,$$

where ";" denotes the covariant derivative with respect to the Levi-Civita connection of α.

3. Lifting of Randers Metrics from G to TG

Let G be a Lie group equipped with a left-invariant Randers metric F defined by the underlying left-invariant Riemannian metric α and the left-invariant vector field U such that $\|U\|_\alpha < 1$, i.e.,

$$F(x,y) = \sqrt{\alpha_x(y,y)} + \alpha_x(U,y), \qquad \forall (x,y) \in TM. \tag{11}$$

In [2], Asgari-Salimi Moghaddam defined a left-invariant Riemannian metric $\tilde{\alpha}$ on TG by

$$\tilde{\alpha}(X^c, Y^c) = \alpha(X,Y), \qquad \tilde{\alpha}(X^v, Y^v) = \alpha(X,Y), \qquad \tilde{\alpha}(X^c, Y^v) = 0, \tag{12}$$

where X and Y are arbitrary vector fields on G. Then, they find its Levi-Civita connection as follows.

Lemma 3.1. ([2]) Let $\widetilde{\nabla}$ be the Levi-Civita connection induced by $\tilde{\alpha}$ and ∇ be the Levi-Civita connection of α. Then, for any two X and Y are left-invariant vector fields on G, the following hold

$$\widetilde{\nabla}_{X^c} Y^c = (\nabla_X Y)^c, \qquad \widetilde{\nabla}_{X^v} Y^v = \left(\nabla_X Y - \frac{1}{2}[X,Y]\right)^c,$$
$$\widetilde{\nabla}_{X^c} Y^v = \left(\nabla_X Y + \frac{1}{2} ad^*_Y X\right)^v, \qquad \widetilde{\nabla}_{X^v} Y^c = \left(\nabla_X Y + \frac{1}{2} ad^*_X Y\right)^v, \tag{13}$$

where ad^*_X denotes the adjoint of ad_X with respect to α.

Following [1], we denote the complete and vertical lifts of a left-invariant Randers metric $F = \alpha + \beta$ on G to TG by F^c and F^v, respectively and defined them as follows:

$$F^c\big((x,y), \tilde{z}\big) := \sqrt{\tilde{\alpha}(\tilde{z}, \tilde{z})} + \tilde{\alpha}\big(U^c(x,y), \tilde{z}\big), \tag{14}$$
$$F^v\big((x,y), \tilde{z}\big) := \sqrt{\tilde{\alpha}(\tilde{z}, \tilde{z})} + \tilde{\alpha}\big(U^v(x,y), \tilde{z}\big), \tag{15}$$

where $x \in G$, $y \in T_xG$, $\tilde{z} \in T_{(x,y)}TG$ and $U = \beta^\sharp$. By a simple calculation, we get

$$\|U^c\|_{\tilde{\alpha}} = \|U^v\|_{\tilde{\alpha}} = \|U\|_\alpha < 1.$$

It is easy to see that F^c and F^v are left-invariant Randers metrics on TG.

3.1. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. More precisely, we will study the lifting of the class of Douglas or generalized Douglas-Weyl Randers metrics $F = \alpha + \beta$ on G defined by (11). We suppose that the complete lift and vertical lift of F are given by (14) and (15), respectively.

In [3], Atshafrouz-Najafi characterized the class of left-invariant Randers metrics of generalized Douglas-Weyl type. They proved that a left-invariant Randers metric $F(x,y) = \sqrt{\alpha_x(y,y)} + \alpha_x(U,y)$ on a Lie group G is a generalized Douglas-Weyl metric if and only if for any left-invariant vector fields X, Y and Z on G the following holds

$$\Big\langle U,\ [\nabla_X Y, Z] + [Y, \nabla_X Z] \Big\rangle = \frac{1}{n-1} \langle X, Y \rangle \sum_{i=1}^{n} \Big\langle U,\ [\nabla_{X_i} X_i, Z] + [X_i, \nabla_{X_i} Z] \Big\rangle,$$
$$- \frac{1}{n-1} \langle X, Z \rangle \sum_{i=1}^{n} \Big\langle U,\ [\nabla_{X_i} X_i, Y] + [X_i, \nabla_{X_i} Y] \Big\rangle, \tag{16}$$

where $\{X_i\}_{i=1}^n$ is an orthonormal basis of the Lie algebra $\mathfrak{g} = T_eG$ with respect to α.

By using the above result, we get the following.

Lemma 3.2. *Let G be an n-dimensional Lie group equipped with a left-invariant Randers metric $F(x,y) = \sqrt{\alpha_x(y,y)} + \alpha_x(U,y)$ defined by the underlying left-invariant Riemannian metric α and the left-invariant vector field U such that $\|U\|_\alpha < 1$. Then F^c is a generalized Douglas-Weyl metric on TG if and only if the following holds*

$$\Big\langle U,\ \Big[\nabla_X Y - \frac{1}{2}[X,Y], Z\Big]\Big\rangle = \frac{1}{2n-1}\langle X,Y\rangle \sum_{i=1}^{n} \Big\langle U,\ 2[\nabla_{X_i}X_i, Z] + [X_i, \nabla_{X_i}Z]\Big\rangle, \tag{17}$$

$$\Big\langle U, [[X,Z],Y]\Big\rangle + \Big\langle U, [[X,Y],Z]\Big\rangle = 0, \tag{18}$$

where X, Y and Z are arbitrary left-invariant vector fields, and $\{X_i\}_{i=1}^n$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$ with respect to α.

Proof. Suppose that $\{X_i\}_{i=1}^n$ is an orthonormal basis for the Lie algebra $\mathfrak{g}$ with respect to α. Then $\{\tilde{X}_i\}_{i=1}^{2n} = \{X_i^c, X_i^v\}_{i=1}^n$ is an orthonormal basis of the Lie algebra $\tilde{\mathfrak{g}}$ with respect to $\tilde{\alpha}$. Since F^c is a left-invariant Randers metric on the Lie group TG, for any vector fields $\tilde{X}, \tilde{Y}, \tilde{Z} \in \tilde{\mathfrak{g}}$ we have

$$\tilde{\alpha}\Big(U^c,\ [\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}] + [\tilde{Y}, \tilde{\nabla}_{\tilde{X}}\tilde{Z}]\Big) = \frac{1}{2n-1}\Big\{\tilde{\alpha}(\tilde{X},\tilde{Y})\tilde{\alpha}\big(U^c, \mathcal{A}(\tilde{Z})\big) - \tilde{\alpha}(\tilde{X},\tilde{Z})\tilde{\alpha}\big(U^c, \mathcal{A}(\tilde{Y})\big)\Big\}, \tag{19}$$

where

$$\mathcal{A}(\tilde{Z}) := \sum_{i=1}^{2n}[\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_i, \tilde{Z}] + [\tilde{X}_i, \tilde{\nabla}_{\tilde{X}_i}\tilde{Z}].$$

Taking into account (12) and (13), one can easily obtain

$$\tilde{\alpha}\big(U^c, \mathcal{A}(Z^c)\big) = \sum_{i=1}^{n}\Big\langle U, 2\nabla_{X_i}X_i + [X_i, \nabla_{X_i}Z]\Big\rangle, \quad \tilde{\alpha}\big(U^c, \mathcal{A}(Z^v)\big) = \sum_{i=1}^{n}\langle U, 2\nabla_{X_i}X_i\rangle. \tag{20}$$

Letting $\tilde{X} = X^c$, $\tilde{Y} = Y^c$ and $\tilde{Z} = Z^c$ in (19) and using (20), we get

$$\begin{aligned}\Big\langle U,\ [\nabla_X Y, Z] + [Y, \nabla_X Z]\Big\rangle &= \frac{1}{2n-1}\Big\{\langle X,Y\rangle \sum_{i=1}^{n}\langle U,\ 2[\nabla_{X_i}X_i, Z] + [X_i, \nabla_{X_i}Z]\rangle\Big\} \\ &\quad - \frac{1}{2n-1}\Big\{\langle X,Z\rangle \sum_{i=1}^{n}\langle U,\ 2[\nabla_{X_i}X_i, Y] + [X_i, \nabla_{X_i}Y]\rangle\Big\},\end{aligned} \tag{21}$$

Let us put $\tilde{X} = X^v$, $\tilde{Y} = Y^v$ and $\tilde{Z} = Z^c$ in (19). Then we get (17). Similarly, we have

$$\Big\langle U,\ \Big[\nabla_X Z - \frac{1}{2}[X,Z], Y\Big]\Big\rangle = \frac{1}{2n-1}\langle X,Z\rangle \sum_{i=1}^{n}\Big\langle U,\ 2[\nabla_{X_i}X_i, Y] + [X_i, \nabla_{X_i}Y]\Big\rangle. \tag{22}$$

Adding (17) and (22), then subtracting the result from (21), we obtain (18).

□

A 2-step nilpotent Lie group is a non-abelian connected Lie group such that the lower central series of its Lie algebra terminates at second step, i.e., $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$. By Lemma 3.2, we conclude the following.

Corollary 3.3. *Let G be an n-dimensional two step nilpotent Lie group. Then F^c is a generalized Douglas-Weyl metric if and only if the following holds*

$$\langle U,\ [\nabla_X Y, Z]\rangle = \frac{1}{2n-1}\langle X,Y\rangle \sum_{i=1}^{n}\Big\langle U,\ 2[\nabla_{X_i}X_i, Z] + [X_i, \nabla_{X_i}Z]\Big\rangle. \tag{23}$$

Now, we characterize F^v of generalized Douglas-Weyl type.

Lemma 3.4. *Let G be an n-dimensional two step nilpotent Lie group. Then F^v is a generalized Douglas-Weyl metric on TG if and only if the following holds*

$$\mathcal{C}(X,Y,Z) - \frac{1}{2}\Big\langle U, [[X,Y],Z] + [Y,[X,Z]]\Big\rangle = \frac{1}{2n-1}\Big\{\langle X,Y\rangle\big\langle U,\mathcal{B}(Z)\big\rangle - \langle X,Z\rangle\big\langle U,\mathcal{B}(Y)\big\rangle\Big\}, \tag{24}$$

$$\mathcal{C}(X,Y,Z) + \frac{1}{2}\Big\langle U, [Y, ad^*_Z X]\Big\rangle = \frac{1}{2n-1}\langle X,Y\rangle\Big\langle U,\mathcal{B}(Z)\Big\rangle, \tag{25}$$

$$\mathcal{C}(X,Y,Z) + \frac{1}{2}\Big\langle U,\ [ad^*_X Y, Z] + [Y, ad^*_X Z]\Big\rangle = 0, \tag{26}$$

where $\mathcal{B}$ and $\mathcal{C}$ defined by (2) and (3), respectively, and $X,Y,Z \in \mathfrak{g}$.

Proof. Since F^v is a left-invariant Randers metric on the Lie group TG, then for any vector fields $\tilde{X},\tilde{Y},\tilde{Z} \in \tilde{\mathfrak{g}}$ we have the following

$$\tilde{\alpha}\Big(U^v,\ [\tilde{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}] + [\tilde{Y},\tilde{\nabla}_{\tilde{X}}\tilde{Z}]\Big) = \frac{1}{2n-1}\Big\{\tilde{\alpha}(\tilde{X},\tilde{Y})\tilde{\alpha}\big(U^v,\mathcal{A}(\tilde{Z})\big) - \tilde{\alpha}(\tilde{X},\tilde{Z})\tilde{\alpha}\big(U^v,\mathcal{A}(\tilde{Y})\big)\Big\}. \tag{27}$$

Taking into account (12) and (13), one can easily obtain

$$\tilde{\alpha}\big(U^v,\mathcal{A}(Z^c)\big) = 0, \quad \tilde{\alpha}\big(U^v,\mathcal{A}(Z^v)\big) = \big\langle U,\mathcal{B}(Z)\big\rangle. \tag{28}$$

Letting $\tilde{X} = X^v$, $\tilde{Y} = Y^v$ and $\tilde{Z} = Z^v$ in (27), we get (24). Similarly, putting $\tilde{X} = X^c$, $\tilde{Y} = Y^c$ and $\tilde{Z} = Z^v$ in (27) implies (25). Finally, putting $\tilde{X} = X^v$, $\tilde{Y} = Y^c$ and $\tilde{Z} = Z^c$ in (27), we have (26).

□

3.2. 5-Dimensional Two Step Nilpotent Lie Groups

As we mentioned, a 2-step nilpotent Lie group is a non-abelian connected Lie group which its Lie algebra lower central series vanishes at second step, i.e., $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = 0$. The basic example is the Heisenberg group. Any simply connected nilpotent Lie group is diffeomorphic to Euclidean space. Nilpotent groups arise in Galois theory, as well as in the classification of groups.

Here, we are going to consider 5-dimensional two step nilpotent Lie groups. Let G be a connected 5-dimensional two step nilpotent Lie group with Lie algebra $\mathfrak{g}_i$ whose center is i-dimensional for $i = 1,2,3$. There exists an orthonormal basis $\{X_1,\ldots,X_5\}$ for $\mathfrak{g}_i$ with respect to an inner product $<,>$ on $\mathfrak{g}_i$. These spaces were classified in [7]. Also, their invariant connections were calculated in [15]. Let us recall some useful facts from [15].

Case 1. (1-dimensional center) Let $\mathfrak{g}_1$ be the Lie algebra such that its non-zero brackets are as follows:

$$[X_1,X_2] = \lambda X_5, \qquad [X_3,X_4] = \mu X_5, \tag{29}$$

where $\lambda \geq \mu > 0$ and $\{X_5\}$ is the basis of the center of $\mathfrak{g}_1$. Suppose that α is the left invariant Riemannian metric on the Lie group G induced from $<,>$. By using Kozsul's formula for the Levi-Civita connection of α, we have the following table

	X_1	X_2	X_3	X_4	X_5
∇_{X_1}	0	$\frac{1}{2}\lambda X_5$	0	0	$-\frac{1}{2}\lambda X_2$
∇_{X_2}	$-\frac{1}{2}\lambda X_5$	0	0	0	$\frac{1}{2}\lambda X_1$
∇_{X_3}	0	0	0	$\frac{1}{2}\mu X_5$	$-\frac{1}{2}\mu X_4$
∇_{X_4}	0	0	$-\frac{1}{2}\mu X_5$	0	$\frac{1}{2}\mu X_3$
∇_{X_5}	$-\frac{1}{2}\lambda X_2$	$\frac{1}{2}\lambda X_1$	$-\frac{1}{2}\mu X_4$	$\frac{1}{2}\mu X_3$	0

Table 1: 1-dimensional center (Taken from [15])

Case 2. (2-dimensional center) The Lie algebra structure of $\mathfrak{g}_2$ is as the following form

$$[X_1, X_2] = \lambda X_4, \qquad [X_1, X_3] = \mu X_5, \tag{30}$$

where $\lambda \geq \mu > 0$ and $\{X_4, X_5\}$ is the basis of the center of $\mathfrak{g}_2$. In this case, for the Levi-Civita connection of α we have the following

	X_1	X_2	X_3	X_4	X_5
∇_{X_1}	0	$\frac{1}{2}\lambda X_4$	$\frac{1}{2}\mu X_5$	$-\frac{1}{2}\lambda X_2$	$-\frac{1}{2}\mu X_3$
∇_{X_2}	$-\frac{1}{2}\lambda X_4$	0	0	$\frac{1}{2}\lambda X_1$	0
∇_{X_3}	$-\frac{1}{2}\mu X_5$	0	0	0	$\frac{1}{2}\mu X_1$
∇_{X_4}	$-\frac{1}{2}\lambda X_2$	$\frac{1}{2}\lambda X_1$	0	0	0
∇_{X_5}	$-\frac{1}{2}\mu X_3$	0	$\frac{1}{2}\mu X_1$	0	0

Table 2: 2-dimensional center (Taken from [15])

Case 3. (3-dimensional center) The Lie algebra structure of this case is as follows

$$[X_1, X_2] = \lambda X_3, \tag{31}$$

where $\lambda > 0$ and $\{X_3, X_4, X_5\}$ is a basis of $\mathfrak{g}_3$. The Levi-Civita connection of α is given by the following table

	X_1	X_2	X_3	X_4	X_5
∇_{X_1}	0	$\frac{1}{2}\lambda X_3$	$-\frac{1}{2}\lambda X_2$	0	0
∇_{X_2}	$-\frac{1}{2}\lambda X_3$	0	$\frac{1}{2}\lambda X_1$	0	0
∇_{X_3}	$-\frac{1}{2}\lambda X_1$	$\frac{1}{2}\lambda X_2$	0	0	0
∇_{X_4}	0	0	0	0	0
0	0	0	0	0	0

Table 3: 3-dimensional center (Taken from [15])

Now, we can give an application of Theorem 1.1.

Theorem 3.5. *Let G be a 5-dimensional two step nilpotent Lie group equipped with a left invariant Randers metric $F(x, y) = \sqrt{\alpha_x(y, y)} + \alpha_x(U, y)$ such that U is in the center of $\mathfrak{g} = T_eG$. Then (TG, F^c) a generalized Douglas-Weyl Randers metric if and only if the center of $\mathfrak{g}$ is 3-dimensional. In this case, (TG, F^v) is also a generalized Douglas-Weyl Randers metric.*

Proof. We divide the proof into three cases:

Case 1) Note that, in this case we have

$$\nabla_{X_i} X_i = 0.$$

We assume that $U = aX_5$ with $|a| < 1$. For each $j, k, l = 1, \ldots, 5$, the following holds

$$\Big\langle X_5,\ \Big[\nabla_{X_j} X_k - \frac{1}{2}[X_j, X_k], X_l\Big]\Big\rangle = \frac{1}{9}\langle X_j, X_k\rangle \sum_{i=1}^{5} \Big\langle X_5,\ [X_i, \nabla_{X_i} X_l]\Big\rangle. \tag{32}$$

But for $j = k = 1$ and $l = 5$ the left hand side of (32) vanishes while the right hand side is the non-zero scalar $-(\mu^2 + \lambda^2/2)$. Thus, F^c is not a generalized Douglas-Weyl Randers metric.

Case 2) We consider $U = aX_4 + bX_5$ where $\sqrt{a^2 + b^2} < 1$. Note that in this case we have $\nabla_{X_i} X_i = 0$. Putting it in (23) yields

$$\Big\langle aX_4 + bX_5,\ \Big[\nabla_{X_j} X_k - \frac{1}{2}[X_j, X_k], X_l\Big]\Big\rangle = \frac{1}{9}\langle X_j, X_k\rangle \sum_{i=1}^{5} \Big\langle aX_4 + bX_5,\ [X_i, \nabla_{X_i} X_l]\Big\rangle. \tag{33}$$

For $j = k = 1$ and $l = 5$, (33) reduces to following

$$0 = -\frac{1}{2} b\mu^2.$$

Hence, we suppose that $b = 0$. Now, let $j = 3$, $k = 5$ and $l = 2$. Then, the right hand side of (33) vanishes while the left hand side is the scalar $\frac{1}{2}a\mu\lambda$ which yields $a = 0$. This is a contradiction. Then F^c is not a generalized Douglas-Weyl Randers metric.

Case 3) We consider $U = aX_3 + bX_4 + cX_5$ where $\sqrt{a^2 + b^2 + c^2} < 1$. Note that in this case we have $\nabla_{X_i} X_i = 0$. Putting it in (23) yields

$$\Big\langle aX_3 + bX_4 + cX_5,\ \Big[\nabla_{X_j} X_k - \frac{1}{2}[X_j, X_k], X_l\Big]\Big\rangle = \frac{1}{9}\langle X_j, X_k\rangle \sum_{i=1}^{5} \Big\langle aX_3 + bX_4 + cX_5,\ [X_i, \nabla_{X_i} X_l]\Big\rangle. \tag{34}$$

Setting $j = k = 1$ and $l = 3$ in (34) implies that

$$0 = -\frac{1}{9}\lambda^2 a.$$

Thus, we have $a = 0$. Since for any i, k, l the vector $\Big[\nabla_{X_j} X_k - \frac{1}{2}[X_j, X_k], X_l\Big]$ is a multiple of X_3, then it is perpendicular to $bX_4 + cX_5$. Thus the left hand side of (34) is always zero. Letting $a = 0$ implies that the right hand side of (34) is also always zero. This means that F^c is a generalized Douglas-Weyl Randers metric.

The same argument shows that for $U = bX_4 + cX_5$ the three equations (24), (25) and (26) hold trivially. Thus F^v is also a generalized Douglas-Weyl Randers metric. This completes the proof.

□

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