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Approximate symmetries and invariant solutions for the generalizations of the Burgers-Korteweg-de Vries model

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ABSTRACT: In this paper the generalizations of the Burgers-Korteweg-de Vries model with small parameter derived by Kudryshov et al[N.A. Kudryashov, D.I. Sinelshchikov. Extended models of non-linear waves in liquid with gas bubbles, International Journal of Non-Linear Mechanics 63 (2014) 31-38] is studied. A comprehensive study on the approximate symmetry analysis of the waves models is presented. First, we obtain approximate symmetry for the equation. Subsequently, in a physical application, using the first-order approximate symmetries, corresponding approximate invariant solutions to the perturbed non-linear models are obtained.

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1. Introduction

In this paper, we investigate approximate symmetries and approximate solutions for the generalizations of the Burgers-Korteweg-de Vries equation given by

$$\begin{aligned} & v_t + \alpha v v_x + \beta v_{xxx} - \mu v_{xx} + \\ & \epsilon\Big[(6\alpha\beta - \beta_2 + 3\beta\lambda_2 + 6\beta\lambda_1 - 2\alpha\lambda_3)v_x v_{xx} \\ & (2\alpha\beta - \beta_1 + 3\beta\lambda_2)v v_{xxx} + \frac{\alpha}{2}(2\lambda_1 + \lambda_2)v^2 v_x - \\ & (2\mu\lambda_1 + \mu\alpha + \nu)v_x^2 \\ & \mu^2 v_{xxx} + (\gamma - 3\beta\mu)v_{xxxx} + 2\beta^2 v_{xxxxx}\Big] = 0 \end{aligned} \tag{1}$$

Here $\alpha,\beta,\beta_1,\beta_2,\mu,\nu,\gamma,\lambda_1,\lambda_2$ and λ_3 are constants and is small parameter. This equation has been obtained by Kudryashov and Sinelshchikov in [1] for the first time. This equation describes non-linear waves in a bubbly liquid. In the purely dispersive case $\mu = \nu = \gamma = 0$, Eq. (1) is the generalization of the Korteweg-de Vries equation.

Recentely, in paper [2] by using Painlev´e test, it is shown that the perturbed Burgers-Korteweg-de Vries equation is not Painlev´e integrable.

The outline of the paper is arranged as follows. In Section 2, we present the first-order approximate symmetry analysis of the equation for two cases. Subsequently, in Section 3, the approximate symmetries will be used to obtain the the first-order approximate invariant solutions of the equation of study. The last section is a conclusion of our results.

We use symbolic software package ASP [3] for all approximate symmetries and approximate invariant solutions computations.

*Approximate symmetry analysis for the non-linear waves equations with small parameter

First, we recall some principles associated with approximate symmetry approach proposed by Baikov *et al* [4, 5].

The approximate symmetry of the model (1) is of the form

$$\begin{aligned} X &= X^0 + \epsilon X^1 \\ &\equiv (\tau_0(t,x,v) + \epsilon\tau_1(t,x,v))\partial_t + (\xi_0(t,x,v) + \\ &\epsilon\xi_1(t,x,v))\partial_x + (\eta_0(t,x,v) + \epsilon\eta_1(t,x,v))\partial_v. \end{aligned} \tag{2}$$

We know that Eq. (1) is approximately invariant under the approximate transformation group generated by (2) if and only if

$$\Big[X^0 F_0(z) + \epsilon(X^1 F_0(z) + X^0 F_1(z))\Big]\Big|_{(1)} = o(\epsilon),$$

with

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$$\left[X^0F_0(z)+\epsilon(X^1F_0(z)+X^0F_1(z))\right]\Big|_{(1)}=o(\epsilon),$$

$$\begin{aligned}F_0(z) &= (v_t+\alpha vv_x+\beta v_{xxx}-\mu v_{xx}),\\ F_1(z) &= (6\alpha\beta-\beta_2+3\beta\lambda_2+6\beta\lambda_1-\\ &2\alpha\lambda_3)v_xv_{xx}+(2\alpha\beta-\beta_1+3\beta\lambda_2)vv_{xxx}\\ &+\frac{\alpha}{2}(2\lambda_1+\lambda_2)v^2v_x-(2\mu\lambda_1+\mu\alpha+\nu)v_x^2-\\ &(2\mu\lambda_2+\mu\alpha+\nu)vv_{xx}\end{aligned}$$

By solving

$$\left[X^0F_0(z)\right]\Big|_{F_0(z)=0}=0, \tag{3}$$

and *determining equation for deformations*

$$\left[X^1F_0(z)\right]\Big|_{F_0(z)=0}+H=0, \tag{4}$$

where H is the *auxiliary function*

$$H=\frac{1}{\epsilon}\left[X^0(F_0(z)+\epsilon F_1(z))\Big|_{F_0(z)+\epsilon F_1(z)=0}\right], \tag{5}$$

the operators X^0 and X^1 are obtained. We refer to [4, 5, 6, 7, 8] for further details.

Now we study approximate symmetry groups and Lie algebras associated with the extended model for two different cases:

Case 1. *$\alpha,\beta,\beta_1,\beta_2,\mu,\nu,\gamma,\lambda_1,\lambda_2$ and λ_3 are arbitrary.*

Here we solve Eq. (3) for the exact symmetries X^0 of the equation

$v_t+\alpha vv_x+\beta v_{xxx}-\mu v_{xx}=0.$(6)

By solving Eq. (3), we find the system of overdetermined: $\tau0,t=\tau0,x=\tau0,v=\xi0,x=\xi0,v=\eta0,t=\eta0,x=\eta0,v=0,\xi0,t=\alpha\eta0$.

So we have

$\tau_0=B_1$, $\xi_0=B_2\alpha t+B_3$, $\eta_0=B_2$,(7)

with B_1,B_2 and B_3 arbitrary constants. Hence,

$X^0=B_1\partial_t+(B_2\alpha t+B_3)\partial_x+B_2\partial_v$.

Therefore, we get a three-dimensional Lie algebra for Eq. (6) from

$$X_1^0=\partial_t,\quad X_2^0=\alpha t\partial_x+\partial_v,\quad X_3^0=\partial_x. \tag{8}$$

By substituting X^0 into Eq. (5) we obtain

$H=B_2((\alpha\lambda_2+2\alpha\lambda_1)vv_x-(2\mu\lambda_2+\alpha\mu+\nu)v_{xx}+(3\beta\lambda_2+2\alpha\beta-\beta_1)v_{xxx})$.

Now Eq. (4) has the form

$$X^1(v_t+\alpha vv_x+\beta v_{xxx}-\mu v_{xx})\Big|_{(6)}+B_2((\alpha\lambda_2+2\alpha\lambda_1)vv_x-(2\mu\lambda_2+\alpha\mu+\nu)v_{xx}+(3\beta\lambda_2+2\alpha\beta-\beta_1)v_{xxx})=0, \tag{9}$$

where X^1 is the third prolongation of the operator $X1=\tau1\partial t+\xi1\partial x+\eta1\partial v$.

By calculating the terms in (9) we have

$\tau_{1,x}=\tau_{1,v}=\xi_{1,xx}=\xi_{1,v}=\eta_{1,t}=\eta_{1,x}=\eta_{1,vv}=0$, $\beta(\tau_{1,t}-3\xi_{1,x})+B_2(2\alpha\beta+3\beta\lambda_2-\beta_1)=0$, $B_2(\alpha\mu+2\lambda_2\mu+\nu)+\mu(\tau_{1,t}-2\xi_{1,x})=0$,

$\alpha(2\lambda1B2+\lambda2B2-\xi1,x+\tau1,t)\nu+\alpha\eta1-\xi1,t=0$. So we get

$$\begin{aligned}\tau_1 &= (\alpha-\frac{3\nu}{\mu}-\frac{2\beta_1}{\beta})B_2t+C_1,\\ \xi_1 &= (\alpha-\frac{\nu}{\mu}-\frac{\beta_1}{\beta}+\lambda_2)B_2x+C_2\alpha t+C_3\\ \eta_1 &= (\frac{2\nu}{\mu}-2\lambda_1+\frac{\beta_1}{\beta})B_2v+C_2,\end{aligned} \tag{10}$$

with C_1,C_2 and C_3 arbitrary constants. Hence

$$X^1=\Big((\alpha-\frac{3\nu}{\mu}-\frac{2\beta_1}{\beta})B_2t+C_1\Big)\partial_t+\Big((\alpha-\frac{\nu}{\mu}-\frac{\beta_1}{\beta}+\lambda_2)B_2x+C_2\alpha t+C_3\Big)\partial_x+\Big((\frac{2\nu}{\mu}-2\lambda_1+\frac{\beta_1}{\beta})B_2v+C_2\Big)\partial_v.$$

Substituting (10) and (7) into (2), we get the following approximate symmetries for Eq. (1):

$$\begin{aligned}X_1 &= \partial_t,\quad X_2=\partial_x,\quad X_3=\epsilon\partial_t,\\ X_4 &= \epsilon\partial_x,\quad X_5=\epsilon(\alpha t\partial_x+\partial_v),\\ X_6 &= \epsilon t\Big(\alpha-\frac{3\nu}{\mu}-\frac{2\beta_1}{\beta}\Big)\partial_t+\\ &\Big(\alpha t+\epsilon(\alpha-\frac{\nu}{\mu}-\frac{\beta_1}{\beta}+\lambda_2)x\Big)\partial_x+\\ &\Big(1+\epsilon(\frac{2\nu}{\mu}-2\lambda_1+\frac{\beta_1}{\beta})v\Big)\partial_v.\end{aligned} \tag{11}$$

Now, we have the following table of commutators.

So the vector fields (11) generate a six-parameter approximate transformations group. Note that the operators $X_1,X_2,\cdots,X_6$ form approximate Lie algebra in the first-order of precision [7, 9].

From equations (11) one can easily see that all symmetries (8) of Eq. (6) are stable. What we come to the corresponding conclusions about the stability of the symmetries is that the perturbed model (1) inherits the symmetries of Eq. (6).

Case 2. $\mu=\nu=\gamma=0$.

From these assumptions we obtain

$$\begin{aligned}&v_t+\alpha vv_x+\beta v_{xxx}+\epsilon\Big[(6\alpha\beta-\beta_2+3\beta\lambda_2+6\beta\lambda_1-\\ &2\alpha\lambda_3)v_xv_{xx}+(2\alpha\beta-\beta_1+3\beta\lambda_2)vv_{xxx}+\\ &\frac{\alpha}{2}(2\lambda_1+\lambda_2)v^2v_x+2\beta^2v_{xxxxx}\Big]=0.\end{aligned} \tag{12}$$

Model (12) was introduced in [1, 10]. Special solutions for

Table 1: Approximate commutators

$[,]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	αX_4	$\alpha X_2 + (\alpha - \beta^{-1}\mu^{-1}[3\nu\beta + 2\beta_1\mu])\, X_3$
X_2		0	0	0	0	$(\alpha + \lambda_2 - \beta^{-1}\mu^{-1}[\nu\beta + \beta_1\mu])X_4$
X_3			0	0	$o(\epsilon)$	$\alpha X_4 + o(\epsilon)$
X_4				0	0	$o(\epsilon)$
X_5					0	$o(\epsilon)$
X_6						0

Table 2: Approximate commutators

$[,]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	0	0	0	αX_4	$X_4 - \beta^{-1}(2\alpha\beta + 3\beta\lambda_2 - \beta_1)\, X_3$	$3(2\alpha\beta + 3\beta\lambda_2 - \beta_1)\, X_3$
X_2		0	0	0	0	0	$(2\alpha\beta + 3\beta\lambda_2 - \beta_1)X_3$
X_3			0	0	$o(\epsilon)$	$\alpha X_4 + o(\epsilon)$	$o(\epsilon)$
X_4				0	0	0	$o(\epsilon)$
X_5					0	$o(\epsilon)$	$o(\epsilon)$
X_6						0	$-2(2\alpha\beta + 3\beta\lambda_2 - \beta_1)\, X_5$
X_7							0

canonical form of this model were obtained in there.

Here we solve Eq. (3) for the exact symmetries X^0 of the equation

$$v_t + \alpha v v_x + \beta v_{xxx} = 0 \tag{13}$$

From Eq. (3), we find the system of overdetermined:

$$\tau_{0,x} = \tau_{0,v} = \xi_{0,v} = \eta_{0,t} = \eta_{0,x} = \eta_{0,vv} = 0,$$
$$\tau_{0,t} = -\frac{3\eta_{0,v}}{2}, \quad \xi_{0,t} = -\alpha(v\eta_{0,v} - \eta_0), \quad \xi_{0,x} = -\frac{\eta_{0,v}}{2}.$$

So we have

$$\tau_0 = 3B_1 t + B_2, \quad \xi_0 = B_3\alpha t + B_1 x + B_4, \quad \eta_0 = B_3 - 2B_1 v, \tag{14}$$

with B_1,B_2,B_3 and B_4 arbitrary constants. Hence,

$$X^0 = (3B_1 t + B_2)\partial_t + (B_3\alpha t + B_1 x + B_4)\partial_x + (-2B_1 v + B_3)\partial_v.$$

Therefore, we obtain a four-dimensional Lie algebra for Eq. (13) from

$$X_1^0 = 3t\partial_t + x\partial_x - 2v\partial_v, \quad X_2^0 = \partial_t,$$
$$X_3^0 = \alpha t\partial_x + \partial_v, \quad X_4^0 = \partial_x. \tag{15}$$

By substituting X^0 into Eq. (5) we have

$$H = B_3\alpha(2\lambda_1 + \lambda_2)vv_x - B_1\alpha(2\lambda_1 + \lambda_2)v^2 v_x - 2B_1\beta(2\alpha + 3\lambda_2)vv_{xxx} - 2B_1(2\alpha(3\beta + \lambda_3) + 3\beta(2\lambda_1 + \lambda_2))v_x v_{xx} + B_3(\beta(2\alpha + 3\lambda_2) - \beta_1)v_{xxx} + 2B_1\beta_1 vv_{xxx} + 2B_1\beta_2 v_x v_{xx} - 4B_1\beta_2 v_{xxxxx}.$$

The determining deformations equation yields that $B_1 = 0$. So, we find

$$\tau_{1,x} = \tau_{1,v} = \xi_{1,xx} = \xi_{1,v} = \eta_{1,t} = \eta_{1,x} = \eta_{1,vv} = 0,$$
$$\beta\tau_{1,t} - 3\beta\xi_{1,x} + B_3(2\alpha\beta + 3\lambda_2\beta - \beta_1) = 0,$$
$$\alpha v\tau_{1,t} - \alpha v\xi_{1,x} - \xi_{1,t} + \alpha\eta_1 + \alpha B_3(\lambda_2 + 2\lambda_1)v = 0.$$

So we have

$$\xi_1 = C_4 + \frac{(((2\alpha + 3\lambda_2)B_3 + C_1)x + 3\alpha C_3 t)\beta - B_3\beta_1 x}{3\beta}, \quad \tau_1 = C_1 t + C_2$$
$$\eta_1 = C_3 + \frac{((2\alpha - 6\lambda_1)B_3 - 2C_1)\beta v - B_3\beta_1 v}{3\beta}, \tag{16}$$

with C_1,C_2,C_3 and C_4 arbitrary constants. Substituting (16) and (14) into (2), we get

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \epsilon\partial_t, \quad X_4 = \epsilon\partial_x, \quad X_5 = \epsilon(\alpha t\partial_x + \partial_v),$$
$$X_6 = \epsilon t\Big(\frac{\beta_1}{\beta} - 2\alpha - 3\lambda_2\Big)\partial_t + \alpha t\partial_x + \Big(1 + \epsilon v(2\alpha - 2\lambda_1 + 2\lambda_2 - \frac{\beta_1}{\beta})\Big)\partial_v \tag{17}$$
$$X_7 = \epsilon\,(2\alpha\beta + 3\beta\lambda_2 - \beta_1)(3t\partial_t + x\partial_x - 2v\partial_v).$$

There for, we have the following table of commutators.

We see that the vector fields (17) generate a seven-parameter approximate transformations group. We note that the operators X_1,X_2,···,X_7 form an approximate Lie algebra in the first-order of precision.

From equations (17) we arrive to the fact that not all generators (15) of Eq. (13) are stable. In fact, the vector field

$$X_1^0 = 3t\partial_t + x\partial_x - 2v\partial_v,$$

from (15) is unstable. So, the perturbed model (12) does not inherit the symmetries of Eq. (13).

2. Approximately Invariant Solutions for the non-linear waves equations

Now, by using the approximate symmetries obtained in the previous section, we construct the first-ordr approximate invariant solutions for the non-linear model [7, 8, 11].

Case 1. *Invariance under* $X = (1 + \epsilon\alpha t)\partial_x + \epsilon\partial_v$.

Here, we investigate the solutions invariant under the operator X of the Lie algebra (11). Solving the characteristic equations for the invariants of X, we find :

$$w(r) = v(t, x) - \frac{\epsilon x}{\alpha\epsilon t + 1}, \qquad r = t.$$

So, the reduced equation is

$$(r\alpha\epsilon + 1)^2 w_r + \left[\frac{\alpha\epsilon}{2}(2\lambda_1 + \lambda_2)(r\alpha\epsilon + 1)w^2 + (r\alpha^2\epsilon + \alpha)w - \epsilon^2(\alpha\mu + 2\lambda_1\mu + \nu)\right]\epsilon = 0 \tag{18}$$

From Eq.(18) we obtain

$$w(r) = \frac{-R}{\epsilon(2\lambda_1 + \lambda_2)} \cdot \frac{2CY_0(R) - J_0(R)}{2CY_1(R) - J_1(R)},$$

where $R = (-2\alpha^{-1}\epsilon^3(\alpha\epsilon r + 1)^{-1}(\alpha\mu + 2\lambda_1\mu + \nu)(2\lambda_1 + \lambda_2))^{1/2}$, C is arbitrary constant, $J_0(R)$, $J_1(R)$ are the Bessel functions of the first kind, and $Y_0(R)$, $Y_1(R)$ are the Bessel functions of the second kind. So we obtain:

$$v(t, x) = \frac{x\epsilon}{\alpha\epsilon t + 1} - \frac{R}{\epsilon(2\lambda_1 + \lambda_2)\sqrt{\alpha\epsilon t + 1}} \cdot \frac{2CY_0(R) - I_0(R)}{2CY_1(R) - I_1(R)}, \tag{19}$$

where R, $I_0(R)$, $I_1(R)$ are the Bessel functions of the first kind, and $Y_0(R)$, $Y_1(R)$ are the Bessel functions of the second kind. By substituting this solution into the left side of equation (1) we have

$$v_t + \alpha v v_x + \beta v_{xxx} - \mu v_{xx} + \epsilon\Big[(6\alpha\beta - \beta_2 + 3\beta\lambda_2 + 6\beta\lambda_1 - 2\alpha\lambda_3)v_x v_{xx} + (2\alpha\beta - \beta_1 + 3\beta\lambda_2)v v_{xxx} + \frac{\alpha}{2}(2\lambda_1 + \lambda_2)v^2 v_x - (2\mu\lambda_1 + \mu\alpha + \nu)v^{2,x} - (2\mu\lambda_2 + \mu\alpha + \nu)v v_{xx} + \mu^2 v_{xxx} + (\gamma - 3\beta\mu)v_{xxxx} + 2\beta^2 v_{xxxxx}\Big] = o(\epsilon^4).$$

Therefore solution (19) is a approximate invariant solution for the generalization of the Burgers-Korteweg-de Vries model with small parameter (1).

Case 2. *Invariance under*?.

Now, we invastigate solutions invariant under the operator X of the Lie algebra (17). For the operator X we obtain two invariants:

$$w(r) = v - \frac{\epsilon x}{\alpha\epsilon t + 1}, \qquad r = t.$$

So, the reduced equation is

$$(2\alpha\epsilon r + 2)w_r + \epsilon\alpha(2 + \epsilon(2\lambda_1 + \lambda_2)w)w = 0. \tag{20}$$

From (20) we find

$$w(r) = \frac{2}{\epsilon(2C\alpha r - 2\lambda_1 - \lambda_2) + 2C},$$

where C is constant. So

$$v(t, x) = \frac{\epsilon^2 x(2C\alpha t - 2\lambda_1 - \lambda_2) + 2\epsilon(Cx + \alpha t) + 2}{(\epsilon(2C\alpha t - 2\lambda_1 - \lambda_2) + 2C)(\alpha\epsilon t + 1)}. \tag{21}$$

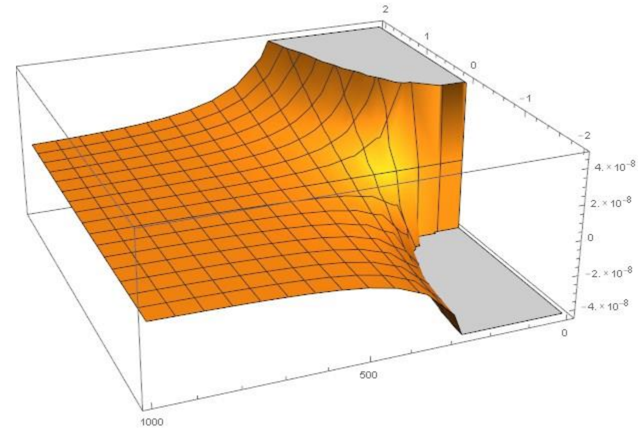

Fig. 1: Approximate invariant solution associated with $\alpha = 2,\ \lambda_1 = 3,\ \lambda_2 = 4,\ C = 4,\ \epsilon = 10^{-1}$.

By substituting (21) into the left side of equation (12) we have

$$v_t + \alpha v v_x + \beta v_{xxx} + \epsilon\Big[(6\alpha\beta - \beta_2 + 3\beta\lambda_2 + 6\beta\lambda_1 - 2\alpha\lambda_3)v_x v_{xx} + (2\alpha\beta - \beta_1 + 3\beta\lambda_2)v v_{xxx} + \frac{\alpha}{2}(2\lambda_1 + \lambda_2)v^2 v_x + 2\beta^2 v_{xxxxx}\Big]$$

Thus solution (21) is a approximate invariant solution for the generalization of the Korteweg-de Vries model with small parameter (12). A plot of solution (21) is shown in Fig. (1). For a physical meaning of the solution, we explain that this solution represent a two-crest wave.

We have presented waves with two crests in a bubbly liquid for the first time.

3. Conclusions

By the approximate symmetry approach proposed by Baikov, Gazizov and Ibragimov, we have studied the first-order approximate symmetries for the generalizations of the Burgers-Korteweg-de Vries equation with small parameter. We have used the approximate symmetry method for finding approximate invariant solutions of the non-linear model. We have shown waves with two crests in a bubbly liquid.

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