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Control of parabolic PDE systems with time varying spatial boundary conditions using a novel nonlinear backstepping scheme

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ABSTRACT: In this paper, an extension of backstepping controller for parabolic PDE systems (Heat Transfer Process) with time-varying spatial boundary is studied. The PDE system dynamics is transformed to an exponentially stable target system via a new nonlinear backstepping transformation. The exponential stability of the closed-loop system is established by using a proper Lyapunov function. Finally, numerical simulation is provided to support the effectiveness of the proposed controller.

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1- INTRODUCTION

In distributed-parameter systems described by partial differential equations (PDEs), the state variables depend on time and spatial coordinates in which changes in the shape and material properties characterized by phenomena such as material deformation may result in system models represented by moving boundary parabolic PDEs. Both infinite dimensionality and changes in the domain of these models impose further complexities and limitations to both analysis and design of such systems [2,3].

Control analysis of these systems have not been received much of attention. Most of the proposed control schemes are based on discretization by means of suitable approximation techniques such as modal decomposition or Galerkin's method which yield approximate finite-dimensional models [1,7]. Another more complicated approach introduced in [10] is the use of backstepping concept for boundary observation of PDE systems. In this methodology, an invertible Volterra integral transformation is used to transform the estimation error dynamics into a suitably selected stable distributed parameter target system. Backstepping observer design is extended to linear parabolic PDEs with spatially and time-varying reaction parameters [5], parabolic PDEs with nonlinear reactive-convective terms [4] and finally, to semi-linear parabolic PDEs [13].

In case of a moving boundary parabolic PDE, even if the process parameters are time-invariant, the system is inherently nonautonomous [12]. For such problems, as infinite-dimensional systems, an early-lumping approach, the Galerkin's method is used for an eigenfunctions-based observer design in [4] for the boundary control of a 2D heat equation with time-dependent spatial domain. Furthermore, the observer design for the parabolic PDEs with time-varying domain was fully studied in [8]. An extension of the backstepping observer design of parabolic PDEs by time-varying domain with an application to the Czochralski crystal growth process was proposed in [9].

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In this paper, the boundary control of parabolic PDEs with time varying spatial domain using a backstepping transformation is studied. The contribution of our approach relies on new backstepping transformation without rescaling the space domain into a fixed boundary. In fact, we use an extension of backstepping transformation which was used in [11] for linear PDE on a fixed domain. The proposed controller provides exponential stability in the sense of H1 norm.

Problem Statement

Consider heat transfer problem along a beam of length L shown in Fig.1.

Where $\mathbf{u(t)}, \theta(\zeta,\mathbf{t}), \theta^{\mathbf{ref}}, \mathbf{f(t)}, \mathbf{f^{ref}}$ will be introduced later.

Our goal is to find a position at which phase transition from solid to liquid occurs and vice versa. Hence, the length of the beam $0 \le \zeta \le L$ can be divided into two sub-domains. The first $0 \le \zeta \le f(t)$ is a part of the beam melted and the second part $f(t) \le \zeta \le L$ is the solid part, where $f(t) \in \mathbb{R}^+$ is the time-dependent length of the domain and $t \in [0,\infty)$ represents the time.

The control input, namely, the heat affects the moving boundary ($f(t)$) and is manipulated at $\zeta = 0$. The dynamic

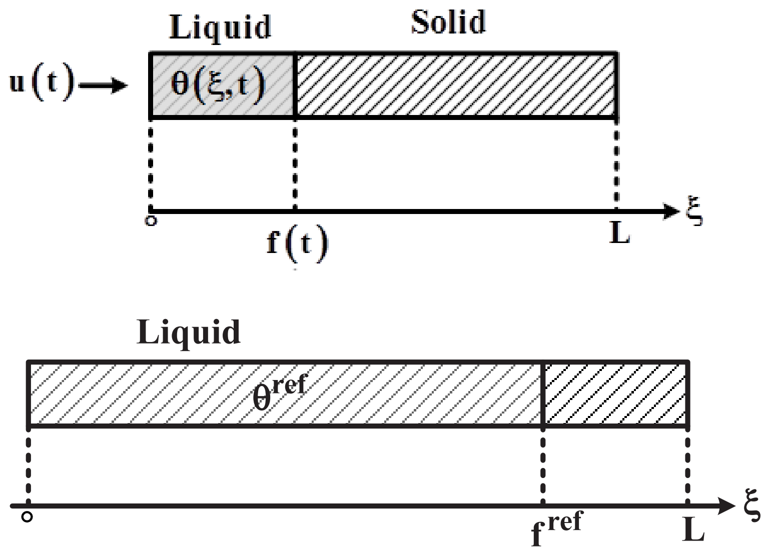

Fig. 1. a. Schematic of Hear Transfer Process, b. Steady-State of Heat Transfer Process

equations of the system consists of two part

1) Parabolic Partial Differential Equation that describes melted part of the beam is

$$\frac{\partial\theta(\zeta,t)}{\partial t}=\alpha\frac{\partial^2\theta(\zeta,t)}{\partial\zeta^2}\qquad 0\le\zeta\le f(t)<L \tag{1}$$

where $\theta(\zeta,t)\in[0,L]\times R^+\to R$ is the state variable (the temperature distribution along the beam) and α is the process parameter.

We assume the following boundary conditions for the PDE system which is applicable to the temperature stabilization in the heat transfer process:

$$\frac{\partial}{\partial\zeta}\theta(0,t)=u(t) \tag{2}$$

$$\theta(f(t),t)=\theta^{ref} \tag{3}$$

Where $u(t)$ is the control input applied at the boundary $\zeta=0$ to stabilize the system state. Our objective is to make $\theta(\zeta,t)\to\theta^{ref}$ and $f(t)\to f^{ref}$ asymptotically where $f^{ref}<L$ is the maximum length of the beam to be melted.

2) The dynamical equation of the moving boundary is an ordinary differential equation of the form

$$\frac{\partial f(t)}{\partial t}=-q\frac{\partial}{\partial\zeta}\theta(f(t),t) \tag{4}$$

where q accounts for the process parameter.

To develop the controller, the following assumptions are made.

Assumption 1.

$\theta(\zeta,t)\ge\theta^{ref};0\le\zeta\le f(t)$ which means temperature in melted part of the beam is higher than θ^{ref}.

Assumption 2.

$\frac{\partial f(t)}{\partial t}\ge 0;\forall t\ge 0$ which is plausible to assume the moving interface $f(t)$ increases as heat inserted through the beam $(u(t)>0)$

The block diagram of the system depicted in Fig. 2.

As depicted in Fig. 2, the proposed control scheme utilizes the measurement on $f(t)$, the moving boundary of melted part, and $\theta(\zeta,t)$ to design the control signal $u(t)$.

We define two error signals as

$$v(\zeta,t)=\theta(\zeta,t)-\theta^{ref} \tag{5}$$

$$X(t)=f(t)-f^{ref} \tag{6}$$

Our objective is to find a proper control input signal $(u(t)>0)$ such that the errors approach zero in steady state.

The following control law stabilizes the PDE system (1)-(4)

$$u(t)=\frac{\gamma}{\alpha}\int_0^{f(t)}(\theta(\zeta,t)-\theta^{ref})d\zeta+\frac{\gamma}{q}(f(t)-f^{ref}) \tag{7}$$

Where $\gamma>0$ is the control gain.

Towards this end, consider the following backstepping transformation

$$\omega(\zeta,t)=v(\zeta,t)+\int_\zeta^{f(t)}k(\zeta,\eta)v(\eta,t)d\eta+\Psi(f(t)-\zeta)X(t) \tag{8}$$

Which transforms the system (1)-(4) to the following exponentially stable target PDE system

$$\frac{\partial\omega(\zeta,t)}{\partial t}=\alpha\frac{\partial^2\omega(\zeta,t)}{\partial\zeta^2}+\frac{\gamma}{q}X(t)\frac{\partial}{\partial t}f(t) \tag{9}$$

$$\omega(f(t),t)=0 \tag{10}$$

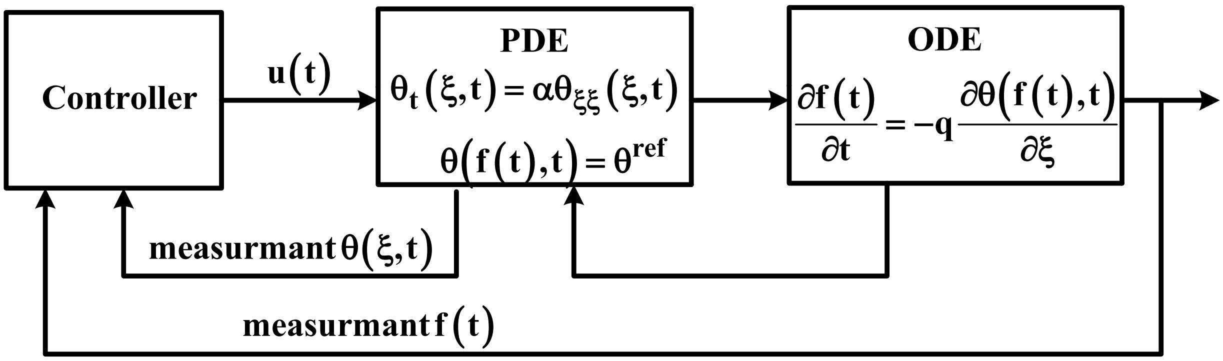

Fig. 2. Block diagram of heat transfer moving boundary system

$$\frac{\partial}{\partial \zeta}\omega(0,t) = \frac{\partial}{\partial \zeta}v(0,t) - \frac{\gamma}{\alpha}\int_{\zeta}^{f(t)} -v(\eta,t)d\eta - \frac{\gamma}{q}X(t) = 0 \tag{11}$$

$$\frac{\partial}{\partial t}f(t) = \frac{\partial}{\partial t}X(t) = -\gamma X(t) - q\frac{\partial}{\partial \zeta}\omega(f(t),t) \tag{12}$$

where the gain kernel $k(\zeta,\eta)$ and $\Psi(f(t)-\zeta)$ are obtained as:

$$k(\zeta,\eta) = \frac{\gamma}{\alpha}(\eta-\zeta) \tag{13}$$

$$\Psi(f(t)-\zeta) = \frac{\gamma}{q}(f(t)-\zeta) \tag{14}$$

Proof.

By recalling liepnitz formula

$$\frac{\partial}{\partial x}\int_{u}^{v} f(x,y)dy = f(x,v)\frac{\partial v}{\partial x} - f(x,u)\frac{\partial u}{\partial x} + \int_{u}^{v}\left(\frac{\partial}{\partial x}f(x,y)\right)dy \tag{15}$$

and noting that $v(f(t),t) = 0$, $\frac{\partial v(\zeta,t)}{\partial t} = \frac{\partial \theta(\zeta,t)}{\partial t}$ and $\frac{\partial^2 v(\zeta,t)}{\partial \zeta^2} = \frac{\partial^2 \theta(\zeta,t)}{\partial \zeta^2}$

we take derivative of (8) with respect to space and time to obtain

$$\begin{aligned}
&\frac{\partial \omega(\zeta,t)}{\partial t} = \frac{\partial v(\zeta,t)}{\partial t} + \int_{\zeta}^{f(t)} k(\zeta,\eta)\frac{\partial}{\partial t}v(\eta,t)d\eta + \\
&X(t)\frac{\partial}{\partial t}f(t)\Psi'(f(t)-\zeta) + \Psi(f(t)-\zeta)\frac{\partial}{\partial t}X(t) \\
&\frac{\partial \omega(\zeta,t)}{\partial \zeta} = \frac{\partial v(\zeta,t)}{\partial \zeta} - k(\zeta,\zeta)v(\zeta,t) + \\
&\int_{\zeta}^{f(t)} \frac{\partial k(\zeta,\eta)}{\partial \zeta}v(\eta,t)d\eta - \Psi'(f(t)-\zeta)X(t)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\frac{\partial^2 \omega(\zeta,t)}{\partial \zeta^2} = \frac{\partial^2 v(\zeta,t)}{\partial \zeta^2} - v(\zeta,t)\frac{d}{d\zeta}k(\zeta,\zeta) - \\
&k(\zeta,\zeta)\frac{\partial v(\zeta,\eta)}{\partial \zeta} - \frac{\partial k(\zeta,\eta)}{\partial \zeta}v(\zeta,t) \\
&+\int_{\zeta}^{f(t)} \frac{\partial^2 k(\zeta,\eta)}{\partial \zeta^2}v(\eta,t)d\eta + \Psi''(f(t)-\zeta)X(t)
\end{aligned} \tag{17}$$

where

$$\frac{d}{d\zeta}k(\zeta,\zeta) = \frac{\partial k(\zeta,\zeta)}{\partial \zeta} + \frac{\partial k(\zeta,\zeta)}{\partial \eta} \tag{18}$$

obviously, (13) and (14) are easily obtained.

Exponential stability of the target system (9) –(12) implies exponential stability of the original system (1) – (4) knowing the fact that (8) has a unique inverse transformation as

$$v(\zeta,t) = \omega(\zeta,t) + \int_{\zeta}^{f(t)} l(\zeta,\eta)\omega(\eta,t)d\eta + \Phi(f(t)-\zeta)X(t) \tag{19}$$

By taking derivative of (19) with respect to space and time we obtain

$$\begin{aligned}
&\frac{\partial v(\zeta,t)}{\partial t} = \frac{\partial \omega(\zeta,t)}{\partial t} + \int_{\zeta}^{f(t)} l(\zeta,\eta)\frac{\partial}{\partial t}\omega(\eta,t)d\eta + \\
&X(t)\frac{\partial}{\partial t}f(t)\Phi'(f(t)-\zeta) + \Phi(f(t)-\zeta)\frac{\partial}{\partial t}X(t) \\
&\frac{\partial v(\zeta,t)}{\partial \zeta} = \frac{\partial \omega(\zeta,t)}{\partial \zeta} - l(\zeta,\zeta)\omega(\zeta,t) + \\
&\int_{\zeta}^{f(t)} \frac{\partial l(\zeta,\eta)}{\partial \zeta}\omega(\eta,t)d\eta - \Phi'(f(t)-\zeta)X(t)
\end{aligned} \tag{20}$$

$$\frac{\partial^2 \nu(\zeta,t)}{\partial\zeta^2} = \frac{\partial^2 \omega(\zeta,t)}{\partial\zeta^2} - l(\zeta,\zeta)\frac{\partial\omega(\zeta,\eta)}{\partial\zeta} - \omega(\zeta,t)\frac{d}{d\zeta}l(\zeta,\zeta) - \frac{\partial l(\zeta,\zeta)}{\partial\zeta}\omega(\zeta,t) + \int_{\zeta}^{f(t)} \frac{\partial^2 l(\zeta,\eta)}{\partial\zeta^2}\omega(\eta,t)d\eta + \Psi''(f(t)-\zeta)X(t) \quad (21)$$

After similar calculations the kernel $l(\zeta,\eta)$ and function $\Phi(f(t)-\zeta)$ would take the form

$$l(\zeta,\eta) = -\sqrt{\frac{\gamma}{\alpha}}\sin\left(\sqrt{\frac{\gamma}{\alpha}}(\eta-\zeta)\right) \quad (22)$$

$$\Phi(f(t)-\zeta) = -\sqrt{\frac{\gamma\alpha}{q}}\sin\left(\sqrt{\frac{\gamma}{\alpha}}(f(t)-\zeta)\right) \quad (23)$$

Physically, positive control signal $(u(t)>0)$ leads to $\theta(\zeta,t)>\theta^{ref};\forall\zeta\in(0,f(t))$ and $\frac{\partial f(t)}{\partial t}>0$ while negative control signal may lead to a freezing process.

Now assume

$$f(0)<f(t)<f^{ref};\forall t>0 \quad (24)$$

and pursue the stability analysis.

Theorem. The target system (9) – (12) is exponentially stable in the sense of H_1 - norm.

Proof. Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2}\left(\|\omega(\zeta,t)\|_{H_1}^2 + \lambda X^2(t)\right) \quad (25)$$

where λ is a positive parameter, which is chosen later, and H_1 - norm of $\omega(\zeta,t)$ is defined as

$$\|\omega(\zeta,t)\|_{H_1} = \left(\int_0^{f(t)}\omega^2(\zeta,t)d\zeta + \int_0^{f(t)}\left(\frac{\partial\omega(\zeta,t)}{\partial\zeta}\right)^2 d\zeta\right)^{\frac{1}{2}} \quad (26)$$

Taking the time derivative of (25) and replacing from (9)-(11) to obtain:

$$\dot{V}(t) = \int_0^{f(t)}\omega(\zeta,t)\left[\alpha\frac{\partial^2\omega(\zeta,t)}{\partial\zeta^2} + \frac{\gamma}{q}X(t)\frac{\partial}{\partial t}f(t)\right]d\zeta + \frac{1}{2}\frac{\partial}{\partial t}f(t)\omega^2(f(t),t) + \int_0^{f(t)}\frac{\partial\omega(\zeta,t)}{\partial\zeta}\frac{\partial^2\omega(\zeta,t)}{\partial\zeta\partial t}d\zeta + \lambda X(t)\left[-\gamma X(t) - q\frac{\partial}{\partial\zeta}\omega(f(t),t)\right] \quad (27)$$

Note:

Poincare inequalities takes the following form for any variable $\omega(\zeta,t)$ defined on the time varying spaces $0\le\zeta\le f(t)$ (see Appendix A)

$$\int_0^{f(t)}\omega^2(\zeta,t)d\zeta \le 2f(t)\omega^2(0,t) + 4f^2(t)\int_0^{f(t)}\left(\frac{\partial\omega(\zeta,t)}{\partial\zeta}\right)^2 d\zeta \quad (28)$$

$$\int_0^{f(t)}\omega^2(\zeta,t)d\zeta \le 2f(t)\omega^2(s(t),t) + 4f^2(t)\int_0^{f(t)}\left(\frac{\partial\omega(\zeta,t)}{\partial\zeta}\right)^2 d\zeta \quad (29)$$

Using integration by parts, applying (9) – (12) , imposing poincare inequality and the fact that $f(t)<f^{ref};\forall t>0$, we end up with:

$$\dot{V} \le -\frac{\alpha}{1+4\left(f^{ref}\right)^2}\left(\int_0^{f(t)}\omega^2(\zeta,t)d\zeta + \int_0^{f(t)}\left(\frac{\partial\omega(\zeta,t)}{\partial\zeta}\right)^2 d\zeta\right) - \lambda\left(\gamma - \frac{\lambda q^2 f^{ref}}{\alpha}\right)X^2(t) + \frac{\gamma\alpha}{2q}\frac{\partial f(t)}{\partial t}\left(\int_0^{f(t)}\omega^2(\zeta,t)d\zeta + \left(1+\frac{\gamma\alpha}{q}\right)X^2(t)\right) \quad (30)$$

Applying these, Young's and Cauchy- Schwartz inequalities [10] and choosing,

$$a = \min\left\{\frac{2\alpha}{1+4\left(f^{ref}\right)^2}, 2\left(\gamma - \frac{\lambda q^2 f^{ref}}{\alpha}\right)\right\} \quad (31)$$

$$b = \frac{\alpha\gamma}{2q}\max\left\{1, \frac{1}{\lambda}\left(1+\frac{\alpha\gamma}{q}\right)\right\} \quad (32)$$

$$\lambda < \frac{\alpha\gamma}{q^2 f^{ref}} \quad (33)$$

Therefore

$$\dot{V} \le -aV + b\frac{\partial f(t)}{\partial t}V \quad (34)$$

Since $\frac{\partial f(t)}{\partial t}>0$ then;

$$V(t) \le e^{b(f(t)-f(0))}V(0)e^{-at} \quad (35)$$

finally, using $f(t)<f^{ref}$ we arrive at

Table 1 . Physical Properties of Zinc

Description/symbol	Value
ρ /Density	$6570 kg/m^3$
ΔH^* /Latent heat of fusion	$111{,}961 J/kg$
C_p /Heat Capacity	$389.5678 J/(kg.K$
k /Thermal conductivity	$111 w/m$

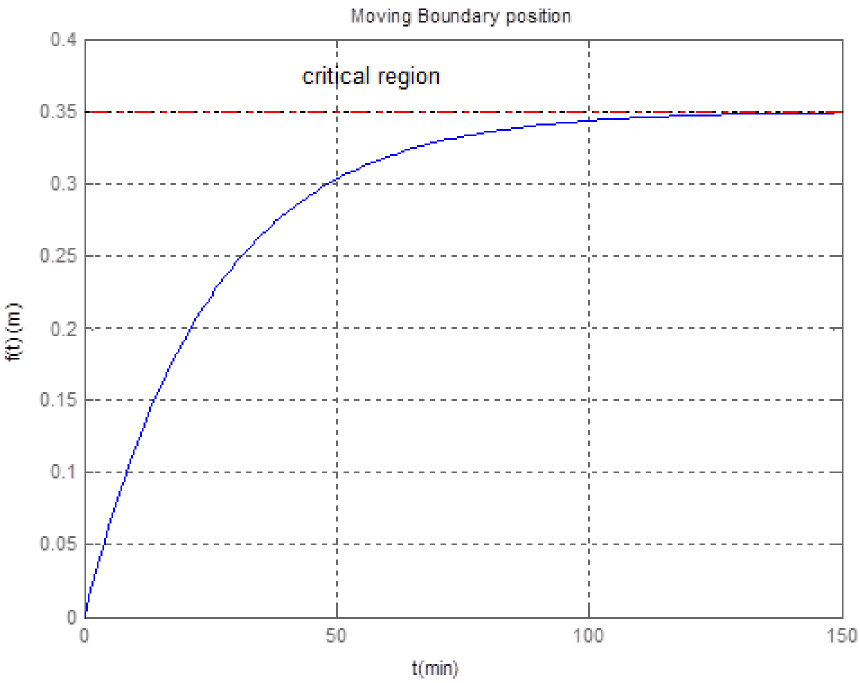

Fig. 3. Time Evolution of the moving boundary position

$$V(t) \le e^{b\left(f^{ref} - f(0)\right)} V(0) e^{-at} \tag{36}$$

2- SIMULATION RESULTS

To verify the performance of the controller (7) we use numerical value of zinc beam [14]. The definition of α, q in (1) and (4) which related to heat transfer process are $\alpha = \frac{k}{\rho C_p}$ and $q = \frac{k}{\rho \Delta H^*}$ respectively. The physical properties of zinc are given in Table 1.

The steady state of melted part is $f^{ref} = 35cm$. The control gain in (7) is $\gamma = 0.01$.

The dynamic of the moving boundary f(t) is depicted in Fig. 3. The closed loop stability is evidence.

Time evolution of the positive control signal is depicted in Fig. 4.

.The simulation of coupled system (1)- (4) shows that the interface converges to its setpoint while keeping $\frac{\partial f(t)}{\partial t} \ge 0; \forall t \ge 0$ and $f(t) < f^{ref}$ with a positive control signal $u(t) > 0$ as we expected from theoretical result.

3- CONCLUSION

In this paper, we studied an extension of backstepping controller of parabolic PDEs (Heat Transfer Process) with time-dependent spatial domain. The PDE system dynamics was transformed to an exponentially stable target system by a nonlinear backstepping transformation. Exponential stability of the closed-loop system was discussed via Lyapunov's method.

4- APPENDIX A.

Proof of Poincare inequalities for the time varying spaces

The proof to the conservative form of Poincar´e inequality (28for the time-varying space $0 \le \zeta \le f(t)$ is given here, and (29) can be shown similarly. We start with the following relation:

$$\omega^2(\zeta,t) = -\frac{\partial}{\partial\zeta}\left[\left(f(t)-\zeta\right)\omega^2(\zeta,t)\right] + 2\left(f(t)-\zeta\right)\omega(\zeta,t)\frac{\partial\omega(\zeta,t)}{\partial\zeta} \tag{A.1}$$

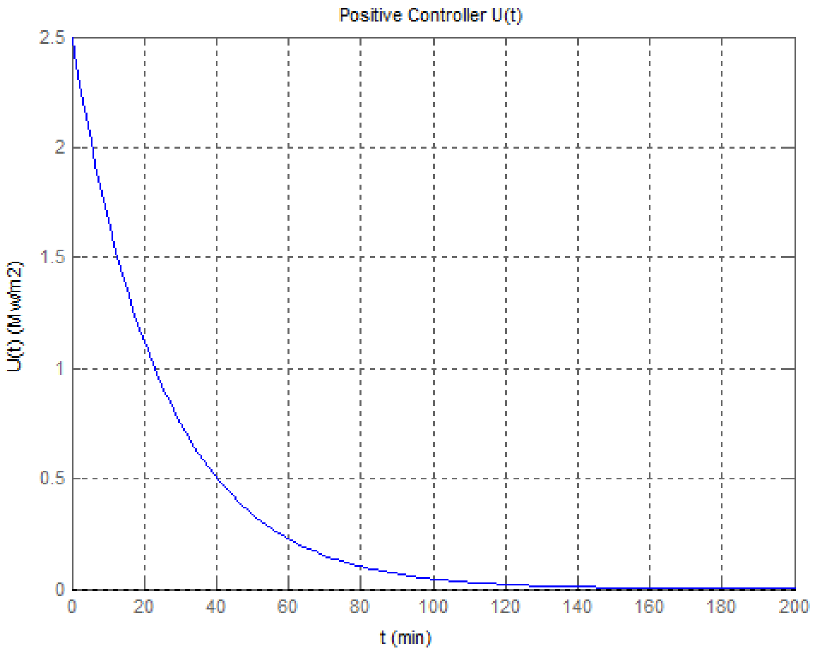

Fig. 4. Positive controller

By integrating both sides of this equation with respect to ξ from 0 to *f(t)* one obtains:

$$\int_0^{f(t)} \omega^2(\zeta,t)d\zeta = f(t)\omega^2(0,t) + 2\int_0^{f(t)} \left(f(t)-\zeta\right)\omega(\zeta,t)\frac{\partial\omega(\zeta,t)}{\partial\zeta}d\zeta \tag{A.2}$$

By using Cauchy-Schwartz and Young's inequalities, respectively, One can readily shown that

$$\begin{aligned}\int_0^{f(t)} \omega^2(\zeta,t)d\zeta &\le f(t)\omega^2(0,t) + \left(\int_0^{f(t)} \omega^2(\zeta,t)d\zeta\right)^{\frac{1}{2}}\left(\int_0^{f(t)} 4(f(t)-\zeta)^2\left(\frac{\partial\omega(\zeta,t)}{\partial\zeta}\right)^2 d\zeta\right)^{\frac{1}{2}} \\ &\le f(t)\omega^2(0,t) + \frac{1}{2}\int_0^{f(t)} \omega^2(\zeta,t)d\zeta + 2\int_0^{f(t)} (f(t)-\zeta)^2\left(\frac{\partial\omega(\zeta,t)}{\partial\zeta}\right)^2 d\zeta\end{aligned} \tag{A.3}$$

The last integral in (A.3) is majorized by

$$\sup_{0\le\zeta\le f(t)} \left(f(t)-\zeta\right)\int_0^{f(t)}\left(\frac{\partial\omega(\zeta,t)}{\partial\zeta}\right)^2 d\zeta = f^2(t)\int_0^{f(t)}\left(\frac{\partial\omega(\zeta,t)}{\partial\zeta}\right)^2 d\zeta \tag{A.4}$$

resulting in the inequality (28).

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