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Some fundamental problems in global Finsler geometry

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ABSTRACT: The geometry and analysis on Finsler manifolds is a very important part of Finsler geometry. In this survey article, we introduce some important and fundamental topics in global Finsler geometry and discuss the related properties and the relationships in them. In particular, we optimize and improve the various definitions of Lie derivatives on Finsler manifolds. Further, we also obtain an estimate of lower bound for the non-zero eigenvalues of the Finsler Laplacian under the condition that $\mathrm{Ric}_N \geq K > 0$.

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1. Preliminaries

Let M be a connected manifold of dimension n and $\pi : TM_0 \to M$ be the natural projective map, where $TM_0 := TM \setminus \{0\}$. π pulls back TM to a vector bundle π^*TM over TM_0. The fiber at a point $(x, y) \in TM_0$ is defined by

$$\pi^*TM|_{(x,y)} := \{(x, y, v) \mid v \in T_xM\} \cong T_xM.$$

In other words, $\pi^*TM|_{(x,y)}$ is just a copy of T_xM. Similarly, we define the pull-back cotangent bundle π^*T^*M whose fiber at (x, y) is a copy of T^*_xM. That is,

$$\pi^*T^*M|_{(x,y)} := \{(x, y, \theta) \mid \theta \in T^*_xM\} \cong T^*_xM.$$

π^*T^*M can be viewed as the dual vector bundle of π^*TM by setting

$$(x, y, \theta)(x, y, v) := \theta(v), \quad \theta \in T^*_xM,\ v \in T_xM.$$

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Take a standard local coordinate system (x^i, y^i) in TM. Let $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ and $\{dx^i, dy^i\}$ be the natural local frame and coframe for $T(TM_0)$ and $T^*(TM_0)$ respectively. Let

$$\partial_i := \left(x, y, \frac{\partial}{\partial x^i}|_x\right).$$

Then $\{\partial_i\}$ is a local frame for π^*TM. Dually, put

$$dx^i := \left(x, y, dx^i|_x\right).$$

Then $\{dx^i\}$ is a local coframe for π^*T^*M.

The vertical tangent bundle of M is defined by $VTM := span\{\frac{\partial}{\partial y^i}\}$. VTM is a well-defined subbundle of $T(TM_0)$ and we can obtain a decomposition $T(TM_0) = \pi^*TM \oplus VTM$.

For a Finsler manifold (M, F), let

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are defined by

$$G^i = \frac{1}{4}\left\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\right\}.$$

We call G the spray induced by F and G^i the spray coefficients of F. Define $N^i_j := \frac{\partial G^i}{\partial y^j}$ and let

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}.$$

Then $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$ form a local frame for $T(TM)$. Further, $HTM := span\left\{\frac{\delta}{\delta x^i}\right\}$ is a well-defined subbundle of $T(TM_0)$ and is called the horizontal tangent bundle of M. Then we obtain a decomposition for $T(TM_0)$, $T(TM_0) = HTM \oplus VTM$.

The following maps are natural and are important for our discussions below.

(1) Define a vector bundle map $\rho : T(TM_0) \to \pi^*TM$ by

$$\rho\left(\frac{\partial}{\partial x^i}|_{(x,y)}\right) = \partial_i, \quad \rho\left(\frac{\partial}{\partial y^i}|_{(x,y)}\right) = 0.$$

It is clear that $\ker \rho = VTM$.

(2) Define a linear map $\mathcal{H} : \pi^*TM \to HTM$ with the following properties

$$\mathcal{H}(\partial_i) := \frac{\delta}{\delta x^i}.$$

Obviously, $\mathcal{H}$ is an isomorphism.

2. Gradient vector fields and Laplacian on Finsler manifolds

Let M be an n-dimensional manifold. A Finsler metric F on M is a non-negative function on TM such that F is C^∞ on $TM\backslash\{0\}$ and the restriction $F_x := F|_{T_xM}$ is a Minkowski function on T_xM for all $x \in M$. For Finsler metric F on M, there is a Finsler co-metric F^* on M which is a non-negative function on the cotangent bundle T^*M given by

$$F^*(x, \xi) := \sup_{y \in T_xM\backslash\{0\}} \frac{\xi(y)}{F(x, y)}, \quad \forall \xi \in T^*_xM.$$

We call F^* the dual Finsler metric of F. Finsler metric F and its dual Finsler metric F^* satisfy the following relation.

Lemma 2.1. (Lemma 3.1.1, [13]) *Let F be a Finsler metric on M and F^* its dual Finsler metric. For any vector $y \in T_xM \setminus \{0\}$, $x \in M$, the covector $\xi = g_y(y, \cdot) \in T^*_xM$ satisfies*

$$F(x, y) = F^*(x, \xi) = \frac{\xi(y)}{F(x, y)}. \tag{2.1}$$

Conversely, for any covector $\xi \in T^*_xM\backslash\{0\}$, there exists a unique vector $y \in T_xM\backslash\{0\}$ such that $\xi = g_y(y, \cdot) \in T^*_xM$.

Naturally, by Lemma 2.1, we define a map $\mathcal{L} : TM \to T^*M$ by

$$\mathcal{L}(y) := \begin{cases} g_y(y, \cdot), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

It follows from (2.1) that

$$F(x, y) = F^*(x, \mathcal{L}(y)).$$

Thus $\mathcal{L}$ is a norm-preserving transformation. We call $\mathcal{L}$ the Legendre transformation on Finsler manifold (M, F).

Take a basis $\{\mathbf{b}_i\}_{i=1}^n$ for TM and its dual basis $\{\theta^i\}_{i=1}^n$ for T^*M. Express $\xi = \mathcal{L}(y) = \xi_i \theta^i$, we have

$$\xi_i = g_{ij}(x, y) y^j,$$

where $g_{ij}(x, y) := \frac{1}{2} \left[F^2\right]_{y^i y^j}(x, y)$. The Jacobian of $\mathcal{L}$ is given by

$$\frac{\partial \xi_i}{\partial y^j} = g_{ij}(x, y).$$

Thus $\mathcal{L}$ is a diffeomorphism from $TM \setminus \{0\}$ onto $T^*M \setminus \{0\}$. Let

$$g^{*kl}(x, \xi) := \frac{1}{2} \left[F^{*2}\right]_{\xi_k \xi_l}(x, \xi).$$

For any $\xi = \mathcal{L}(y)$, differentiating $F^2(x, y) = F^{*2}(x, \mathcal{L}(y))$ with respect to y^i yields

$$\frac{1}{2} \left[F^2\right]_{y^i}(x, y) = \frac{1}{2} \left[F^{*2}\right]_{\xi_k}(x, \xi) g_{ik}(x, y), \tag{2.2}$$

which implies

$$g^{*kl}(x, \xi)\xi_l = \frac{1}{2} \left[F^{*2}\right]_{\xi_k}(x, \xi) = \frac{1}{2} g^{ik}(x, y) \left[F^2\right]_{y^i}(x, y) = y^k. \tag{2.3}$$

Then, it is clear that

$$g^{*kl}\xi_l \frac{\partial g_{ik}}{\partial y^j} = y^k \frac{\partial g_{ik}}{\partial y^j} = 0.$$

Differentiating (2.2) with respect to y^j gives

$$\begin{aligned} g_{ij}(x, y) &= g^{*kl}(x, \xi) g_{ik}(x, y) g_{jl}(x, y) + g^{*kl}(x, \xi)\xi_l \frac{\partial g_{ik}}{\partial y^j}(x, y) \\ &= g^{*kl}(x, \xi) g_{ik}(x, y) g_{jl}(x, y). \end{aligned}$$

Therefore, we get

$$g^{*kl}(x, \xi) = g^{kl}(x, y). \tag{2.4}$$

Given a smooth function f on M, the differential df_x at any point $x \in M$,

$$df_x = \frac{\partial f}{\partial x^i}(x) dx^i$$

is a linear function on T_xM. We define the gradient vector $\nabla f(x)$ of f at $x \in M$ by $\nabla f(x) := \mathcal{L}^{-1}(df(x)) \in T_xM$. In a local coordinate system, by (2.3) and (2.4), we can express ∇f as

$$\nabla f(x) = \begin{cases} g^{ij}(x, \nabla f) \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, & x \in M_f, \\ 0, & x \in M \setminus M_f, \end{cases} \tag{2.5}$$

where $M_f = \{x \in M | df(x) \neq 0\}$. Further, by Lemma 2.1, we have the following

$$df_x(v) = g_{\nabla f_x}(\nabla f_x, v), \quad \forall v \in T_xM$$

and

$$F(x, \nabla f_x) = F^*(x, df_x) = \frac{df_x(\nabla f_x)}{F(x, \nabla f_x)}.$$

By definition, a smooth measure m on M is a measure locally given by a smooth n-form

$$dm = \sigma(x)dx^1 \cdots dx^n.$$

The restriction m_x of m to T_xM is a Haar measure on T_xM. For every Finsler manifold (M, F), there are several associated measures, including Busemann-Hausdorff measure m_{BH} and Holmes-Thompson measure m_{HT}. A Finsler manifold (M, F) equipped with a smooth measure m is called a Finsler measure space and denoted by (M, F, dm).

Let us consider an oriented manifold M equipped with a measure m. We can view dm as an n-form (volume form) on M. Let X be a vector field on M. Define an $(n-1)$-form $X\rfloor dm$ on M by

$$X\rfloor dm(X_2, \cdots, X_n) := dm(X, X_2, \cdots, X_n).$$

Define

$$d(X\rfloor dm) = div(X)dm.$$

We call $div(X)$ the divergence of X. Clearly, $div(X)$ depends only on the volume form dm. In a local coordinate system (x^i), express $dm = \sigma(x)dx^1 \cdots dx^n$. Then for a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M,

$$div(X) = \frac{1}{\sigma}\frac{\partial}{\partial x^i}\left(\sigma X^i\right) = \frac{\partial X^i}{\partial x^i} + \frac{X^i}{\sigma}\frac{\partial \sigma}{\partial x^i}.$$

Applying the Stokes theorem to $\eta = X\rfloor dm$, we obtain

$$\int_M div(X)dm = \int_M d(X\rfloor dm) = 0, \quad if\ \partial M = \emptyset, \tag{2.6}$$

$$\int_M div(X)dm = \int_M d(X\rfloor dm) = \int_{\partial M} X\rfloor dm, \quad if\ \partial M \neq \emptyset.$$

One can also define $divX$ in the weak form by following divergence formula:

$$\int_M \phi\ divX\ dm = -\int_M d\phi(X)\ dm$$

for all $\phi \in \mathcal{C}_c^\infty(M)$, where $\mathcal{C}_c^\infty(M)$ denotes the set of $\mathcal{C}^\infty$-functions on M with compact support.

Now we introduce the Laplacian on a Finsler measure space (M, F, dm). There are some different definitions on Laplacian in Finsler geometry (e.g. see [1][5][15]). The following definition is from [4].

Given a smooth measure $dm = \sigma(x)dx^1 \cdots dx^n$ and C^k $(k \geq 2)$ function f on M, define the Finsler Laplacian $\triangle f$ of f by

$$\triangle f := div(\nabla f).$$

By (2.5), the Laplacian of f is expressed by

$$\triangle f = \frac{1}{\sigma}\frac{\partial}{\partial x^i}\left(\sigma \nabla^i f\right) = \frac{1}{\sigma}\frac{\partial}{\partial x^i}\left(\sigma g^{ij}(x, \nabla f)\frac{\partial f}{\partial x^j}\right), \tag{2.7}$$

where $\nabla^i f := g^{ij}(x, \nabla f)\frac{\partial f}{\partial x^j} = g^{*ij}(x, df)\frac{\partial f}{\partial x^j}$. From (2.7), Finsler Laplacian is a nonlinear elliptic differential operator of the second order.

Remark 2.2. The following are some remarks on Finsler Laplacian.

(i) As we know, for any smooth function φ on M,

$$div(\varphi \nabla f) = \varphi \triangle f + d\varphi(\nabla f).$$

If $\partial M = \emptyset$, applying the divergence formula (2.6) to the above identity yields

$$\int_M \varphi \triangle f dm = -\int_M d\varphi(\nabla f)dm. \tag{2.8}$$

Actually, (2.8) gives the definition of non-linear Laplacian $\triangle f$ on the whole M in the distribution sense.

(ii) Let $dm = e^\rho dm_F$ and $\triangle_F$ denote the Laplacian associated with the Finsler measure dm_F. Then

$$\triangle f = \triangle_F f + d\rho(\nabla f).$$

Note that $\triangle f$ and $\triangle_F f$ are the divergences of the gradient ∇f with respect to m and m_F respectively.

(iii) The Finsler p-Laplacian $\triangle_p f$ of f is formally defined by

$$\triangle_p f := div\left[F^{p-2}(x,\nabla f)\nabla f\right].$$

In the distribution sense, the definition of Finsler p-Laplacian $\triangle_p f$ is given by the following identity

$$\int_M \varphi \triangle_p f d\mu = -\int_M F^{p-2}(x,\nabla f) d\varphi(\nabla f) d\mu, \quad \forall \varphi \in C_0^\infty(M).$$

When $p=2$, $\triangle_p$ is exactly the usual Finsler Laplacian.

3. Energy functionals and eigenvalues

The variational problem of the canonical energy functional also gives rise to the Laplacian. Let H^1 denote the Hilbert space of all L^2 functions f such that $df \in L^2$. Denote by H_0^1 the space of functions $u \in H^1$ with $\int_M udm = 0$ if $\partial M = \emptyset$ and with $u_{|\partial M} = 0$ if $\partial M \neq \emptyset$. The canonical energy functional $\mathcal{E}$ on H_0^1 is defined by

$$\mathcal{E}(u) := \frac{\int_M \left[F^*(x,du)\right]^2 dm}{\int_M u^2 dm}.$$

For functions $u, \varphi \in H_0^1$, by (2.3), we have

$$\frac{d}{d\varepsilon}\left[F^{*2}(x, du+\varepsilon d\varphi)\right]|_{\varepsilon=0} = \frac{\partial[F^{*2}]}{\partial \xi_i}(x,du)\frac{\partial \varphi}{\partial x^i} = 2\nabla^i u(x,du)\frac{\partial \varphi}{\partial x^i} = 2d\varphi(\nabla u).$$

Thus, for any $u \in H_0^1$ with $\int_M u^2 dm = 1$,

$$d_u\mathcal{E}(\varphi) = \frac{d}{d\epsilon}\left[\mathcal{E}(u+\epsilon\varphi)\right]|_{\epsilon=0} = 2\int_M d\varphi(\nabla u)dm - 2\lambda\int_M u\varphi dm, \quad \forall \varphi \in H_0^1, \tag{3.1}$$

where $\lambda = \mathcal{E}(u)$. From (2.8), we can rewrite (3.1) as follows

$$\frac{1}{2}d_u\mathcal{E}(\varphi) = -\int \left[\triangle u + \lambda u\right]\varphi \, dm, \quad \forall \varphi \in H_0^1.$$

Hence, it follows that a function $u \in H_0^1$ satisfies $d_u\mathcal{E} = 0$ with $\lambda = \mathcal{E}(u)$ if and only if

$$\triangle u + \lambda u = 0.$$

In this case, λ and u are called an eigenvalue and an eigenfunction of (M,F,dm), respectively. Thus an eigenfunction u corresponding to an eigenvalue λ satisfies the following equation

$$\frac{1}{\sigma(x)}\frac{\partial}{\partial x^i}\left(\sigma(x)\nabla^i u(x)\right) + \lambda u = 0,$$

where $\nabla^i u(x) = g^{ij}(x,\nabla u)\frac{\partial u}{\partial x^j} = g^{*ij}(x,du)\frac{\partial u}{\partial x^j}$.

Denote by $\mathcal{E}_\lambda$ the union of the zero function and the set of all eigenfunctions corresponding to λ. We call $\mathcal{E}_\lambda$ the eigencone corresponding to λ.

Assume that M is compact without boundary. Let

$$\lambda_1(M) := \inf_{u\in C^\infty(M)} \frac{\int_M \left[F^*(x,du)\right]^2 dm}{\inf_{\lambda\in R}\int_M |u-\lambda|^2 dm}.$$

From [4] and [13], we can find the proof on the fact that $\lambda_1(M)$ is the minimum of the energy functional $\mathcal{E}$. In the following, we give a different proof for this fact.

Proposition 3.1. *$\lambda_1 := \lambda_1(M)$ is the smallest eigenvalue of (M,F,dm), that is, $\lambda_1 = \inf_{u\in H_0^1}\mathcal{E}(u)$.*

Proof. Write

$$\begin{aligned}\int_M |u-\lambda|^2 dm &= \int_M u^2 dm - 2\lambda\int_M udm + \lambda^2\int_M dm \\ &:= a - 2\lambda b + \lambda^2 c,\end{aligned}$$

where $a := \int_M u^2 dm,\ b := \int_M u dm, c := \int_M dm = m(M)$. Let $f(\lambda) := a - 2\lambda b + \lambda^2 c$. By $f'(\lambda) = -2b + 2c\lambda$, we have the following

$$\begin{aligned}
&\inf_{\lambda\in R}\int_M |u-\lambda|^2 dm = \left(\int_M |u-\lambda|^2 dm\right)\Big|_{\lambda=\frac{b}{c}}\\
&= a - 2m(M)^{-1}b^2 + m(M)^{-1}b^2\\
&= \int_M u^2 dm - m(M)^{-1}\left(\int_M u dm\right)^2 \leq \int_M u^2 dm.
\end{aligned}$$

Thus

$$\lambda_1(M) = \inf_{u\in H_0^1(M)} \frac{\int_M [F^*(x,du)]^2 dm}{\int_M u^2 dm} = \inf_{u\in H_0^1(M)} \mathcal{E}(u).$$

□

We call λ_1 the first eigenvalue of (M, F, dm). They are natural problems to determine the lower bound of the first (nonzero) eigenvalue of Laplacian on Finsler manifolds and to study the structure of the first eigencone for a general Finsler metric ([14][16][17]).

4. Hessian

Let (M, F) be a Finsler manifold of dimension n and $\pi : TM \setminus \{0\} \to M$ be the projective map. The pull-back π^*TM admits a unique linear connection, which is called the Chern connection. The Chern connection D is determined by the following equations

$$\begin{aligned}
&D_X^V Y - D_Y^V X = [X, Y],\\
&Z g_V(X, Y) = g_V(D_Z^V X, Y) + g_V(X, D_Z^V Y) + 2C_V(D_Z^V V, X, Y)
\end{aligned}$$

for $V \in TM \setminus \{0\}$ and $X, Y, Z \in TM$, where

$$C_V(X, Y, Z) := C_{ijk}(x, V)X^iY^jZ^k = \frac{1}{4}\frac{\partial^3 F^2(x, V)}{\partial V^i \partial V^j \partial V^k} X^iY^jZ^k$$

is the Cartan tensor of F and $D_X^V Y$ is the covariant derivative with respect to the reference vector V.

Let (M, F) be a Finsler manifold. There are two ways to define the Hessian of a C^2 function on M. Let f be a C^2 function on M. Firstly, the Hessian of f can be defined as a map $D^2 f : TM \to R$ by

$$D^2 f(y) := \frac{d^2}{ds^2}(f \circ c)|_{s=0}, \quad y \in T_xM, \tag{4.1}$$

where $c : (-\varepsilon, \varepsilon) \to M$ is the geodesic with $c(0) = x,\ \dot{c}(0) = y \in T_xM$ (see [13]). In local coordinates,

$$\begin{aligned}
D^2 f(y) &= \frac{\partial^2 f}{\partial x^i \partial x^j}(x)\dot{c}^i(0)\dot{c}^j(0) + \frac{\partial f}{\partial x^i}(x)\ddot{c}^i(0)\\
&= \frac{\partial^2 f}{\partial x^i \partial x^j}(x)y^iy^j - 2\frac{\partial f}{\partial x^i}(x)G^i(x, y)\\
&= \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(x) - \frac{\partial f}{\partial x^m}\Gamma_{ij}^m(x, y)\right)y^iy^j.
\end{aligned}$$

Here, $\Gamma_{ij}^k(x, y)$ denote the Chern connection coefficients of F, which depends on the tangent vector $y \in T_xM$.

There is another definition of the Hessian in Finsler geometry, by which the Hessian of a C^2 function u on M is corresponding to a symmetric matrix $\left(u_{|i|j}(x, \nabla u)\right)$, where "$|$" denotes the horizontal covariant derivative with respect to the Chern connection of the metric. Concretely, the Hessian $\nabla^2 u$ of u is defined by

$$\nabla^2 u(X, Y) := g_{\nabla u}\left(D_X^{\nabla u}\nabla u, Y\right) \tag{4.2}$$

for any $X, Y \in TM$ ([11][16][20]). In a local coordinate system, let $X = X^i \frac{\partial}{\partial x^i},\ Y = Y^j \frac{\partial}{\partial x^j}$. By the definition,

$$D_X^{\nabla u}\nabla u = \left\{\frac{\partial(\nabla^i u)}{\partial x^j}X^j + (\nabla^k u)\Gamma_{jk}^i(x, \nabla u)X^j\right\}\frac{\partial}{\partial x^i}.$$

Thus, we have

$$\begin{aligned} g_{\nabla u}\left(D_X^{\nabla u}\nabla u, Y\right) &= \left(\frac{\partial(\nabla^i u)}{\partial x^j} + (\nabla^k u)\Gamma^i_{jk}(x,\nabla u)\right) X^j Y^l g_{il}(x,\nabla u) \\ &= (\nabla^i u)_{|j}(x,\nabla u) g_{il}(x,\nabla u) X^j Y^l. \end{aligned} \tag{4.3}$$

Here, we have used the facts that $\nabla^i u = g^{ij}(x,\nabla u)\frac{\partial u}{\partial x^j}$ and

$$\begin{aligned} \frac{\partial(\nabla^i u)}{\partial y^m}(x,\nabla u) &= -2C_{klm}(x,\nabla u) g^{ik}(x,\nabla u) g^{jl}(x,\nabla u)\frac{\partial u}{\partial x^j} \\ &= -2C_{klm}(x,\nabla u)(\nabla^l u) g^{ik}(x,\nabla u) = 0. \end{aligned}$$

Further, let $(\nabla u)_i(x,\nabla u) := g_{ij}(x,\nabla u)(\nabla^j u)$. Then

$$(\nabla u)_i(x,\nabla u) = g_{ij}(x,\nabla u) g^{jk}(x,\nabla u)\frac{\partial u}{\partial x^k} = \frac{\partial u}{\partial x^i} = u_{|i}(x).$$

Thus it follows (4.3) that

$$g_{\nabla u}\left(D_X^{\nabla u}\nabla u, Y\right) = u_{|i|j}(x,\nabla u) X^i Y^j.$$

Hence, by (4.2), we have the following proposition.

Proposition 4.1. Let u be a C^2 function on Finsler manifold (M,F). Then, for any $X, Y \in TM$, we have

$$\nabla^2 u(X,Y) = u_{|i|j}(x,\nabla u) X^i Y^j. \tag{4.4}$$

It follows from (4.4) that the Hessian $\nabla^2 u$ of a C^2 function u is determined completely by the following symmetric matrix

$$\mathrm{Hess}(u) := \left(u_{|i|j}(x,\nabla u)\right). \tag{4.5}$$

Remark 4.2. *When F is a Riemann metric, for any C^2 function f on M, the Hessians of f defined by (4.1) and (4.2) respectively are identical.*

Theorem 4.3. ([13][20]) *On $M_f = \{x \in M \mid \nabla f|_x \neq 0\}$, we have*

$$\Delta f = \sum_a \nabla^2 f(e_a, e_a) - \mathbf{S}(\nabla f) := \mathrm{tr}_{\nabla f}\nabla^2 f - \mathbf{S}(\nabla f),$$

where $e_1, \ldots, e_n$ is a local $g_{\nabla f}$-orthonormal frame on M_f and $\mathbf{S}$ denotes the S-curvature.

5. Lie derivatives on Finsler manifolds

Lie derivatives have close connections with Laplacians and Hessians of smooth functions on the manifolds and they are also important tools for studies on Ricci soliton and Ricci flow on Finsler manifolds. However, up to now, there are not yet exact definitions for various Lie derivatives on Finsler manifolds. Further, some wrong computations about Lie derivatives on Finsler manifolds can be found in some literatures. These cases motivate us to optimize and improve the various definitions of Lie derivatives on Finsler manifolds.

Let $V = V^i\frac{\partial}{\partial x^i}$ be a vector field on M and $\{\varphi_t\}$ the local 1-parameter transformation group of M generated by V, $V(x) = \frac{d\varphi_t(x)}{dt}|_{t=0}$. The Lie derivative of a tensor in the direction of V is defined as the first-order term in a suitable Taylor expansion of the tensor when it is moved by the flow of V. The precise formula, however, depends on what type of tensor we use ([12]).

In the following, we mainly consider the Lie derivative on a Finsler manifold (M,F) of dimension n. For each φ_t, it is naturally extended to a transformation $\tilde{\varphi}_t: TM \to TM$ defined by

$$\tilde{\varphi}_t(x,y) := (\varphi_t(x), (\varphi_t)_*(y)) = \left(\varphi_t(x), y^i\frac{\partial\varphi_t(x)}{\partial x^i}\right).$$

It is easy to check that $\{\tilde{\varphi}_t\}$ is a local 1-parameter transformation group of TM. Further,

$$\begin{aligned} \left.\frac{d\tilde{\varphi}_t(x,y)}{dt}\right|_{t=0} &= \left.\left(\frac{d\varphi_t(x)}{dt}, y^m\frac{\partial^2\varphi_t}{\partial t\partial x^m}\right)\right|_{t=0} = \left.\left(V(x), y^m\frac{\partial}{\partial x^m}\left(\frac{d\varphi_t(x)}{dt}\right)\right)\right|_{t=0} \\ &= \left(V(x), y^m\frac{\partial V(x)}{\partial x^m}\right). \end{aligned}$$

Then $\hat{V} := V^i(x)\frac{\partial}{\partial x^i} + y^m\left(\frac{\partial V^i}{\partial x^m}\right)\frac{\partial}{\partial y^i}$ is the vector field on TM induced by $\{\tilde{\varphi}_t\}$. We call $\hat{V}$ the complete lift of V.

If $f : TM \to \mathrm{R}$ is a function, then $f\left(\tilde{\varphi}_t(x,y)\right) = f(x,y) + t\left(\mathcal{L}_{\hat{V}}f\right)(x,y) + o(t)$ or

$$\left(\mathcal{L}_{\hat{V}}f\right)(x,y) = \lim_{t\to 0}\frac{f\left(\tilde{\varphi}_t(x,y)\right) - f(x,y)}{t}.$$

Thus the Lie derivative $\mathcal{L}_{\hat{V}}f$ is simply the directional derivative $df(\hat{V})$, that is,

$$\mathcal{L}_{\hat{V}}f = \hat{V}f.$$

When we have a vetor field $Y \in T(TM)$ things get a litle more complicated as $Y|_{\tilde{\varphi}_t(x,y)}$ can't be compared directly to $Y|_{(x,y)}$ since the vectors live in different tangent spaces. Thus we consider the curve $t \mapsto \tilde{\varphi}_t^*\left(Y|_{\tilde{\varphi}_t(x,y)}\right)$ that lies in $T_y(TM)$, here $\tilde{\varphi}_t^* := (\tilde{\varphi}_t)_*^{-1} = (\tilde{\varphi}_{-t})_*$. In other words we define

$$\left(\mathcal{L}_{\hat{V}}Y\right)|_{(x,y)} = \lim_{t\to 0}\frac{\tilde{\varphi}_t^*\left(Y|_{\tilde{\varphi}_t(x,y)}\right) - Y|_{(x,y)}}{t}.$$

This Lie derivative turns out to be the Lie bracket ([12]),

$$\mathcal{L}_{\hat{V}}Y = [\hat{V}, Y]. \tag{5.1}$$

If $\xi \in \mathcal{T}_k^0(TM)$ is a tensor of $(0,k)-$type over TM, its Lie derivative $\mathcal{L}_{\hat{V}}\xi$ with respect to $\hat{V}$ is the tensor of the same type given by

$$(\mathcal{L}_{\hat{V}}\xi)(Y_1,\cdots,Y_k) := \hat{V}\left(\xi(Y_1,\cdots,Y_k)\right) - \sum_{i=1}^{k}\xi(Y_1,\cdots,\mathcal{L}_{\hat{V}}Y_i,\cdots,Y_k),$$

that is,

$$(\mathcal{L}_{\hat{V}}\xi)(Y_1,\cdots,Y_k) = \mathcal{L}_{\hat{V}}\left(\xi(Y_1,\cdots,Y_k)\right) - \sum_{i=1}^{k}\xi(Y_1,\cdots,\mathcal{L}_{\hat{V}}Y_i,\cdots,Y_k),$$

where $Y_i \in T(TM_0),\ 1 \le i \le k$.

Let $\eta \in \mathcal{T}_k^1(TM)$ be a tensor of $(1,k)-$type over TM. The Lie derivative $\mathcal{L}_{\hat{V}}\eta$ of η with respect to $\hat{V}$ is defined by ([9])

$$(\mathcal{L}_{\hat{V}}\eta)(Y_1,\cdots,Y_k) := \left[\hat{V}, \eta(Y_1,\cdots,Y_k)\right] - \sum_{i=1}^{k}\eta(Y_1,\cdots,\mathcal{L}_{\hat{V}}Y_i,\cdots,Y_k) \tag{5.2}$$

for $Y_i \in T(TM_0),\ 1 \le i \le k$. Clearly, $\mathcal{L}_{\hat{V}}\eta$ is still a tensor of $(1,k)-$type over TM. Obviously, (5.2) can be also rewritten as

$$(\mathcal{L}_{\hat{V}}\eta)(Y_1,\cdots,Y_k) = \mathcal{L}_{\hat{V}}\left(\eta(Y_1,\cdots,Y_k)\right) - \sum_{i=1}^{k}\eta(Y_1,\cdots,\mathcal{L}_{\hat{V}}Y_i,\cdots,Y_k).$$

As natural applications of the definitions above, when the tensors that we discuss are restricted to the vector bundle π^*TM and its dual π^*T^*M on TM, we firstly give the following convention: if $X \in \pi^*TM_0$, the Lie derivative of X with respect to $\hat{V}$ is given by

$$\mathcal{L}_{\hat{V}}X = \rho\left[\hat{V}, X\right] \in \pi^*TM_0. \tag{5.3}$$

Here, our convention is different from that in [9].

Based on convention (5.3), firstly, we give Lie derivatives of some fundamental tensors on π^*TM and its dual π^*T^*M. For $y = y^i\frac{\partial}{\partial x^i} \in T_xM$, let $\mathcal{Y} = (x,y,y) = y^i\partial_i|_{(x,y)} \in \pi^*TM_0$ and $\xi = \mathcal{L}(\mathcal{Y}) = y_i dx^i|_{(x,y)} \in \pi^*T^*M_0$, where $y_i := g_{ij}(x,y)y^j$. Further, let

$$\mathcal{L}_{\hat{V}}\mathcal{Y} := (\mathcal{L}_{\hat{V}}y^i)\partial_i, \quad \mathcal{L}_{\hat{V}}\xi := (\mathcal{L}_{\hat{V}}y_k)dx^k$$

and

$$\mathcal{L}_{\hat{V}}g := (\mathcal{L}_{\hat{V}}g_{ij})dx^i \otimes dx^j.$$

Here, $g = g_{ij}(x,y)dx^i \otimes dx^j$ is the the inner product on π^*TM_0. Then, it is easy to show that

$$\mathcal{L}_{\hat{V}}y^i = 0, \tag{5.4}$$
$$\mathcal{L}_{\hat{V}}y_k = y^m(V_{k|m} + V_{m|k}), \tag{5.5}$$
$$\mathcal{L}_{\hat{V}}g_{ij}(x,y) = V_{j|i}(x,y) + V_{i|j}(x,y) + 2y^m V^l_{|m}(x,y)C_{lij}(x,y). \tag{5.6}$$

where $V_i := g_{ij}(x,y)V^j$. Here we have used the fact that $\frac{\partial V^m}{\partial x^k}g_{jm} = V_{j|k} - g_{jl}\Gamma^l_{ik}V^i$. Further, the Lie derivative of the Finsler metric F is given by

$$\mathcal{L}_{\hat{V}}F = \hat{V}F = F^{-1}V_{0|0}.$$

More general, the Lie derivative of an arbitrary tensor field Υ_I on π^*TM with respect to the complete lift $\hat{V}$ of a vector field $V = V^i(x)\frac{\partial}{\partial x^i}$ on M is defined by

$$(\mathcal{L}_{\hat{V}}\Upsilon_I)|_{(x,y)} := \lim_{t\to 0}\frac{\tilde{\varphi}_t^*(\Upsilon_I|_{\tilde{\varphi}_t(x,y)}) - \Upsilon_I|_{(x,y)}}{t},$$

where I denotes a mixed multi-index. In this case, in the local coordinates (x^i, y^i) on TM, the Lie derivative of an arbitrary mixed tensor field, for example, a tensor T of $(1,2)$-type with the components $T^i_{jk}(x,y)$ on π^*TM, is given by

$$\mathcal{L}_{\hat{V}}T := \left(\mathcal{L}_{\hat{V}}T^i_{jk}\right)\partial_i \otimes dx^j \otimes dx^k.$$

Here,

$$\begin{aligned}
\mathcal{L}_{\hat{V}}T^i_{jk} &= \hat{V}\left(T^i_{jk}\right) + \frac{\partial V^a}{\partial x^j}T^i_{ak} + \frac{\partial V^a}{\partial x^k}T^i_{ja} - \frac{\partial V^i}{\partial x^a}T^a_{jk} \\
&= V^m T^i_{jk|m} + y^m(V^l_{|m})\frac{\partial T^i_{jk}}{\partial y^l} - T^m_{jk}V^i_{|m} + T^i_{mk}V^m_{|j} + T^i_{jm}V^m_{|k},
\end{aligned}$$

where "|" denotes the horizontal covariant derivative with respect to the Chern connection (see [2][8]).

Now let us define the Lie derivative of Chern connection D with respect to $\hat{V}$. For $\zeta \in T(TM_0)$ and $X \in \pi^*TM$, define $\mathcal{L}_{\hat{V}}D$ by ([9][12])

$$\begin{aligned}
\left(\mathcal{L}_{\hat{V}}D\right)_\zeta X &= \left(\mathcal{L}_{\hat{V}}D\right)(\zeta, X) \\
&:= \mathcal{L}_{\hat{V}}\left(D_\zeta X\right) - D_{\mathcal{L}_{\hat{V}}\zeta}X - D_\zeta(\mathcal{L}_{\hat{V}}X) \\
&= \mathcal{L}_{\hat{V}}\left(D_\zeta X\right) - D_{[\hat{V},\zeta]}X - D_\zeta(\mathcal{L}_{\hat{V}}X).
\end{aligned} \tag{5.7}$$

Write Chern connection 1-form $\omega_j^{\ i} = \Gamma^i_{jk}(x,y)dx^k$. Put

$$(\mathcal{L}_{\hat{V}}D)(\partial_j, \partial_k) := (\mathcal{L}_{\hat{V}}\Gamma^i_{jk})\partial_i. \tag{5.8}$$

By (5.3), we have

$$\mathcal{L}_{\hat{V}}\partial_j = \rho[\hat{V}, \partial_j] = -\frac{\partial V^i}{\partial x^j}\partial_i.$$

Further,

$$\mathcal{L}_{\hat{V}}(D_{\partial_j}\partial_k) = \mathcal{L}_{\hat{V}}(\Gamma^i_{kj}\partial_i) = (\hat{V}\Gamma^i_{kj})\partial_i - \Gamma^m_{kj}\frac{\partial V^i}{\partial x^m}\partial_i, \tag{5.9}$$
$$D_{\mathcal{L}_{\hat{V}}\partial_j}\partial_k = -\Gamma^i_{kl}\frac{\partial V^l}{\partial x^j}\partial_i, \tag{5.10}$$
$$D_{\partial_j}(\mathcal{L}_{\hat{V}}\partial_k) = -\left(\frac{\partial^2 V^i}{\partial x^k \partial x^j} + \Gamma^i_{mj}\frac{\partial V^m}{\partial x^k}\right)\partial_i. \tag{5.11}$$

From (5.7) and (5.9)-(5.11), we obtain

$$\begin{aligned}
(\mathcal{L}_{\hat{V}}D)(\partial_j, \partial_k) &= \mathcal{L}_{\hat{V}}(D_{\partial_j}\partial_k) - D_{\mathcal{L}_{\hat{V}}\partial_j}\partial_k - D_{\partial_j}(\mathcal{L}_{\hat{V}}\partial_k) \\
&= \left((\hat{V}\Gamma^i_{kj}) - \Gamma^m_{kj}\frac{\partial V^i}{\partial x^m} + \Gamma^i_{kl}\frac{\partial V^l}{\partial x^j} + \frac{\partial^2 V^i}{\partial x^k \partial x^j} + \Gamma^i_{mj}\frac{\partial V^m}{\partial x^k}\right)\partial_i.
\end{aligned}$$

From (5.8), we obtain

$$\mathcal{L}_{\hat{V}}\Gamma^i_{jk} = \frac{\partial^2 V^i}{\partial x^j \partial x^k} + \Gamma^i_{lj}\frac{\partial V^l}{\partial x^k} + \Gamma^i_{kl}\frac{\partial V^l}{\partial x^j} - \Gamma^l_{kj}\frac{\partial V^i}{\partial x^l} + V^l \frac{\partial \Gamma^i_{kj}}{\partial x^l} + y^s \frac{\partial V^l}{\partial x^s} P_k{}^i{}_{jl}, \tag{5.12}$$

where $P_k{}^i{}_{jl} := \frac{\partial \Gamma^i_{kj}}{\partial y^l}$ determine the Landsberg curvature of F.

Note that

$$V^i_{|j} = \frac{\delta V^i}{\delta x^j} + V^m \Gamma^i_{mj} = \frac{\partial V^i}{\partial x^j} + V^m \Gamma^i_{mj}.$$

We get

$$\begin{aligned}\frac{\delta V^i_{|j}}{\delta x^k} &= \frac{\delta}{\delta x^k}\left(\frac{\partial V^i}{\partial x^j}\right) + \frac{\delta V^m}{\delta x^k}\Gamma^i_{mj} + V^m \frac{\delta \Gamma^i_{mj}}{\delta x^k} \\ &= \frac{\partial^2 V^i}{\partial x^k \partial x^j} + \frac{\partial V^m}{\partial x^k}\Gamma^i_{mj} + V^m \frac{\delta \Gamma^i_{mj}}{\delta x^k}.\end{aligned}$$

Further, we have

$$\begin{aligned} V^i_{|j|k} &= \frac{\delta V^i_{|j}}{\delta x^k} + V^m_{|j}\Gamma^i_{mk} - V^i_{|m}\Gamma^m_{jk} \\ &= \frac{\partial^2 V^i}{\partial x^k \partial x^j} + V^m \frac{\delta \Gamma^i_{mj}}{\delta x^k} + V^a \Gamma^m_{aj}\Gamma^i_{mk} - V^a \Gamma^i_{am}\Gamma^m_{jk} \\ &\quad + \frac{\partial V^m}{\partial x^k}\Gamma^i_{mj} + \frac{\partial V^m}{\partial x^j}\Gamma^i_{mk} - \frac{\partial V^i}{\partial x^m}\Gamma^m_{jk}. \end{aligned} \tag{5.13}$$

Then, by comparing (5.12) and (5.13), we can get

$$V^i_{|j|k} = \mathcal{L}_{\hat{V}}\Gamma^i_{jk} - \frac{\partial V^l}{\partial x^s} y^s P_j{}^i{}_{kl} + V^m \frac{\delta \Gamma^i_{mj}}{\delta x^k} + V^r \Gamma^l_{rj}\Gamma^i_{lk} - V^r \Gamma^i_{rl}\Gamma^l_{jk} - V^l \frac{\partial \Gamma^i_{kj}}{\partial x^l}.$$

On the other hand,

$$\begin{aligned} R_j{}^i{}_{km} V^m &= V^m \left(\frac{\delta \Gamma^i_{mj}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^m} + \Gamma^l_{jm}\Gamma^i_{kl} - \Gamma^i_{lm}\Gamma^l_{jk}\right) \\ &= V^m \left(\frac{\partial \Gamma^i_{mj}}{\partial x^k} - N^r_k \frac{\partial \Gamma^i_{mj}}{\partial y^r} - \frac{\partial \Gamma^i_{jk}}{\partial x^m} + N^r_m \frac{\partial \Gamma^i_{jk}}{\partial y^r} + \Gamma^l_{jm}\Gamma^i_{kl} - \Gamma^i_{lm}\Gamma^l_{jk}\right). \end{aligned}$$

Here, $R_j{}^i{}_{km} = R_j{}^i{}_{km}(x,y)$ are the coefficients of Riemann curvature tensor with respect to Chern connection. Thus we can get

$$\begin{aligned} V^i_{|j|k} - R_j{}^i{}_{km} V^m &= \mathcal{L}_{\hat{V}}\Gamma^i_{jk} - \frac{\partial V^l}{\partial x^m} y^m P_j{}^i{}_{kl} - V^m N^r_m \frac{\partial \Gamma^i_{jk}}{\partial y^r} \\ &= \mathcal{L}_{\hat{V}}\Gamma^i_{jk} - \left(\frac{\partial V^r}{\partial x^s} + V^m \Gamma^r_{ms}\right) y^s P_j{}^i{}_{kr} \\ &= \mathcal{L}_{\hat{V}}\Gamma^i_{jk} - y^m V^r_{|m} P_j{}^i{}_{kr}. \end{aligned}$$

Hence, we have the following

Theorem 5.1. *The Lie derivative of Chern connection with respect to the complete lift $\hat{V}$ of a vector field $V = V^i(x)\frac{\partial}{\partial x^i}$ on M is determined by*

$$\mathcal{L}_{\hat{V}}\Gamma^i_{jk} = R_j{}^i{}_{mk} V^m + V^i_{|j|k} + y^m V^l_{|m} P_j{}^i{}_{kl}. \tag{5.14}$$

Remark 5.2. *Let $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ be the spray induced by Finsler metric F, where $G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x,y) y^j y^k$. By (5.1), we have*

$$\mathcal{L}_{\hat{V}} G = [\hat{V}, G] = -y^i y^j \left\{ \frac{\partial^2 V^k}{\partial x^i \partial x^j} + V^m \frac{\partial \Gamma^k_{ij}}{\partial x^m} + 2\frac{\partial V^m}{\partial x^j}\Gamma^k_{mi} - \Gamma^m_{ij}\frac{\partial V^k}{\partial x^m} \right\} \frac{\partial}{\partial y^k}. \tag{5.15}$$

Let

$$\mathcal{L}_{\hat{V}}G := (\mathcal{L}_{\hat{V}}G^k)\frac{\partial}{\partial y^k}.$$

From (5.12) and (5.15), we have

$$\begin{aligned}\mathcal{L}_{\hat{V}}G^k &= -y^iy^j\left\{\frac{\partial^2 V^k}{\partial x^i\partial x^j} + V^m\frac{\partial\Gamma^k_{ij}}{\partial x^m} + 2\frac{\partial V^m}{\partial x^j}\Gamma^k_{mi} - \Gamma^m_{ij}\frac{\partial V^k}{\partial x^m}\right\}\\ &= -y^iy^j\mathcal{L}_{\hat{V}}\Gamma^k_{ij}.\end{aligned}$$

Further, by (5.14), we know the following

$$\mathcal{L}_{\hat{V}}G^k = -(R^k{}_mV^m + V^k_{|0|0}).$$

Obviously, $\mathcal{L}_{\hat{V}}G^k \neq \frac{1}{2}\left(\mathcal{L}_{\hat{V}}\Gamma^k_{ij}\right)y^iy^j$.

It is natural to establish a connection between Lie derivative and Hessian (or Laplacian) of a C^2 function on the manifold. Firstly, given a C^2 function $u = u(x)$ on M, let us determine the Lie derivative of the fundamental tensor with respect to the complete lift $\widehat{\nabla u}$ of the gradient vector field ∇u. Recall that $\nabla u = g^{ij}(x,\nabla u)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}$ and

$$\begin{aligned}\nabla^i u(x) = g^{ij}(x,\nabla u)\frac{\partial u}{\partial x^j} &= g^{*ij}(x,du)\frac{\partial u}{\partial x^j},\\ (\nabla u)_i = g_{ij}(x,\nabla u)(\nabla^j u) &= u_{|i}(x).\end{aligned}$$

The complete lift $\widehat{\nabla u}$ of ∇u is given by $\widehat{\nabla u} = (\nabla^i u)(x)\frac{\partial}{\partial x^i} + y^m\left(\frac{\partial(\nabla^i u)}{\partial x^m}\right)\frac{\partial}{\partial y^i}$. Let

$$(\widehat{\nabla u})_i := g_{im}(x,y)(\nabla^m u)(x) = g_{im}(x,y)\left(g^{mk}(x,\nabla u)\frac{\partial u}{\partial x^k}\right).$$

By (5.6), we have

$$\mathcal{L}_{\widehat{\nabla u}}g_{ij}(x,y) = (\widehat{\nabla u})_{i|j}(x,y) + (\widehat{\nabla u})_{j|i}(x,y) + 2y^m(\nabla^l u)_{|m}(x,y)C_{lij}(x,y). \tag{5.16}$$

Here, by the definitions, we have

$$\begin{aligned}(\widehat{\nabla u})_{i|j}(x,y) &= g_{im}(x,y)\left(g^{mk}(x,\nabla u)\right)_{|j}(x,y)\frac{\partial u}{\partial x^k}\\ &\quad + g_{im}(x,y)g^{mk}(x,\nabla u)u_{|k|j}(x,y)\\ (\nabla^l u)_{|m}(x,y) &= \left(g^{lr}(x,\nabla u)\frac{\partial u}{\partial x^r}\right)_{|m}(x,y) = \left(g^{lr}(x,\nabla u)\right)_{|m}(x,y)\frac{\partial u}{\partial x^r}\\ &\quad + g^{lr}(x,\nabla u)u_{|r|m}(x,y).\end{aligned}$$

Further,

$$\begin{aligned}\left(g^{mk}(x,\nabla u)\right)_{|j}(x,y) &= -g^{mr}(x,\nabla u)g^{ks}(x,\nabla u)\left(g_{rs}(x,\nabla u)\right)_{|j}(x,y)\\ &= -g^{mr}(x,\nabla u)g^{ks}(x,\nabla u)\Big\{\frac{\partial g_{rs}}{\partial x^j}(x,\nabla u) + 2C_{rsl}(x,\nabla u)\frac{\partial(\nabla^l u)}{\partial x^j}\\ &\quad - g_{ls}(x,\nabla u)\Gamma^l_{rj}(x,y) - g_{rl}(x,\nabla u)\Gamma^l_{sj}(x,y)\Big\}\\ &= -g^{mr}(x,\nabla u)g^{ks}(x,\nabla u)\Big\{\frac{\partial g_{rs}}{\partial x^j}(x,\nabla u) + 2C_{rsl}(x,\nabla u)\frac{\partial(\nabla^l u)}{\partial x^j}\Big\}\\ &\quad + g^{mr}(x,\nabla u)\Gamma^k_{rj}(x,y) + g^{ks}(x,\nabla u)\Gamma^m_{sj}(x,y).\end{aligned}$$

In particular, it is easy to see that

$$(\widehat{\nabla u})_{i|j}(x,\nabla u) = u_{|i|j}(x,\nabla u).$$

Thus, by (5.16), we get

$$\mathcal{L}_{\widehat{\nabla u}}g_{ij}(x,\nabla u) = 2u_{|i|j}(x,\nabla u) + 2(\nabla^m u)u_{|r|m}(x,\nabla u)C^r_{ij}(x,\nabla u).$$

Then, by (4.5), we know that the Hessian in Finsler geometry can also be determined by Lie derivative.

Proposition 5.3. *Let $u = u(x)$ be a C^2 function on Finsler manifold (M, F). Then the Hessian of u is determined by*

$$\mathrm{Hess}(u) = \frac{1}{2}\mathcal{L}_{\widehat{\nabla u}}g(x, \nabla u) - \left((\nabla^m u)u_{|r|m}(x, \nabla u)C^r_{ij}(x, \nabla u)\right),$$

where $g(x, \nabla u) := (g_{ij}(x, \nabla u))$.

Corollary 5.4. ([7]) *Let $u = u(x)$ be a C^2 function on Riemannian manifold (M, g). Then the Hessian of u is determined by*

$$\mathrm{Hess}(u) = \frac{1}{2}\mathcal{L}_{\nabla u}g.$$

6. Estimates of eigenvalues on compact Finsler manifolds

For an n-dimensional Finsler manifold (M, F, dm) equipped with a smooth measure m and for any $v \in T_xM\backslash\{0\}$, let $\eta : (-\varepsilon, \varepsilon) \longrightarrow M$ be the geodesic with $\dot{\eta}(0) = v$ and decompose the measure m along η as

$$dm = \mathrm{e}^{-\psi_\eta}\sqrt{\det\left(g_{ij}(\eta, \dot{\eta})\right)}\, dx^1dx^2 \cdots dx^n,$$

where $\psi_\eta = \psi_\eta\left(\eta(t), \dot{\eta}(t)\right) : (-\varepsilon, \varepsilon) \longrightarrow R$ is a C^∞ function. Then, for $N \in R\backslash\{n\}$, define the weighted Ricci curvature ([10])

$$\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + \psi''_\eta(0) - \frac{\psi'_\eta(0)^2}{N - n}.$$

As the limits of $N \to \infty$ and $N \downarrow n$, we define the weighted Ricci curvatures as follows.

$$\mathrm{Ric}_\infty(v) := \mathrm{Ric}(v) + \psi''_\eta(0),$$
$$\mathrm{Ric}_n(v) := \begin{cases} \mathrm{Ric}(v) + \psi''_\eta(0) & \text{if } \psi'_\eta(0) = 0, \\ -\infty & \text{if } \psi'_\eta(0) \neq 0. \end{cases}$$

We say that $\mathrm{Ric}_N \geq K$ for $K \in R$ if $\mathrm{Ric}_N(v) \geq KF^2(x, v)$ for all $x \in M$ and $v \in T_xM$. Notice that the quantity $\psi'_\eta(0) = \mathbf{S}(x, v)$ is just the S-curvature with respect to the measure m and $\psi''_\eta(0) = \dot{\mathbf{S}}(x, v) = \mathbf{S}_{|m}(x, v)v^m$, where "|" denotes the horizontal covariant derivative with respect to the Chern connection ([3][10]). Hence we can rewrite $\mathrm{Ric}_N(v)$ as

$$\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + \dot{\mathbf{S}}(x, v) - \frac{\mathbf{S}(x, v)^2}{N - n}.$$

The following is a new observation for weighted Ricci curvature on Finsler maniflods. Let (M, F, m) be an n-dimensional Finsler manifold with $dm = \sigma(x)dx^1 \cdots dx^n$. Let Y be a C^∞ geodesic field on an open subset $U \subset M$ and $\hat{g} = g_Y$. Let

$$dm := e^{-\psi}Vol_{\hat{g}}, \quad Vol_{\hat{g}} = \sqrt{det\left(g_{ij}\left(x, Y_x\right)\right)}dx^1 \cdots dx^n,$$

where ψ is given by

$$\psi(x) = \ln\frac{\sqrt{\det\left(g_{ij}\left(x, Y_x\right)\right)}}{\sigma(x)} = \tau\left(x, Y_x\right) \text{ (distortion)}.$$

For $y = Y_x \in T_xM$ (that is, Y is a geodesic extension of $y \in T_xM$), we know that ([13])

$$\mathbf{S}(x, y) = y[\tau(x, Y)] = d\psi(y)$$

and

$$\dot{\mathbf{S}}(x, y) = y[S(x, Y)] = y[Y(\psi)] = \mathrm{Hess}\psi(y).$$

Hence

$$\mathrm{Ric}_N(y) = \mathrm{Ric}(y) + \mathrm{Hess}\psi(y) - \frac{d\psi(y)^2}{N - n}.$$

In Riemann geometry, Bochner formula is a bridge to use the analytic tools to study the geometry and topology of a manifold. In Finsler geometry, the Bochner-Weitzenböck formula and the corresponding Bochner inequality on Finsler manifolds have been established by Ohta-Sturm in [11]. For our aim, we need the following formula.

Lemma 6.1. (Integrated Bochner-Weitzenböck formula, [11]) *Given* $u \in H^2_{\mathrm{loc}}(M) \cap \mathcal{C}^1(M)$ *such that* $\Delta u \in H^1_{\mathrm{loc}}(M)$, *we have*

$$-\int_M d\phi \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) dm \geq \int_M \phi \left\{ d(\Delta u)(\nabla u) + \mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N} \right\} dm$$

for $N \in [n, \infty]$ *and all nonnegative functions* $\phi \in H^1_c(M) \cap L^\infty(M)$.

By using above integrated Bochner-Weitzenböck formula, we can get the estimate of eigenvalues on compact Finsler manifolds as follows.

Theorem 6.2. *Let* (M, F, dm) *be a compact Finsler manifold and satisfy* $Ric_N \geq K > 0$ *for some* $N \in [n, \infty]$. *Then, for any non-zero eigenvalue* $\lambda > 0$ *corresponding to the eigenfunction* $u \in H^2(M) \cap \mathcal{C}^1(M)$, $\Delta u = -\lambda u$, *we have*

$$\lambda \geq \frac{KN}{N-1}.$$

When $N = \infty$, *we have* $\lambda \geq K$.

Proof. By the assumption and Lamma 6.1, we have

$$-\int_M d\phi \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) dm \geq \int_M \phi \left\{ d(\Delta u)(\nabla u) + KF^2(\nabla u) + \frac{(\Delta u)^2}{N} \right\} dm.$$

Taking the test function $\phi = 1$ in the above inequality, we get

$$\int_M \left\{ d(\Delta u)(\nabla u) + KF^2(\nabla u) + \frac{(\Delta u)^2}{N} \right\} dm \leq 0.$$

Since $\Delta u = -\lambda u$, we have

$$\int_M d(\Delta u)(\nabla u) dm = -\int_M (\Delta u)^2 dm = -\lambda^2 \int_M u^2 dm$$

and

$$-\lambda^2 \int_M u^2 dm + K \int_M F^2(\nabla u) dm + \frac{\lambda^2}{N} \int_M u^2 dm \leq 0,$$

which implies that

$$K \int_M F^2(\nabla u) dm \leq \lambda^2 \frac{N-1}{N} \int_M u^2 dm.$$

Then, we obtain

$$\lambda^2 \geq \frac{KN}{N-1} \frac{\int_M F^2(\nabla u) dm}{\int_M u^2 dm}.$$

By the assumption again, $\lambda = \mathcal{E}(u) = \frac{\int_M F^2(\nabla u) dm}{\int_M u^2 dm} > 0$. Thus we conclude the following

$$\lambda \geq \frac{KN}{N-1}.$$

In particular, we can get $\lambda \geq K$ when $N = \infty$. This completes the proof of Theorem 6.2. □

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