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On Finsler metrics with weakly isotropic S-curvature

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ABSTRACT: In this paper, we focus on a class of Finsler metrics which are called general (α,β) - metrics: $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. We examine the metrics as weakly isotropic S-curvature.

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(*Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration*)

1. Introduction

The S-curvature plays an essential role in Finsler geometry. It has been introduced by Z. Shen while he was studying on the volume form in Finsler geometry, [10]. Therefore, many authors have studied on this idea and obtained some important results, [8], [9], [12] and [13]. A Finsler metric of an isotropic S-curvature is defined as follows:

$$\mathbf{S} = (n+1)\mathfrak{c}F, \tag{1.1}$$

$\mathfrak{c} = \mathfrak{c}(x)$ is a scalar function on M.

The E-curvature $\mathbf{E} = E_{ij}dx^i \otimes dx^j$ is another Riemannian quantity which has been obtained from the S-curvature. In fact, it is introduced as follows:

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial y^i \partial y^j}. \tag{1.2}$$

A Finsler metric F of an isotropic E-curvature defined as follows: there is a scalar function $\mathfrak{c} = \mathfrak{c}(x)$ on M such that

$$\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}\mathfrak{h}, \tag{1.3}$$

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$\mathfrak{h}$ is a family of bilinear forms $\mathfrak{h}_y = \mathfrak{h}_{ij} dx^i \otimes dx^j$, which are defined by $\mathfrak{h}_{ij} := F F_{y^i y^j}$.

By (1.2), one can easily realize that Finsler metric of isotropic E-curvature is of isotropic S-curvature. However, the converse is still an open problem. In [1], Cheng-Shen have proved that (1.1) is equivalent to (1.3) for Randers metrics. Then, X. Chun-Huan, X. Cheng, I.Y. Lee and M.H. Lee have obtained a similar result for some special (α,β)-metrics [3], [6]. Najafi-Tayebi have obtained a condition on (α,β)-metrics which has been verified that (1.1) and (1.3) are equivalent [7]. All these studying inspire us to focus on the idea for the general (α,β)-metrics. There are some progress and results on the general (α,β)-metrics, (see, [18], [19], [20]). The general (α,β)-metrics has been introduced by C. Yu and H. Zhu, [15]. These class of metrics are defined as follow:

$$F = \alpha\phi(b^2, s),$$

$\alpha := \sqrt{a_{ij}(x)y^i y^j}$ and $\beta := b_i(x)y^i$ $(b := ||\beta||_\alpha)$ are Riemannian metric and 1-form, respectively. Here, also $\phi(b^2, s)$ is a positive smooth function. In 2011, Yu and Zhu have obtained a sufficient condition for general (α,β)-metrics to be locally projectively flat [15]. Then, they have completely classified the general (α,β)-metrics with constant flag curvature under some suitable conditions. Moreover, They have constructed many new projectively flat Finsler metrics that these metrics are of constant flag curvature which are $1, 0$ and -1, [16]. Many authors have obtained some important result and classification on the general (α,β) - metrics, (see, [4], [5], [14], [17], [21], [22]).

For a general (α,β) - metric, we use

$$\begin{aligned} Q &= \frac{\phi_2}{\phi - s\phi_2}, \\ \Delta &= 1 + sQ + (b^2 - s^2)Q_2, \\ \Phi &= -(Q - sQ_2)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q_{22}, \end{aligned}$$

and

$$\Xi = (s + b^2 Q)\frac{\Phi}{\Delta^2}.$$

Focusing on the method in [7, 20], we study the general (α,β)-metric to be a weakly isotropic S-curvature.

We give the following theorem:

Theorem 1.1. *Let $F = \alpha\phi(b^2, s)$ be a general (α,β)-metric on M^n. Suppose that Ξ and b are not constant. F is to be of a weakly isotropic S-curvature if and only if F is to be of an isotropic S-curvature.*

B. Najafi and A.Tayebi have proved that if F is an (α,β) - metrics of isotropic S-curvature, then b is a constant term, [7]. However, If F is a general (α,β) metric, then b is not necessarily to be a constant term. Moreover, if b is a constant term, then it has been obtained that the general (α,β)-metrics have reduced to (α,β)-metrics. According to these discussion, we suppose that b is not a constant term.

2. Preliminaries

F be a Finsler metric on M^n. Every Finsler metric F induces a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$. The spray coefficients G^i are defined by

$$G^i := \frac{1}{4} g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\},$$

where g^{ij} is the inverse of the fundamental tensor $g_{ij} := [\frac{1}{2}F^2]_{y^i y^j}$. For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as $G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$.

S - curvature, is given by

$$\mathbf{S} \;=\; \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\Big[\ln \sigma_{BH}\Big],$$

$dV_F = \sigma_F(x) dx^1 \wedge \cdots \wedge dx^n$ is the Busemann-Hausdorff volume form.

E-curvature $\mathbf{E} = E_{ij} dx^i \otimes dx^j$ of F is defined by

$$E_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right).$$

Definition 2.1. *[11] Let F be a Finsler metric on M^n. Then*

(a) F is to be a weakly isotropic S-curvature if $\mathbf{S} = (n+1)\mathfrak{c}F + \eta$,

(b) F is to be a isotropic S-curvature if $\mathbf{S} = (n+1)\mathfrak{c}F$,

$\mathfrak{c} = \mathfrak{c}(x)$ is a scalar function on M, $\eta = \eta_i(x)y^i$ is a $1-$form on M.

It is obvious that if F is of isotropic E-curvature iff F is of weakly isotropic S-curvature.

We introduce the well known identities as follows:

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),\ r_{00} = r_{ij}y^iy^j,\ s^i{}_0 = a^{ij}s_{jk}y^k,$$
$$r_i = b^jr_{ji},\ s_i = b^js_{ji},\ r_0 = r_iy^i,\ s_0 = s_iy^i,\ r^i = a^{ij}r_j,\ s^i = a^{ij}s_j,\ r = b^ir_i,$$

where $(a^{ij}) = (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$.

To prove the main theorem, we give some essential facts given below:

Lemma 2.2. *[15] The spray coefficients G^i of a general (α,β)-metric $F = \alpha\phi(b^2, s)$ are related to the spray coefficients ${}^\alpha G^i$ of α and given by*

$$G^i = {}^\alpha G^i + \alpha Qs^i{}_0 + \{\Theta(-2\alpha Qs_0 + r_{00} + 2\alpha^2Rr) + \alpha\Omega(r_0 + s_0)\}\frac{y^i}{\alpha}$$
$$+ \{\Psi(-2\alpha Qs_0 + r_{00} + 2\alpha^2Rr) + \alpha\Pi(r_0 + s_0)\}b^i - \alpha^2R(r^i + s^i),$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2},\quad R = \frac{\phi_1}{\phi - s\phi_2},\quad \Theta = \frac{Q - sQ_2}{2\Delta},\quad \Psi = \frac{Q_2}{2\Delta},$$
$$\Pi = \frac{R_2 - 2sRQ_2 + sQR_2}{\Delta},\quad \Omega = \frac{2R - sR_2 + 2b^2RQ_2 - b^2QR_2}{\Delta},$$

where $\Delta := 1 + sQ + (b^2 - s^2)Q_2$.

S-curvature of general (α,β)-metrics has been obtained by H. Zhu, [20]:

$$\mathbf{S} = (2\Psi + T - 2g)(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Qs_0) + \alpha Pr,$$

where

$$\Phi := -(Q - sQ_2)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q_{22}, \tag{2.1}$$
$$T := (n+1)\Omega + s\Pi + \Pi_2(b^2 - s^2) - 2R + sR_2, \tag{2.2}$$
$$P := 2(n+1)\Theta R + 4s\Psi R + 2(\Psi_2R + \Psi R_2)(b^2 - s^2) + \Pi - R_2, \tag{2.3}$$

$g(b^2) := \dfrac{f'(b^2)}{f(b^2)}$. Moreover, the classification of general (α,β)-metric of isotropic S-curvature has given as follows, [20]:

Lemma 2.3. *Let $F = \alpha\phi(b^2, s)$ be a general (α,β)-metric on M^n. Suppose that b is not a constant. Then F is of isotropic S-curvature if and only if one of the following satisfies*

1) ϕ satisfies

$$\frac{\Phi}{2\Delta^2}\mathfrak{d}(b^2 - s^2) + [(n+1)E + H_2(b^2 - s^2) + 2sH - 2sg](\mathfrak{K} + \mathfrak{d}b^2) = (n+1)\mathfrak{c}\phi, \tag{2.4}$$

$$g(b^2) := \frac{f'(b^2)}{f(b^2)},$$

$$E := \frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}H,$$
$$H := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\}}.$$

Moreover, α and β satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_ib_j,$$
$$s_i = 0,$$

where $\mathfrak{K} = \mathfrak{K}(x), \mathfrak{d} = \mathfrak{d}(x)$ and $\mathfrak{K} + \mathfrak{d}b^2 \neq 0$.

2) ϕ satisfies (2.4) and

$$(1 + \tau b^2)(2\Psi + T - 2g) - \frac{\Phi}{\Delta^2}(\tau s - Q) = 0.$$

Moreover, α and β satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_ib_j + \tau(b_is_j + b_js_i),$$

where $s_i \neq 0$ and $1 + \tau b^2 \neq 0$, where $\tau = \tau(b^2)$.

3) ϕ satisfies (2.4) and

$$\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T - 2g) = 0.$$

Moreover, α and β satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_ib_j + b_i\theta_j + b_j\theta_i, \quad s_i = 0,$$

where $\theta = \theta_i(x)y^i \neq 0$ is a $1-$form which is orthogonal to β.

Lemma 2.4. *[20] A general (α, β)-metric is a Riemannian metric if and only if $\Phi = 0$.*

3. Weakly isotropic *S*-curvature

Firstly, we prove the following essential Lemma. The Lemma helps to prove the main theorem of this paper. We classify the general (α, β)-metrics of weakly isotropic S-curvature, therefore we follows the following Lemma:

Lemma 3.1. *Let $F = \alpha\phi(b^2, s)$ be a general (α, β)-metric M^n. F is of weakly isotropic S-curvature $\mathbf{S} = (n + 1)\mathfrak{c}F + \eta$ if and only if the following equality holds*

$$\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0) - (2\Psi + T)(r_0 + s_0) - \alpha P r = -(n + 1)\mathfrak{c}F + \vartheta, \tag{3.1}$$

where

$$\vartheta := -2g(r_0 + s_0) - \eta,$$

and

$$g := \frac{f'(b^2)}{f(b^2)}.$$

Proof. Combining (2.1) and the definition of weakly isotropic S-curvature, $\mathbf{S} = (n + 1)\mathfrak{c}F + \eta$, we prove the lemma.

□

Simplifying (3.1), we need to use the special coordinate as follows: $\psi : (s, u^a) \to (y^i)$ by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^a = u^a, \tag{3.2}$$

where

$$\bar{\alpha} = \sqrt{\sum_{a=2}^{n}(u^a)^2}.$$

Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

Take a special coordinate system at an arbitrary x as in (3.2). It is easy to get

$$r_1 = br_{11}, \quad r_a = br_{1a}, \quad r = b^2 r_{11}, \quad s_1 = 0, \quad s_a = bs_{1a}.$$

Let

$$\bar{r}_{10} := \sum_{a=2}^{n} r_{1a}y^a, \quad \bar{r}_{00} := \sum_{a,b=2}^{n} r_{ab}y^a y^b, \quad \bar{r}_0 := \sum_{a=2}^{n} r_a y^a,$$

$$\bar{s}_{10} := \sum_{a=2}^{n} s_{1a}y^a, \quad \bar{s}_0 := \sum_{a=2}^{n} s_a y^a.$$

Put

$$\vartheta = t_i y^i - \eta_i y^i.$$

Then t_i are given by

$$t_1 = -2bgr_{11}, \quad t_a = -2bg(r_{1a} + s_{1a}).$$

A direct computation yields

$$r_0 = r_{11}\frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha} + b\bar{r}_{10}, \quad s_0 = \bar{s}_0 = b\bar{s}_{10},$$

and

$$r_{00} = r_{11}\frac{s^2}{b^2 - s^2}\bar{\alpha}^2 + 2\bar{r}_{10}\frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{r}_{00},$$
$$\vartheta = -2bg\frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha} + -2bg(\bar{r}_{10} + \bar{s}_{10}) - \eta.$$

When we plug the expressions obtained above into (3.1), we verify that (3.1) is equivalent to the following equations:

$$\big\{[\frac{s^2\Phi}{2\Delta^2} - sb^2(2\Psi + T) - b^4P]r_{11} + (n+1)\mathfrak{c}b^2\phi - sbt_1\big\}\bar{\alpha}^2 + \frac{\Phi}{2\Delta^2}(b^2 - s^2)\bar{r}_{00} = 0, \tag{3.3}$$
$$[\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T)](r_{1a} + s_{1a}) - (s + b^2Q)\frac{\Phi}{\Delta^2}s_{1a} + b\eta_a - bt_a = 0, \tag{3.4}$$
$$\eta_1 = 0. \tag{3.5}$$

Since F is a non-Riemannian metric, $\Phi \neq 0$ by Lemma 2.4. It is obvious that $\bar{r}_{00}$, and $\bar{\alpha}$ are independent of s. Following (3.3), and (3.4), we see that the following relations hold in a special coordinate system (s, y^a) at a point x:

$$r_{ab} = \mathfrak{K}\delta_{ab}, \tag{3.6}$$
$$[\frac{s^2\Phi}{2\Delta^2} - sb^2(2\Psi + T) - b^4P]r_{11} + (n+1)\mathfrak{c}b^2\phi - sbt_1 + \frac{\mathfrak{K}\Phi}{2\Delta^2}(b^2 - s^2) = 0, \tag{3.7}$$
$$[\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T)](r_{1a} + s_{1a}) - (s + b^2Q)\frac{\Phi}{\Delta^2}s_{1a} + b\eta_a - bt_a = 0, \tag{3.8}$$

$\mathfrak{K} = \mathfrak{K}(x)$ is independent of s. (3.6) satisfies that there is a $1-$ form θ, [20]. Then, we have

$$r_{ij} = \mathfrak{K}(x)a_{ij} + \mathfrak{d}(x)b_i b_j + b_i\theta_j + b_j\theta_i, \tag{3.9}$$

for some scalar function $\mathfrak{K}(x), \mathfrak{d}(x)$ and some $1-$ form θ. In fact, (3.6) is equivalent to

$$r_{00} = \mathfrak{K}(x)\alpha^2, \quad \forall y \in (\beta^{\#})^{\perp}, \tag{3.10}$$

where $(\beta^{\#})^{\perp} := \{(y^i) \in T_xM \mid b_iy^i = 0\}$. Notice that any vector lying in hyperplane $\beta = 0$ can be represented as $b^2y^i - \beta b^i$. Substituting it into (3.10), one can see that (3.9) holds. Also, we always assume that θ is perpendicular to β, i.e., $\theta_ib^i = 0$. That is because if θ is not orthogonal to β, we can represent θ as $\theta' + \frac{\theta^l b_l}{b^2}\beta$, therefore θ' orthogonal to β.

By (3.9), we have

$$r_{11} = \mathfrak{K} + \mathfrak{d}b^2, \quad r_{1a} = b\theta_a. \tag{3.11}$$

Plugging (3.11) into (3.7) and (3.8) yields

$$[\frac{s^2\Phi}{2\Delta^2} - sb^2(2\Psi + T) - b^4P + 2sb^2g](\mathfrak{K} + \mathfrak{d}b^2) + (n+1)\mathfrak{c}b^2\phi + \frac{\mathfrak{K}\Phi}{2\Delta^2}(b^2 - s^2) = 0, \tag{3.12}$$

$$[\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) + 2b^2g](b^2\theta_a + s_a) - (s + b^2Q)\frac{\Phi}{\Delta^2}s_a + b^2\eta_a = 0. \tag{3.13}$$

We state the the following Proposition:

Proposition 3.2. *Let $F = \alpha\phi(b^2, s)$ be a non-Riemannian general (α, β)-metric on M^n. Suppose that b is not a constant. F is to be a weakly isotropic S-curvature if and only if one of the following holds*

1) ϕ satisfies

$$\frac{\Phi}{2\Delta^2}\mathfrak{d}(b^2 - s^2) + [(n+1)E + H_2(b^2 - s^2) + 2sH - 2sg](\mathfrak{K} + \mathfrak{d}b^2) = (n+1)\mathfrak{c}\phi, \tag{3.14}$$

where $g(b^2) := \dfrac{f'(b^2)}{f(b^2)}$,

$$E := \frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}H, \tag{3.15}$$

$$H := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\}}. \tag{3.16}$$

Moreover, α and β satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_ib_j,$$
$$s_i = 0,$$

where $\mathfrak{K} = \mathfrak{K}(x), \mathfrak{d} = \mathfrak{d}(x)$ and $\mathfrak{K} + \mathfrak{d}b^2 \neq 0$.

In that case, $\mathbf{S} = (n+1)\mathfrak{c}\phi$: that is, F is of isotropic S-curvature.

2) ϕ satisfies (3.14) and

$$(1 + \tau b^2)(2\Psi + T - 2g) - \frac{\Phi}{\Delta^2}(\tau s - Q) = \frac{\eta_a}{s_a}.$$

Moreover, α and β satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_ib_j + \tau(b_is_j + b_js_i), \tag{3.17}$$

where $s_i \neq 0$ and $1 + \tau b^2 \neq 0$, where $\tau = \tau(b^2)$ and $\eta = \eta_i(x)y^i$ is a $1-$ form on M.

3) ϕ satisfies (3.14) and

$$\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T - 2g) = -\frac{\eta_a}{\theta_a}. \tag{3.18}$$

Moreover, α and β satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_ib_j + b_i\theta_j + b_j\theta_i,$$
$$s_i = 0,$$

where $\theta = \theta_i(x)y^i \neq 0$ is a $1-$ form which is orthogonal to β, and $\eta = \eta_i(x)y^i$ is a $1-$ form on M.

Proof. Mainly, the sufficient part of the Proposition follows the proof of Proposition 4.1 in [20]. Thus we omit it. Hence, we just need to prove the necessary part. Suppose that F is of weak isotropic S-curvature, then (3.9), (3.12) and (3.13) hold. (3.12) is equivalent to the following

$$\frac{\Phi}{2\Delta^2}\mathfrak{d}(b^2-s^2)+[(n+1)E+H_2(b^2-s^2)+2sH-2sg](\mathfrak{K}+\mathfrak{d}b^2)=(n+1)\mathfrak{c}\phi,$$

where

$$E:=\frac{\phi_2+2s\phi_1}{2\phi}-\frac{s\phi+(b^2-s^2)\phi_2}{\phi}H,\quad H:=\frac{\phi_{22}-2(\phi_1-s\phi_{12})}{2\{\phi-s\phi_2+(b^2-s^2)\phi_{22}\}}.$$

Let us suppose that $\Xi=(s+b^2Q)\frac{\Phi}{\Delta^2}$ is not constant. Since b is not constant, then we divide (3.9) into two cases:

I) If $\theta=\tau(x)s_0$, then according to $b\neq$ constant we have three possible cases in the following

a) $r_{ij}=\mathfrak{K}a_{ij}+\mathfrak{d}b_ib_j$ and $s_0=0$, where $\mathfrak{K}+\mathfrak{d}b^2\neq 0$.
In this case, $\theta_a=0$ and $s_a=0$. It is easy to see that (3.13) is reduced to $b^2\eta_a=0$. Hence, we have $\eta_a=0$. Thus, by (3.5), we get $\eta=0$ and as a result we have that F is of isotropic S-curvature $\mathbf{S}=(n+1)\mathfrak{c}F$.

b) $r_{ij}=\mathfrak{K}a_{ij}+\mathfrak{d}b_ib_j+\tau(b_is_j+b_js_i)$ and $s_0\neq 0$, where $1+\tau b^2\neq 0$. In that case, $\theta_a=\tau s_a$ and $s_a\neq 0$. (3.13) is reduced to

$$s_a\{(1+b^2\tau)[\frac{s\Phi}{\Delta^2}-b^2(2\Psi+T)+2b^2g]-(s+b^2Q)\frac{\Phi}{\Delta^2}\}+b^2\eta_a=0,$$

which is equivalent to (3.17).

c) $r_{ij}=\mathfrak{K}a_{ij}+\mathfrak{d}b_ib_j-\frac{1}{b^2}(b_is_j+b_js_i)$ and $s_0\neq 0$, where $\mathfrak{K}+\mathfrak{d}b^2\neq 0$. In this case, $\theta_a=-\frac{1}{b^2}s_a$ and $s_a\neq 0$. (3.13) is reduced to

$$-(s+b^2Q)\frac{\Phi}{\Delta^2}s_a+b^2\eta_a=0.$$

In fact, (3.19) implies

$$\frac{\Xi}{b^2}s_a=\eta_a. \tag{3.19}$$

Since $s_a\neq 0$, using the last equation, we obtain that $\frac{\Xi}{b^2}$ is a constant. It is a contradiction. This implies that We need to omit this case.

II) $\theta\neq\tau(x)s_0$.
In this case, $r_{ij}=\mathfrak{K}(x)a_{ij}+\mathfrak{d}(x)b_ib_j+b_i\theta_j+b_j\theta_i$ and $s_0=0$, where $\theta_i\neq 0$. Hence, $\theta_a\neq 0$ and $s_a=0$. Therefore, (3.13) is reduced to

$$[\frac{s\Phi}{\Delta^2}-b^2(2\Psi+T)+2b^2g]b^2\theta_a+b^2\eta_a=0, \tag{3.20}$$

which is equivalent to (3.18).

□

4. The proof of Theorem 1.1

It is sufficient to prove that if F is of weakly isotropic S-curvature, then F is of isotropic S-curvature. In fact, it suffices to show that if (3.19) and (3.20) hold, then $\eta_a=0$.

Firstly, Assume that (3.19) hold. We claim that $\eta_a=0$. Let $\eta_a\neq 0$. By (3.19), we get

$$\frac{s_a}{b^2}\{(1+b^2\tau)[\frac{s\Phi}{\Delta^2}-b^2(2\Psi+T)+2b^2g]-(s+b^2Q)\frac{\Phi}{\Delta^2}\}+\eta_a=0, \tag{4.1}$$

Since $s_a\neq 0$ and b is not a constant, by (4.1) it follows that

$$(1+b^2\tau)[\frac{s\Phi}{\Delta^2}-b^2(2\Psi+T)+2b^2g]-(s+b^2Q)\frac{\Phi}{\Delta^2}=0. \tag{4.2}$$

By (4.1) and (4.2), we obtain $\eta_a = 0$.

Now, suppose that (3.20) hold. Let

$$\Upsilon := \big[\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T)\big]_s.$$

We see that $\Upsilon = 0$ if and only if

$$\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) = b^2\mu,$$

where $\mu = \mu(x)$ is independent of s.

$\Upsilon = 0$:
By (3.20), we get

$$b^2[\mu + 2g]\theta_a + \eta_a = 0. \tag{4.3}$$

Since $\theta_a \neq 0$ and b are not constant, it follows from (4.3) that

$$\mu + 2g = 0.$$

By (4.3), we have $\eta_a = 0$.

$\Upsilon \neq 0$:
By (3.20), we get

$$[\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T)]\theta_a = -2b^2 g\theta_a - \eta_a.$$

By the assumption $\Upsilon \neq 0$, it is fact that this is impossible. It is a contradiction. Hence $\eta_a = 0$.

□

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