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# On generalized Berwald manifolds: extremal compatible linear connections, special metrics and low dimensional spaces

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**ABSTRACT:** The notion of generalized Berwald manifolds goes back to V. Wagner [60]. They are Finsler manifolds admitting linear connections on the base manifold such that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition). Presenting a panoramic view of the general theory we are going to summarize some special problems and results.

Spaces of special metrics are of special interest in the generalized Berwald manifold theory. We discuss the case of generalized Berwald Randers metrics, Finsler surfaces and Finsler manifolds of dimension three.

To provide the unicity of the compatible linear connection we are looking for, we introduce the notion of the extremal compatible linear connection minimizing the norm of the torsion tensor point by point. The mathematical formulation is given in terms of a conditional extremum problem for checking the existence of compatible linear connections in general. Explicite computations are presented in the special case of generalized Berwald Randers metrics.

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(*Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration*)

## 1. A panoramic view - new trends, methods and recent results in the theory of generalized Berwald manifolds

*Finsler geometry is a non-Riemannian geometry in a finite number of dimensions. The differentiable structure is the same as the Riemannian one but distance is not uniform in all directions. Instead of the Euclidean spheres in the tangent spaces, the unit vectors form the boundary of general convex sets containing the origin in their interiors* (M. Berger).

Let $M$ be a (connected) differentiable manifold. The Finsler structure is given by a continuous function $F\colon TM \to \mathbb{R}$ measuring the length of tangent vectors. The Finsler metric satisfies some homogeneity and regularity conditions (smoothness on the complement of the zero section, definiteness, positively homogeneity of degree one, convexity). Convexity (in the tangent spaces) means that the Hessian matrix

$$g_{ij} := \frac{\partial^2 E}{\partial y^j \partial y^i}$$

of the energy function $E := (1/2)F^2$ is positive definite. Such a so-called Riemann-Finsler metric makes each tangent space (except the origin) a Riemannian manifold. In Riemannian geometry, the tangent spaces are Euclidean.

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The paper is devoted to the theory of generalized Berwald manifolds. They are Finsler manifolds admitting linear connections on the base manifold such that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition). The classical case of torsion-free compatible linear connections (classical Berwald manifolds) is a widely investigated area in Finsler geometry dominated by Z. I. Szabó's fameous structure theorem on Berwald manifolds [28]. The successful investigation of the general case needs both the classical results to be generalized and some new methods and ideas to be elaborated. The characteristic feature of the subject matter is a very close interaction between Riemannian and Finsler geometry.

A. Average methods and their applications in Finsler geometry.

- Averaged Riemannian metrics on a Finsler manifold, integration of the Riemann-Finsler metric on the indicatrices [36].
- Averaged Randers metrics and their applications: the generalization of Brickell's theorem [46], analytic properties and the asymptotic behavior of the area function of a Funk metric [51].
- Generalized Berwald manifolds: Finsler manifolds admitting compatible linear connections. The generalization of Szabó's metrizability theorem: any compatible linear connection is a metric linear connection with respect to the averaged Riemannian metric [36].
- The intrinsic characterization of semi-symmetric compatible linear connections on Finsler manifolds: the expression of the one-form in the decomposition formula of the torsion tensor in terms of metrics and differential forms given by averaging [43].
- The conformal invariant characterization of generalized Berwald manifolds [43], see also [48].
- Conformal rigidity. The solution of Matsumoto's problem about conformally equivalent Berwald manifolds: the scale function between conformally equivalent Berwald manifolds must be constant unless they are Riemannian [35], see also [37] and [38].
- Asanov's Finsleroid-Finsler metrics as the solutions of a conformal rigidity problem [49] and Asanov's singular Landsberg spaces as generalized Berwald manifolds with semi-symmetric compatible linear connections [52].

B. Alternatives of Riemannian geometry for metric linear connections with non-transitive (closed) holonomy groups.

- Generalized conics' theory and its applications [41]. Remetrization of a closed non-transitive subgroup $G$ in the orthogonal group: there exists a $G$-invariant convex body (esp. a generalized conic) containing the origin in its interior such that it is not a unit ball with respect to any inner product (ellipsoid-problem) and its boundary is a smooth hypersurface (regularity condition). In the context of Riemannian geometry, $G$ is the closure of the holonomy group of a metric linear connection $\nabla$. Constructing a holonomy-invariant convex body in the tangent space at a single point, we can extend it to the entire manifod by parallel transports with respect to $\nabla$. Such a smoothly varying family of convex bodies constitutes a Finsler metric by the induced Minkowski functionals in the tangent spaces instead of the Riemannian inner products. It is clear that the parallel transports with respect to $\nabla$ preserve the Finslerian length of tangent vectors, i.e. we have a generalized Berwald manifold. Generalized conic bodies (vs. classical conics) appear as indicatrices of generalized Berwald spaces (vs. Riemannian spaces).
- Orbits of Euclidean unit elements under a non-transitive, closed subgroup $G$ in the orthogonal group. Minimax and maximin points: the Hausdorff distance from the unit sphere as the measure of non-transitivity. Flat subspaces and the rank of $G$. The case of maximal rank: reducible or finite subgroups [50].

C. Special Finsler spaces.

- Compatible linear connections in Randers spaces: necessary and sufficient conditions, a structure theorem for Randers spaces admitting semi-symmetric compatible linear connections [39], see also [45].
- Compatible linear connections on Finsler surfaces. The comparison of the compatible linear connection with the Lévi-Civita connection of the averaged Riemannian metric. Flat (averaged) Riemannian metrics and the criteria of the existence: the Gauss curvature as the divergence of a vector field and the differential topology of the base manifold (zero Euler characteristic) [57]. A classical approach: the comparison of the compatible linear connection with the canonical nonlinear connection of the Finsler metric. Landsberg surfaces admitting compatible linear connections are classical Berwald surfaces [54]. Polynomial metrics: necessary and sufficient conditions for a Finsler surface with a symmetric (fourth root) polynomial metric to be a generalized Bewald surface [53].

- Totally anti-symmetric compatible linear connections in three-dimensional Finsler spaces: the expression of the torsion tensor in terms of metrics and differential forms given by averaging. The differential topology of the base manifold (Killing vector fields of constant Riemannian length with respect to the averaged Riemannian metric) [58].

D. The unicity of the compatible linear connection in Finsler spaces: extremal compatible linear connections minimizing the norm of the torsion tensor point by point [55]. Extremal compatible linear connections in Randers spaces [56].

The notion of generalized Berwald manifolds goes back to V. Wagner [60]. They are Finsler manifolds admitting linear connections on the base manifold such that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition). The basic problem is the intrinsic characterization of the compatible linear connections including the problem of the unicity. If the linear connection preserving the Finslerian length of tangent vectors is torsion-free then we have a classical Berwald manifold. The intrinsic characterization is well-known because the compatible torsion-free linear connection coincides the canonical connection of the Finsler manifold. In general the intrinsic characterization of the compatible linear connection is based on the so-called averaged Riemannian metric. It is introduced by choosing the Riemann-Finsler metric to be averaged by integration on the indicatrix hypersurfaces point by point. The key result is that if a linear connection is compatible to the Finslerian metric function then it must be metrical with respect to the averaged Riemannian metric. This means that a compatible linear connection on a Finsler manifold is always Riemann-metrizable [36].

In addition to the averaged Riemannian metric, an important associated object given by averaging on a Finsler manifold is the so-called averaged Randers metric. The integration of the contracted-normalized Riemann-Finsler metric on the indicatrix hypersurfaces constitutes the one-form perturbation of the averaged Riemannian metric point by point [46]. If we have a generalized Berwald manifold, then the averaged Randers metric heritages the compatibility property and the investigation of generalized Berwald Randers metrics is motivated in a more general context as well. Some further applications are presented in case of Funk metrics, where the Finslerian unit balls are given by varying the origin in the interior of a convex body $K \subset \mathbb{R}^n$. It can be proved that the perturbating form is the exterior derivative of the area function up to a constant proportional term. The area function is convex, analytic in the interior of the intersection $K \cap (-K)$ and arbitrary large values can be taken provided that we are close enough to the boundary of $K$, see [51]. It is also possible to extend Brickell's theorem to convex bodies with the origin as the minimizer of the area function of the induced Funk metric [46], see also [51].

Finsler manifolds admitting semi-symmetric compatible linear connections form an important class of generalized Berwald manifolds. This means that the torsion tensor is decomposable in a special way. Involving an exact one-form in the decomposition formula we have a so-called Wagner manifold. Hashiguchi and Ichijyo [19] proved that a Finsler manifold is a Wagner manifold if and only if it is conformal to a classical Berwald manifold. In a more explicit way: the logarithm of the scale function is the primitive function of the exact one-form in the decomposition formula for the torsion tensor of the compatible linear connection. Since the conformality is transitive, the unicity problem of the compatible semi-symmetric linear connections with exact one-form in the torsion tensor is closely related to Matsumoto's problem [23] (2001): are there non-homothetic and non-Riemannian conformally equivalent (classical) Berwald spaces? Generalized Berwald manifolds' theory has lots of contacts with the conformal Finsler geometry:

- the conformally invariant characterization of generalized Berwald manifolds, a conformally invariant linear connection on the base manifold [43], see also [48],
- the solution of Matsumoto's problem shows the conformal rigidity of Berwald manifolds in the sense that the scale function between non-Riemannian Berwald manifolds must be constant [35], see also [37] and [38],
- Asanov's Finsleroid-Finsler metrics as the solutions of a conformal rigidity problem [49]. Asanov's singular Landsberg metrics (Unicorns) as generalized Berwald manifolds with semi-symmetric compatible linear connections [52].

The systematic investigation of conformal rigidity problems in Finsler geometry has been started with the conformally equivalent Berwald manifolds (see Matsumoto's problem). The problem persisted, however, in spite of the efforts being made by the classical approach such as Hashiguchi [17] and [18] (transformation formulas between the canonical objects of conformally related Finsler manifolds), Hashiguchi-Ichijyo [19] (conformally Berwald Finsler manifolds and generalized Berwald manifolds' theory, see Wagner spaces), see also Kikuchi [20] (conformal flatness) etc. The new perspectives of the theory of generalized Berwald manifolds have been supported by the solution of Matsumoto's problem of conformally equivalent Berwald manifolds in 2005: the scale function between non-Riemannian (classical) Berwald manifolds must be constant. The proof is based on metrics and differential forms given by averaging [35], see also [37] and [38].

Using average processes is a new and important trend in Finsler geometry; Cs. Vincze [36], [37], [43] and [46], R. G. Torromé [32], T. Aikou [1], M. Crampin [15] and [16], V. S. Matveev (et. al.: H-B. Rademacher, M. Troyanov) [24], [25] and [26]. For further references see also [42], [44] and [47]. The first successful applications were presented in the theory of generalized Berwald manifolds because the method plays the central role of the intrinsic characterization of semi-symmetric compatible linear connections in general [43]. The basic idea is to provide a Riemannian environment for the investigations by the averaged Riemannian metric: if a linear connection is compatible to the Finslerian metric function then it must be metrical with respect to the averaged Riemannian metric [36]. This is the generalization of Szabó's theorem [28] about the Riemann-metrizability of the compatible torsion-free linear connection of a Berwald manifold.

One can be interested in the inverse problem as well. Taking a metric linear connection $\nabla$ on a Riemannian manifold, it is metrizable by a non-Riemannian (Finslerian) metric function if and only if the closure of the holonomy group is not transitive on the Euclidean spheres in the tangent spaces. The non-Riemannian metric provides the alternative of the Riemannian geometry for such a linear connection. The (non-Riemannian) indicatrix hypersurfaces are given as generalized conics [41]. In case of a Riemannian manifold the indicatrices are conics (quadratic hypersurfaces) in the classical sense.

**Example 1.1.** Let $M$ be a flat compact Riemannian manifold and choose a point $p \in M$. Bieberbach's theorem [14] states that the holonomy group of the Lévi-Civita connection $\nabla$ is finite. Therefore we can find a finite system of elements $v_1, \ldots, v_m$ in the tangent space $T_pM$ which is invariant under the holonomy group $G$ and any polyellipsoid[1] defined by equation

$$\frac{d(w, v_1) + \ldots + d(w, v_m)}{m} = \text{const.}$$

is $G$-invariant. Following Z. I. Szabó's idea [28] we can construct a smoothly varying family of compact convex bodies by parallel transports to provide a Finslerian environment for $\nabla$. The Minkowski functionals induced by the polyellipsoids in the tangent spaces constitute a Finslerian fundamental function such that the parallel transports with respect to the Lévi-Civita connection preserve the Finslerian length of tangent vectors.

In general the holonomy group of a metric linear connection is not finite. To adopt the polyellipsoids to the general situation we should develop the theory of conics with infinitely many focal points [41]. The so-called generalized conics are the level sets of a function measuring the average distance (distance-mean) from the elements of a given set of points (focal set). Using integration over the focal set, generalized conics are limits of polyellipsoids (partitions, integral sums). The details of averaging in Finsler geometry and its applications (generalized Berwald manifold theory, the alternatives of Riemannian geometry, conformal rigidity of Berwald manifolds, Asanov's Finsleroid-Finsler metrics and singular Landsberg metrics, generalized Berwald Randers metrics etc.) has been summarized in the special issue of European Journal of Mathematics devoted to new methods and perspectives in Finsler geometry [48].

In what follows we are going to summarize some special problems and results. Spaces of special metrics are of special interest in the generalized Berwald manifold theory. We summarize the basic facts and recent results on generalized Berwald Randers metrics, Finsler surfaces and Finsler manifolds of dimension three. In addition to the special topics we present a general construction of the extremal compatible linear connection minimizing the norm of the torsion tensor point by point. The idea allows us to formulate a conditional extremum problem for checking the existence of compatible linear connections on a Finsler manifold. It is an intrinsic characterization of generalized Berwald manifolds because we have at most one extremal solution [55]. Explicite computations are presented in the special case of generalized Berwald Randers metrics [56].

## 2. Notation and preliminaries

Let $M$ be a (connected) differentiable manifold with local coordinates $u^1$, ..., $u^n$ on $U \subset M$. The induced coordinate system on the tangent manifold consists of the functions $x^1 := u^1 \circ \pi, \ldots, x^n := u^n \circ \pi$ and $y^1 := du^1, \ldots, y^n = du^n$, where $\pi\colon TM \to M$ is the canonical projection.

### *2.1. Finsler manifolds*

The Finsler structure is given by a continuous function $F\colon TM \to \mathbb{R}$ satisfying the following conditions:

(F1) $F$ is smooth on the complement of the zero section (regularity condition),

[1] Polyellipsoids are given as the level sets of a function measuring the arithmetic mean of distances from finitely many elements $v_1, \ldots, v_m$ (focal points).

(F2) $F$ is positively homogeneous of degree one: $F(tv) = tF(v)$ for all $t > 0$ (positive homogeneity), $F \geq 0$ and $F(v) = 0$ if and only if $v$ is the zero element of its tangent space (definiteness),

(F3) the Hessian matrix

$$g_{ij} := \frac{\partial^2 E}{\partial y^j \partial y^i}$$

of the energy function $E := (1/2)F^2$ with respect to the variables $y^1, \ldots, y^n$ is positive definite at each nonzero element $v \in TM$ (strong convexity).

The so-called *Riemann-Finsler metric* $g$ is constituted by the components $g_{ij}$. It is defined on the complement of the zero section because the second order partial differentiability of the energy function at the origin does not follow automatically: if $E$ is of class $C^2$ on the entire tangent manifold $TM$ then, by the positively homogeneity of degree two, it follows that $E$ is quadratic on each tangent space, i.e. the space is Riemannian. The Riemann-Finsler metric makes each tangent space (except at the origin) a Riemannian manifold with standard canonical objects such as the *volume form*

$$d\mu = \sqrt{\det g_{ij}}\ dy^1 \wedge \ldots \wedge dy^n,$$

the *Liouville vector field*

$$C := y^1 \partial/\partial y^1 + \ldots + y^n \partial/\partial y^n$$

and the *induced volume form on the indicatrix hypersurface* $\partial K_p := F^{-1}(1) \cap T_pM \ \ (p \in M)$. The coordinate expression is

$$\mu = \sqrt{\det g_{ij}}\ \sum_{i=1}^{n} (-1)^{i-1} \frac{y^i}{F} dy^1 \wedge \ldots \wedge dy^{i-1} \wedge dy^{i+1} \ldots \wedge dy^n.$$

The Riemann-Finsler metric $g$ is a natural choice for a metric to be averaged. It was the first appearance of averaging and its applications in Finsler geometry, see [36], [37], [43] and [46]. By formula

$$\gamma(v, w) := \int_{\partial K_p} g(v, w)\, \mu = v^i w^j \int_{\partial K_p} g_{ij}\, \mu \tag{2.1}$$

we have an *averaged Riemannian metric* on the base manifold, where $v, w \in T_pM$. For other candidates of Riemannian metrics given by averaging see [25] (Binet-Legendre metrics) and [16]. The following notation and terminology are also frequently used:

$$l_i = \frac{\partial F}{\partial y^i}, \quad g^{ij} = (g_{ij})^{-1}, \quad \mathcal{C}^l_{ij} = g^{lk} \mathcal{C}_{ijk}, \quad \text{where} \quad \mathcal{C}_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$$

is the so-called *first Cartan tensor*. The first Cartan tensor is totally symmetric and $y^k \mathcal{C}_{ijk} = 0$.

### 2.2. Connections

A *connection on a manifold means to assign the direct complements to the vetrical subspaces* spanned by the coordinate vector fields $\partial/\partial y^1, \ldots, \partial/\partial y^n$. Therefore a connection belongs to the general notion of distributions. In case of a linear connection $\nabla$ let $v \in TM$ be a given nonzero tangent vector and consider the parallel vector field $X$ along the curve $c\colon [0, 1] \to M$ such that $v = X(0)$. Then

$$X'(0) = (x^k \circ X)'(0) \frac{\partial}{\partial x^k}(v) + (y^k \circ X)'(0) \frac{\partial}{\partial y^k}(v),$$

where

$$(x^k \circ X)' = c^{k'} \ \text{ and } \ (y^k \circ X)' = X^{k'} = -c^{i'} X^j \Gamma^k_{ij} \circ c \tag{2.2}$$

because of the differential equation for parallel vector fields. Therefore

$$X'(0) = c^{i'}(0) \left( \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \circ \pi \frac{\partial}{\partial y^k} \right)(v). \tag{2.3}$$

The initial tangent vector (2.3) of the parallel vector field $X$ passing through $v$ as a curve in $TM$ belongs to the horizontal subspace at $v$ and vice versa: they span the horizontal subspace as the initial velocity of the curve $c$ is varying in $T_pM$, where $c(0) = p$ is a given base point of the manifold. Especially, we can define the collection of functions

$$G^k_i = y^j \Gamma^k_{ij} \circ \pi \ \ (i, k = 1, \ldots, n)$$

to express the vector fields

$$X_i^h := \frac{\partial}{\partial x^i} - G_i^k \frac{\partial}{\partial y^k} \quad (i = 1, \dots, n)$$

spanning the *horizontal subspaces.* In the general theory of connections there is no any special way to derive $G_i^k$'s. The horizontal distribution (connection) is *homogeneous* if the functions $G_i^k$'s are positively homogeneous of degree one. It is well-known that if a positively one-homogeneous function is of class $C^1$ on the entire tangent manifold $TM$ then it must be linear on each tangent space, i.e. we have a linear connection. Otherwise the connection is non-linear. The *torsion tensor* and the *mixed curvature tensor* of the horizontal distribution (connection) are defined as

$$T_{ij}^k := \frac{\partial G_i^k}{\partial y^j} - \frac{\partial G_j^k}{\partial y^i} \quad (i, j, k = 1, \dots, n)$$

and

$$P_{ijk}^l = -G_{ijk}^l, \quad \text{where} \quad G_{ijk}^l = \frac{\partial G_{ij}^l}{\partial y^k} \quad \text{and} \quad G_{ij}^l = \frac{\partial G_i^l}{\partial y^j} \quad (i, j, k, l = 1, \dots, n), \tag{2.4}$$

respectively. It is well-known that the vanishing of the torsion tensor is a necessary and sufficient condition for the existence of functions $G^k$'s such that

$$\frac{\partial G^k}{\partial y^i} = G_i^k \qquad (i, k = 1, \dots, n).$$

The vanishing of the mixed curvature tensor is a necessary and sufficient condition for the quantities $G_{ij}^k$'s to depend only on the position. In this case they constitute the coefficients of a linear connection $\nabla$ on the base manifold by the formula $G_{ij}^k = \Gamma_{ij}^k \circ \pi$. Conversely, if the horizontal distribution is generated by a linear connection on the base manifold then it is automatically homogeneous and the mixed curvature tensor is automatically zero. To introduce some further related objects to the horizontal distribution we also need a metric structure. Suppose that a horizontal distribution (connection) is given on a Finsler manifold $M$. It is called *conservative* if the derivatives of the fundamental function $F$ vanish along the horizontal directions: $X_i^h F = 0$ $(i = 1, \dots, n)$. An equivalent statement is to require the vanishing of the derivatives of the Finslerian energy $E = (1/2)F^2$ along the horizontal directions: $X_i^h E = 0$ $(i = 1, \dots, n)$. The *second Cartan tensor* or *Landsberg tensor* is

$$P_{ijk} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} - G_i^l \frac{\partial g_{jk}}{\partial y^l} - G_{ij}^l g_{lk} - G_{ik}^l g_{lj}\right). \tag{2.5}$$

The formula says that it is the Lie-derivative of $g_{ij}$ along the horizontal directions[2] up to the constant proportional term 1/2. The *canonical horizontal distribution* on a Finsler manifold is uniquely determined by the following properties:

(H1) it is conservative, i.e. the derivatives of the Finslerian energy/fundamental function vanish along the horizontal directions:

$$X_i^h E = 0, \quad \text{where} \quad E = (1/2)F^2 \quad \text{and} \quad X_i^h := \frac{\partial}{\partial x^i} - G_i^k \frac{\partial}{\partial y^k} \quad (i = 1, \dots, n),$$

(H2) it is homogeneous, i.e. the functions $G_i^k$'s are positively homogeneous of degree 1: by Euler's formula,

$$y^l \frac{\partial G_i^k}{\partial y^l} = G_i^k \quad (i, k = 1, \dots, n),$$

(H3) it is torsion-free, i.e.

$$\frac{\partial G_i^k}{\partial y^j} - \frac{\partial G_j^k}{\partial y^i} = 0 \quad (i, j, k = 1, \dots, n);$$

(H3) is a necessary and sufficient condition for the existence of functions $G^k$'s such that

$$\frac{\partial G^k}{\partial y^i} = G_i^k \quad (i, k = 1, \dots, n).$$

They are called the *geodesic spray coefficients.* The coordinate expression is

$$G^l = \frac{1}{2} g^{jl} \left( y^m \frac{\partial^2 E}{\partial y^j \partial x^m} - \frac{\partial E}{\partial x^j} \right)$$

[2] The first Cartan tensor $C_{ijk}$ is the Lie-derivative of $g_{ij}$ along the vertical directions up to the constant proportional term 1/2.

and the *canonical horizontal distribution* is (locally) spanned by the vector fields

$$X_i^h = \frac{\partial}{\partial x^i} - G_i^l \frac{\partial}{\partial y^l}, \quad \text{where} \quad G_i^l = \frac{\partial G^l}{\partial y^i}.$$

Some direct computations give that the Landsberg tensor (2.5) and the mixed curvature tensor (2.4) of the canonical horizontal distribution are related as follows:

$$P_{ijk} = -\frac{F}{2} l_m P^m_{ijk}. \tag{2.6}$$

Indeed,

$$Fl_m = \frac{\partial E}{\partial y^m}, \quad \frac{\partial E}{\partial y^m} G^m = \frac{1}{2} y^k \frac{\partial E}{\partial x^k}, \quad g_{mi} G^m = \frac{1}{2}\left( y^k \frac{\partial^2 E}{\partial y^i \partial x^k} - \frac{\partial E}{\partial x^i} \right)$$

and (by the vanishing of the derivatives along the horizontal directions)

$$\frac{\partial}{\partial y^i}\left( \frac{\partial E}{\partial y^m} G^m \right) - g_{mi} G^m = \frac{\partial E}{\partial x^i}.$$

Therefore

$$-Fl_m P^m_{ijk} = \frac{\partial E}{\partial y^m} G^m_{ijk} = \frac{\partial}{\partial y^k}\left( \frac{\partial E}{\partial y^m} G^m_{ij} \right) - g_{mk} G^m_{ij} = \frac{\partial}{\partial y^k}\left( \frac{\partial}{\partial y^j}\left( \frac{\partial E}{\partial y^m} G^m_i \right) - g_{mj} G^m_i \right) -$$

$$g_{mk} G^m_{ij} = \frac{\partial}{\partial y^k}\left( \frac{\partial}{\partial y^j}\left( \frac{\partial}{\partial y^i}\left( \frac{\partial E}{\partial y^m} G^m \right) - g_{mi} G^m \right) - g_{mj} G^m_i \right) - g_{mk} G^m_{ij} =$$

$$\frac{\partial}{\partial y^k}\left( \frac{\partial}{\partial y^j}\left( \frac{\partial E}{\partial x^i} \right) - g_{mj} G^m_i \right) - g_{mk} G^m_{ij} = \frac{\partial g_{jk}}{\partial x^i} - 2\mathcal{C}_{jmk} G^m_i - g_{mj} G^m_{ik} - g_{mk} G^m_{ij} =$$

$$\frac{\partial g_{jk}}{\partial x^i} - G^m_i \frac{\partial g_{jk}}{\partial y^m} - g_{mj} G^m_{ik} - g_{mk} G^m_{ij} = 2P_{ijk}$$

as was to be proved.

### 2.3. Generalized Berwald manifolds

**Definition 2.1.** *A Finsler manifold is called a Landsberg manifold if the Landsberg tensor of the canonical horizontal distribution vanishes. The Berwald manifolds are defined by the vanishing of the mixed curvature tensor of the canonical horizontal distribution.*

Formula (2.6) implies that any Berwald manifold is a Landsberg manifold. The converse of this statement is the fameous Unicorn problem in Finsler geometry [4]. In case of a Berwald manifold the quantities $G^k_{ij}$'s depend only on the position. They constitute the coefficients of a (torsion-free) linear connection $\nabla$ on the base manifold by formula $G^k_{ij} = \Gamma^k_{ij} \circ \pi$. The parallel transports with respect to $\nabla$ preserve the Finslerian length of tangent vectors as the following simple argument shows. Let $X$ be a parallel vector field with respect to $\nabla$ along the curve $c\colon [0,1] \to M$. If $F$ is the Finslerian fundamental function then

$$(F \circ X)' = (x^k \circ X)' \frac{\partial F}{\partial x^k} \circ X + (y^k \circ X)' \frac{\partial F}{\partial y^k} \circ X$$

and, by formula (2.2),

$$(F \circ X)' = c^{i\prime}\left( \frac{\partial F}{\partial x^i} - y^j \Gamma^k_{ij} \circ \pi \frac{\partial F}{\partial y^k} \right) \circ X.$$

This means that the parallel transports with respect to $\nabla$ preserve the Finslerian length of tangent vectors (compatibility condition) if and only if

$$\frac{\partial F}{\partial x^i} - y^j \Gamma^k_{ij} \circ \pi \frac{\partial F}{\partial y^k} = 0 \quad (i = 1, \ldots, n), \tag{2.7}$$

where the vector fields

$$\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \circ \pi \frac{\partial}{\partial y^k}$$

span the associated horizontal distribution belonging to $\nabla$.

**Definition 2.2.** *A linear connection $\nabla$ on the base manifold $M$ is called* compatible *to the Finslerian metric if the parallel transports with respect to $\nabla$ preserve the Finslerian length of tangent vectors. Finsler manifolds admitting compatible linear connections are called generalized Berwald manifolds.*

**Corollary 2.3.** *A linear connection $\nabla$ on the base manifold $M$ is compatible to the Finslerian metric function if and only if the induced horizontal distribution is conservative, i.e. the derivatives of the fundamental function $F$ vanish along the horizontal directions with respect to $\nabla$.*

In case of a classical Berwald manifold (with a torsion-free compatible linear connection) the intrinsic characterization is the vanishing of the mixed curvature tensor of the canonical horizontal distribution. In general the intrinsic characterization of the compatible linear connection is based on the averaged Riemannian metric because of the following basic result.

**Theorem 2.4.** [36] *If a linear connection on the base manifold is compatible with the Finslerian metric function then it must be metrical with respect to the averaged Riemannian metric $\gamma$.*

## 3. Generalized Berwald Randers metrics

In this section we formulate a necessary and sufficient condition for a Randers manifold to be a generalized Berwald manifold. Especially, we present a structure theorem for Randers manifolds admitting compatible semi-symmetric linear connections. Let $(M,\alpha)$ be a connected Riemannian manifold and suppose that the one-form $\beta$ in $\wedge^1(M)$ satisfies condition

$$\sup_{\alpha(v,v)=1} \beta(v) < 1.$$

The*Randers functional* on the manifold $M$ is defined as

$$F(v) = \sqrt{\alpha(v,v)} + \beta(v) \tag{3.1}$$

and the pair $(M,F)$ is called a *Randers manifold* with perturbating term $\beta$. In the sense of Definition 2 a Randers manifold is a generalized Berwald manifold if there exists a linear connection $\nabla$ on the base manifold $M$ such that the functional $F$ is invariant under the parallel transports with respect to $\nabla$. In other words, the parallel transports preserve the length of tangent vectors with respect to the Randers functional (3.1).

**Theorem 3.1.** [45] *A Randers manifold is a generalized Berwald manifold if and only if there exists a linear connection $\nabla$ on the manifold $M$ such that $\nabla\alpha = 0$ and $\nabla\beta = 0$.*

The proof is based on the linar "connection" between the tangent spaces. Changing the role of the transleted vectors $\pm v$, the invariance of the Randers functional under the parallel transports implies that $\nabla\alpha = 0$ and $\nabla\beta = 0$. The converse statement is trivial. The following theorem formulates a necessary and sufficient condition for a Randers manifold to be a generalized Berwald manifold in terms of the dual vector field

$$\alpha(\beta^\sharp, X) = \beta(X)$$

of the perturbating term.

**Theorem 3.2.** [45] *A Randers manifold is a generalized Berwald manifold if and only if $\beta^\sharp$ is of constant Riemannian length.*

One of the compatible linear connectons on a generalized Berwald Randers manifold is

$$\nabla_X Y = \nabla^*_X Y + A(X,Y), \quad A(X,Y) = \frac{\alpha(\nabla^*_X\beta^\sharp, Y)\beta^\sharp - \alpha(Y,\beta^\sharp)\nabla^*_X\beta^\sharp}{K^2}, \tag{3.2}$$

where $\beta^\sharp$ is of constant Riemannian length $K$ and $\nabla^*$ is the Lévi-Civita connection of $\alpha$. Randers metrics belong to the more general concept of $(\alpha,\beta)$-metrics, where $\alpha(v) := \sqrt{\alpha(v,v)}$ and $\beta$ is a one-form on the base manifold with bounded supremum norm

$$\sup_{\alpha(v,v)=1} \beta(v) < b_0 < \infty.$$

The general form of the fundamental function is $F(\alpha,\beta) = \alpha\Phi(\beta/\alpha)$, where $\Phi$ is a positive valued smooth function defined on the open interval $(-b_0,b_0)$ satisfying the following regularity property: for any $b \in (-b_0,b_0)\cap\mathbb{R}^+$

$$\Phi(s) - s\Phi'(s) + (b^2 - s^2)\Phi''(s) > 0 \quad (-b \le s \le b).$$

Following the steps of [45], Theorem 3.1 and Theorem 3.2 have been generalized by Tayebi and Barzegari [31] for $(\alpha, \beta)$-metrics satisfying the sign property:

$$A_\Phi(s) := \Phi'(-s)\Phi(s) + \Phi(-s)\Phi'(s) \text{ has a fixed sign.}$$

**Remark 3.3.** The sign property means that $\Phi(-s)/\Phi(s)$ is a strictly monotone function.

**Theorem 3.4.** [31] *An $(\alpha, \beta)$-metric satisfying the sign property is a generalized Berwald manifold if and only if there exists a linear connection $\nabla$ on the manifold $M$ such that $\nabla\alpha = 0$ and $\nabla\beta = 0$.*

**Theorem 3.5.** [31] *An $(\alpha, \beta)$-metric satisfying the sign property is a generalized Berwald manifold if and only if $\beta^\sharp$ is of constant Riemannian length.*

The following results show that the sign property can be weakened as

$$A_\Phi(0) \neq 0 \quad \Leftrightarrow \quad \Phi'(0) \neq 0.$$

**Theorem 3.6.** [48] *An $(\alpha, \beta)$-metric satisfying the regularity property $\Phi'(0) \neq 0$ is a generalized Berwald manifold if and only if there exists a linear connection $\nabla$ on the manifold $M$ such that $\nabla\alpha = 0$ and $\nabla\beta = 0$.*

**Theorem 3.7.** [48] *An $(\alpha, \beta)$-metric satisfying the regularity property $\Phi'(0) \neq 0$ is a generalized Berwald manifold if and only if $\beta^\sharp$ is of constant Riemannian length.*

### *3.1. Randers manifolds with semi-symmetric compatible linear connections*

In what follows we present a structure theorem for Randers manifolds with semi-symmetric compatible linear connections. This means that the torsion tensor is decomposable in the following way:

$$T(X, Y) = \lambda(Y)X - \lambda(X)Y, \tag{3.3}$$

where $\lambda$ is a 1-form on the manifold. Since such a compatible linear connection can be expressed in terms of metrics and differential forms given by averaging [43] it must be uniquely determined. If $\lambda$ is exact, i.e. $\lambda$ is the exterior derivative of a globally well-defined smooth function then we have a (exact) Wagner manifold. For an existence theorem of Wagner manifolds we can refer to [39]. Furthermore, [45] contains the characterization of the local structure of Riemannian manifolds admitting a one-form perturbation such that the Randers manifold is a generalized Berwald manifold with a semi-symmetric compatible linear connection without any special requirements (exactness or closedness) for $\lambda$ in (3.3).

**Theorem 3.8.** [45] *Let $M$ be a connected differentiable manifold equipped with the Randers functional*

$$F(v) = \sqrt{\alpha(v, v)} + \beta(v),$$

*where $\beta$ in $\wedge^1(M)$ is a non-trivial 1-form. If there exists a compatible semi-symmetric linear connection $\nabla$ with torsion* (3.3) *then $M$ is locally isometric to the product manifold $N \times \mathbb{R}$ equipped with the Riemannian metric*

$$e^{C(q,t)}\alpha_q(T\pi(w), T\pi(w)) + \frac{\beta_{(q,t)}(w)\beta_{(q,t)}(w)}{K^2},$$

*where $\alpha$ is a Riemannian metric on the manifold $N$, $C(q, 0) = 0$ for any $q \in N$, $\pi \colon N \times \mathbb{R} \to N$ is the canonical projection, $K < 1$ is a positive constant satisfying*

$$\beta\left(\frac{\partial}{\partial x^n}\right) = K^2,$$

*where the vector field $\partial/\partial x^n$ acts as differentiation of functions on the product manifold with respect to the last coordinate and the pointwise kernels of the 1-form $\beta$ constitute a smooth $(n-1)$-dimensional integrable distribution such that $N$ is a maximal integral manifold. Conversely, the one-form perturbation*

$$F(w) = \sqrt{\alpha(w, w)} + \beta(w)$$

*of such a Riemannan metric admits a uniquely determined compatible semi-symmetric linear connection with torsion tensor*

$$T(X, Y) = \lambda(Y)X - \lambda(X)Y, \quad \textit{where} \quad \lambda^\sharp = \frac{\nabla^*_{\beta^\sharp}\beta^\sharp - f\beta^\sharp}{K^2} \quad \textit{and} \quad f = \frac{div\ \beta^\sharp}{n-1}.$$

The method of finding the local structure of Randers manifolds admitting compatible semi-symmetric linear connections is based on the conformality of the dual vector field $\beta^\sharp$ with respect to the Lorentzian metric

$$\omega(X,Y)=\alpha(X,Y)-\frac{\beta(X)\beta(Y)}{K^2},$$

where $K<1$ is the (positive) constant Riemannian length of the dual vector field. The conformality means that

$$\mathcal{L}_{\beta^\sharp}\omega=2f\omega, \quad \text{where} \quad f=\frac{\text{div } \beta^\sharp}{n-1}.$$

Therefore

$$\mathcal{L}_{\beta^\sharp}\alpha=2f\left(\alpha-\frac{\beta\otimes\beta}{K^2}\right)+\frac{\left(\mathcal{L}_{\beta^\sharp}\beta\right)\otimes\beta+\beta\otimes\left(\mathcal{L}_{\beta^\sharp}\beta\right)}{K^2}. \tag{3.4}$$

Equation (3.4) is a first order differential equation for the Riemannian metric tensor along the integral curves of $\beta^\sharp$. To prove the converse of the statement we also need formula

$$(d\beta)(X,Y)=\frac{\beta(X)\left(\mathcal{L}_{\beta^\sharp}\beta\right)(Y)-\beta(Y)\left(\mathcal{L}_{\beta^\sharp}\beta\right)(X)}{K^2}. \tag{3.5}$$

Remarkable that the special form (3.5) of the exterior derivative of $\beta$ is equivalent to the following property: the pointwise kernels constitute a smooth $(n-1)$-dimensional integrable distribution and $\beta^\sharp$ is of (positive) constant Riemannian length [45]. Equations (3.4) and (3.5) determines the covariant derivative of the dual vector field $\beta^\sharp$ with respect to the Lévi-Civita connection because of the general formulas

$$(d\beta)(X,Y)=\alpha(\nabla^*_X\beta^\sharp,Y)-\alpha(\nabla^*_Y\beta^\sharp,X)$$

and

$$\left(\mathcal{L}_{\beta^\sharp}\alpha\right)(X,Y)=\alpha(\nabla^*_X\beta^\sharp,Y)+\alpha(\nabla^*_Y\beta^\sharp,X).$$

Explicitly,

$$\alpha(\nabla^*_X\beta^\sharp,Y)=\frac{(d\beta)(X,Y)+\left(\mathcal{L}_{\beta^\sharp}\alpha\right)(X,Y)}{2}.$$

Computing $\nabla^*_X\beta^\sharp$ by (3.4) and (3.5), we can directly check that $\nabla\beta=0$, where $\nabla$ is the only metric linear connection with torsion given by

$$T(X,Y)=\lambda(Y)X-\lambda(X)Y, \quad \text{where} \quad \lambda^\sharp=\frac{\nabla^*_{\beta^\sharp}\beta^\sharp-f\beta^\sharp}{K^2} \quad \text{and} \quad f=\frac{\text{div } \beta^\sharp}{n-1}.$$

According to Theorem 3.1, $\nabla$ is a compatible semi-symmetric linear connection. To simplify the situation we can consider the case of $d\beta=0$ which means that (3.4) is reduced to

$$\mathcal{L}_{\beta^\sharp}\alpha=2f\left(\alpha-\frac{\beta\otimes\beta}{K^2}\right).$$

We can also reduce the number of free objects by the choice $\lambda=-\beta$. The special problem has been solved in [39] as an existence theorem of Wagner manifolds - for the terminology see [6]. Vincze [45] presents the generalizations of the results in [39].

## 4. Finsler surfaces

By the fundamental result of the theory [36], a compatible linear connection must be metrical with respect to the averaged Riemannian metric given by integration of the Riemann-Finsler metric on the indicatrix hypersurfaces. Therefore the linear connection is uniquely determined by its torsion tensor. The torsion tensor has a special decomposition in 2D because of

$$T(X,Y)=\left(X^1Y^2-X^2Y^1\right)\left(T^1_{12}\frac{\partial}{\partial u^1}+T^2_{12}\frac{\partial}{\partial u^2}\right)=\rho(X)Y-\rho(Y)X, \tag{4.1}$$

where $\rho_1=T^2_{12}$ and $\rho_2=-T^1_{12}=T^1_{21}$. In higher dimensional spaces such a linear connection is called semi-symmetric. Using some previous results [37], [40], [43] and [48], the torsion tensor of a semi-symmetric compatible linear connection can be expressed in terms of metrics and differential forms given by averaging independently of the dimension of the space, but the compatible linear connection must be of zero curvature in 2D unless the manifold is Riemannian. Therefore we can conclude some topological obstructions because the existence of a metric linear connection of zero curvature is equivalent to the divergence representation of the Gauss curvature of the Riemannian surface. We can prove, for example, that any compact generalized Berwald surface without boundary must have zero Euler characteristic. Therefore the Euclidean sphere does not carry such a geometric structure. An important consequence is that the local conformal flatness is taking to fail in the non-Riemannian differential geometry of surfaces. In some further representative cases (Euclidean plane, hyperbolic plane etc.) we can solve the differential equation of the parallel vector fields to present explicit examples of non-Riemannian two-dimensional generalized Berwald manifolds as well, for details see [57].

### 4.1. The divergence representation of the Gauss curvature

Let $\nabla$ be a linear connection on the base manifold $M$ of dimension 2 and suppose that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition). Theorem 2.4 implies that it is uniquely determined by its torsion tensor of the form

$$T(X,Y)=\rho(X)Y-\rho(Y)X; \tag{4.2}$$

see Formula (4.1). The comparison of $\nabla$ with the Lévi-Civita connection $\nabla^*$ associated with the averaged Riemannian metric (2.1) solves the problem of the intrinsic characterization of the semi-symmetric compatible linear connections for both lower and higher dimensional spaces. The solution is the expression of the 1-form $\rho$ in terms of the canonical data (metrics and differential forms given by averaging) of the Finsler manifold, see [37], [40], [43] and [48]. Let a point $p\in M$ be given and consider the orthogonal group with respect to the averaged Riemannian metric. It is clear that the subgroup $G\subset O(2)$ of the orthogonal transformations leaving the Finslerian indicatrix invariant is finite unless the Finsler surface reduces to a Riemannian one[3], for a more general context of the problem see [50]. If $\nabla$ is a linear connection on the base manifold such that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition) then, by Theorem 2.4, Hol $\nabla\subset G$ is also finite for any $p\in M$ and the curvature tensor of $\nabla$ is zero. We can compute the relation between the curvatures of $\nabla$ and $\nabla^*$, for details see [57]. Taking vector fields with pairwise vanishing Lie brackets on the neighbourhood $U$ of the base manifold, the Christoffel process implies that

$$\gamma(\nabla^*_XY,Z)=\gamma(\nabla_XY,Z)+\frac{1}{2}\left(\gamma(X,T(Y,Z))+\gamma(Y,T(X,Z))-\gamma(Z,T(X,Y))\right),$$

where $\nabla^*$ denotes the Lévi-Civita connection. Since the torsion tensor is of the form (4.2), we have that

$$\nabla^*_XY=\nabla_XY+\rho(Y)X-\gamma(X,Y)\rho^\sharp \quad\Rightarrow\quad \nabla_XY=\nabla^*_XY-\rho(Y)X+\gamma(X,Y)\rho^\sharp,$$

where $\rho^\sharp$ is the dual vector field of $\rho$ defined by $\gamma(\rho^\sharp,X)=\rho(X)$. Some further direct computations show that

$$R(X,Y)Z=R^*(X,Y)Z+$$
$$\Big(\gamma(X,Z)\|\rho^\sharp\|^2-\rho(X)\rho(Z)-\left(\nabla^*_X\rho\right)(Z)\Big)Y+\gamma(Y,Z)\nabla^*_X\rho^\sharp+\gamma(Y,Z)\rho(X)\rho^\sharp+$$
$$\Big(\left(\nabla^*_Y\rho\right)(Z)+\rho(Y)\rho(Z)-\gamma(Y,Z)\|\rho^\sharp\|^2\Big)X-\gamma(X,Z)\nabla^*_Y\rho^\sharp-\gamma(X,Z)\rho(Y)\rho^\sharp.$$

Since the holonomy group of $\nabla$ must be finite in case of a non-Riemannian generalized Berwald surface, we have that $R(X,Y)Z=0$. Taking an orthonormal frame $\gamma(X,Y)=0$, $\gamma(X,X)=\gamma(Y,Y)=1$ at the point $p\in M$ it follows that

$$0=\gamma(R^*(X,Y)Y,X)+\rho^2(X)+\rho^2(Y)-\|\rho^\sharp\|^2+\gamma\left(\nabla^*_X\rho^\sharp,X\right)+\left(\nabla^*_Y\rho\right)(Y),$$

where $\rho^2(X)+\rho^2(Y)-\|\rho^\sharp\|^2=0$ and

$$\left(\nabla^*_Y\rho\right)(Y)=Y\rho(Y)-\rho(\nabla^*_YY)=Y\gamma(\rho^\sharp,Y)-\rho(\nabla^*_YY)=\gamma(\nabla^*_Y\rho^\sharp,Y).$$

Therefore

$$0=\kappa^*(p)+\mathrm{div}^*\rho^\sharp(p)\quad\Rightarrow\quad\kappa^*=-\mathrm{div}^*\rho^\sharp, \tag{4.3}$$

[3]The subgroup $G\subset O(2)$ of the orthogonal transformations leaving the Finslerian indicatrix invariant is obviously compact and, by the closed-subgroup theorem, it is an embedded Lie group of dimension 0 or 1. In case of dimenson 1 the Finslerian indicatrix at $p\in M$ must be invariant under the subgroup $SO(2)$. Therefore it is a quadric. Such a (connected) generalized Berwald surface reduces to a Riemannian surface because we have linear parallel transports between the tangent spaces.

where $\kappa^*$ is the Gauss curvature of the manifold with respect to the averaged Riemannian metric and

$$\operatorname{div}^*\rho^\sharp := \gamma\left(\nabla^*_X\rho^\sharp, X\right) + \gamma\left(\nabla^*_Y\rho^\sharp, Y\right)$$

is the divergence operator. Equation (4.3) is called *the divergence representation of the Gauss curvature.*

**Corollary 4.1.** [57] *A Riemannian surface admits a metric linear connection of zero curvature if and only if its Gauss curvature can be represented as a divergence of a vector field.*

**Corollary 4.2.** [57] *If $M$ is a compact generalized Berwald surface without boundary then it must have zero Euler characteristic.*

**Proof.** Taking the integral of the divergence representation (4.3) we have the zero Euler characteristic due to the Gauss-Bonnet theorem and the divergence theorem.

□

**Corollary 4.3.** [57] *A two-dimensional Euclidean sphere could not carry Finslerian structures admitting compatible linear connections.*

### *4.2. Exact and closed Wagner surfaces*

It is well-known that any Riemannian surface is locally conformally flat. Its Finslerian analogue is that *any non-Riemannian Finsler surface is locally conformal to a locally Minkowski manifold of dimension* 2; a locally Minkowski manifold is a Berwald manifold (torsion-free case, i.e. the compatible linear connection is $\nabla^*$) such that $R^* = 0$. The solution of the so-called Matsumoto's problem in [37], see also [38], proves that the statement is false in the non-Riemannian Finsler geometry.

**Definition 4.4.** *Generalized Berwald manifolds admitting compatible semi-symmetric linear connections with an exact* 1*-form $\rho$ in the torsion* (4.2) *are called exact Wagner manifolds. Generalized Berwald manifolds admitting compatible semi-symmetric linear connections with a closed* 1*-form $\rho$ in the torsion* (4.2) *are called closed Wagner manifolds.*

By Hashiguchi and Ichyjio's classical theorem [19], see also [33] and [34], a Finsler manifold is a conformally Berwald manifold if and only if there exists a semi-symmetric compatible linear connection with an exact 1-form $\rho$ in the torsion (4.2). Especially, it is the exterior derivative of the logarithmic scale function $\alpha$ between the (conformally related) Finslerian fundamental functions $\tilde{F} = e^{\alpha\circ\pi}F$ up to a minus sign. The generalization of Hashiguchi and Ichyjio's classical theorem for closed Wagner manifolds is the statement that *a Finsler manifold is a locally conformally Berwald manifold if and only if it is a closed Wagner manifold.* It is clear from the global version of the theorem that any point of a closed Wagner manifold has a neighbourhood over which it is conformally equivalent to a Berwald manifold, i.e. any closed Wagner manifold is a locally conformally Berwald manifold. What about the converse? Suppose that we have a locally conformally Berwald manifold. The exterior derivatives of the local scale functions constitute a globally well-defined closed 1-form for the torsion (4.2) of a compatible linear connection if and only if they coincide on the intersections of overlapping neighbourhoods. Since the conformal equivalence is transitive it follows that overlapping neighbourhoods carry conformally equivalent Berwald metrics. The problem posed by M. Matsumoto [23] in 2001 is that *are there non-homothetic and non-Riemannian conformally equivalent Berwald spaces*? It has been completely solved by Vincze [37] in 2005, see also [38].

**Theorem 4.5.** [37], see also [38] *The scale function between conformally equivalent Berwald manifolds must be constant unless they are Riemannian.*

**Corollary 4.6.** *A Finsler manifold is a locally conformally Berwald manifold if and only if it is a closed Wagner manifold.*

Using Corollaries 4.3 and 4.6 we have the following result.

**Corollary 4.7.** *A two-dimensional Euclidean sphere could not carry non-Riemannian locally conformally Berwald Finslerian structures. Especially, it can not be a locally conformally flat non-Riemmanian Finsler manifold.*

By the classification of orientable compact surfaces without boundary we can also state that they could not carry Finslerian structures admitting compatible linear connections except the case of genus 1. What about the generic case of the tori $S^1 \times S^1$? It is known [4] that on a compact Riemannian manifold $M$ if $\kappa^*$ is a smooth function of integral zero then there is a smooth solution of the equation $\Delta^*\varphi = \kappa^*$, unique up to the addition of a constant. Therefore $X := -\text{grad}\,\varphi$ is the solution of the divergence representation problem (4.3) of the Gauss curvature. Using formula

$$\text{div}^*(X) = \frac{1}{\sqrt{\det \gamma_{ij}}}\left[\frac{\partial\left(\sqrt{\det \gamma_{ij}}X^1\right)}{\partial u^1} + \frac{\partial\left(\sqrt{\det \gamma_{ij}}X^2\right)}{\partial u^2}\right]$$

it follows that

$$X^2(u^1,u^2) =$$

$$-\frac{1}{\sqrt{\det \gamma_{ij}(u^1,u^2)}}\left(\int_0^{u^2} \kappa^*(u^1,t)\sqrt{\det \gamma_{ij}(u^1,t)} + \frac{\partial\left(\sqrt{\det \gamma_{ij}}X^1\right)}{\partial u^1}(u^1,t)\,dt + c(u^1) + c_0\right)$$

gives the solution of the divergence representation (4.3) of the Gauss curvature in case of the manifolds $\mathbb{R}^2$ and $S^1\times\mathbb{R}$. For more explicit solutions of the divergence representation problem of the curvature in case of the Euclidean and hyperbolic planes see [57]. The divergence representation of the Gauss curvature allows us to introduce the metric linear connection $\nabla$ of torsion (4.2). Since the holonomy group is trivial (at least locally), it is enough to substitute the Riemannian indicatrix with a more general convex closed curve containing the origin in its interior at a single point. Using parallel translations for the extension, such a smoothly varying family of curves provides an alternative way of measuring the length of tangent vectors and we have a non-Riemannian generalized Berwald manifold with $\nabla$ as the compatible linear connection.

### 4.3. Polynomial metrics

Let $m = 2l$ be a positive natural number, $l = 1, 2, \ldots$. A Finslerian metric $F$ is called an $m$-th *root metric* if its $m$-th power $F^m$ is of class $C^m$ on the tangent manifold $TM$. Using homogenity properties, the local expression of an $m$-th root metric is a polynomial of degree $m$ in the variables $y^1, \ldots, y^n$, where $\dim M = n$. $F$ is locally symmetric if each point has a coordinate neighbourhood such that $F^m$ is a symmetric polynomial of degree $m$ in the variables $y^1, \ldots, y^n$ of the induced coordinate system on the tangent manifold. According to the fundamental theorem of symmetric polynomials, the computational processes become more effective and simple. In what follows we present some general observations about locally symmetric $m$-th root metrics. Especially, we are interested in generalized Berwald surfaces with locally symmetric fourth root metrics [53].

**Definition 4.8.** *Let $m = 2l$ be a positive natural number, $l = 1, 2, \ldots$. A Finslerian metric $F$ is called an $m$-th root metric if its $m$-th power $F^m$ is of class $C^m$ on the tangent manifold $TM$.*

Using that $F$ is positively homogeneous of degree one, its $m$-th power is homogeneous of degree $m$. Since it is of class $C^m$ on the tangent manifold $TM$ (including the zero section), its local form must be a polynomial of degree $m$ in the variables $y^1, \ldots, y^n$ as follows:

$$F^m(x,y) = \sum_{i_1+\ldots+i_n=m} a_{i_1\ldots i_n}(x)(y^1)^{i_1}\cdot\ldots\cdot(y^n)^{i_n}. \tag{4.4}$$

Finslerian metrics of the form (4.4) has been introduced by Shimada [27]. They are generalizations of the so-called Berwald-Moór metrics. The geometry of the $m$-th root metrics and some special cases have been investigated by several authors such as M. Matsumoto, K. Okubo, V. Balan, N. Brinzei, L. Tamássy, A. Tayebi and B. Najafi etc. in [2], [3], [13], [21], [29] and [30].

**Example 4.1.** *Riemannian metrics are 2nd root metrics, i.e. $m = 2$.*

**Definition 4.9.** [53] *$F$ is locally symmetric if each point has a coordinate neighbourhood such that $F^m$ is a symmetric polynomial of degree m in the variables $y^1, \ldots, y^n$ of the induced coordinate system on the tangent manifold.*

Suppose that formula (4.4) is a symmetric expression of $F(x,y)$ in the variables $y^1, \ldots, y^n$. Using the fundamental theorem of symmetric polynomials, we can write that

$$F^m(x,y) = P(s^1,\ldots,s^n),$$

[4] S. Donaldson, Geometric Analysis Lecture Notes, http://wwwf.imperial.ac.uk/~skdona/GEOMETRICANALYSIS.PDF

where

$$s^1 = y^1 + \ldots y^n,\ s^2 = y^1y^2 + \ldots + y^{n-1}y^n, \ldots, s^n = y^1 \cdot \ldots \cdot y^n$$

are the so-called elementary symmetric polynomials. The polynomial $P$ with coefficients depending on the position is called the *local characteristic polinomial* of the locally symmetric $m$-th root metric. Using homogenity properties, the reduction of the number of the coefficients depending on the position is

$$F^m(x,y) = \sum_{j_1+2j_2+\ldots+nj_n=m} c_{j_1\ldots j_n}(x)(s^1)^{j_1} \cdot \ldots (s^n)^{j_n}.$$

If $n = 2, 3, 4$ and $m = 4$ (fourth root metrics), the corresponding local characteristic polynomials are of the form

$$P(s^1, s^2) = c_{40}(x)(s^1)^4 + c_{21}(x)(s^1)^2s^2 + c_{02}(x)(s^2)^2,$$
$$P(s^1, s^2, s^3) = c_{400}(x)(s^1)^4 + c_{210}(x)(s^1)^2s^2 + c_{020}(x)(s^2)^2 + c_{101}(x)s^1s^3,$$
$$P(s^1, s^2, s^3, s^4) = c_{4000}(x)(s^1)^4 + c_{2100}(x)(s^1)^2s^2 + c_{0200}(x)(s^2)^2 + c_{1010}(x)s^1s^3 + c_{0001}(x)s^4,$$

respectively *[53]*.

**Corollary 4.10.** *[53] A locally symmetric fourth root metric is locally determined by at most five components of its local characteristic polynomial.*

### *4.4. Generalized Berwald surfaces with locally symmetric 4-rooth metrics*

Introducing the quantities

$$A := F^m(x,y) = \sum_{i_1+\ldots+i_n=m} a_{i_1\ldots i_n}(x)(y^1)^{i_1} \cdot \ldots (y^n)^{i_n}, \quad A_i := \frac{\partial A}{\partial y^i} \text{ and } A_{ij} := \frac{\partial^2 A}{\partial y^i \partial y^j},$$

it is known that the positive definiteness of the Hessian of the energy function with respect to the directional derivatives is equivalent to the positive definiteness of $A_{ij}$. Let $M$ be a two-dimensional Finsler manifold (Finsler surface) with a locally symmetric fourth root metric $F = \sqrt[4]{A}$. Its local characteristic polynomial must be of the form

$$P(s^1, s^2) = A(x,y) = a(x)(y^1 + y^2)^4 + b(x)(y^1 + y^2)^2y^1y^2 + c(x)(y^1y^2)^2, \tag{4.5}$$

where $a(x) = c_{40}(x)$, $b(x) = c_{21}(x)$ and $c(x) = c_{02}(x)$. Differentiating (4.5)

$$A_1 = \frac{\partial A}{\partial y^1} = 4a(x)(y^1 + y^2)^3 + 2b(x)(y^1 + y^2)y^1y^2 + b(x)(y^1 + y^2)^2y^2 + 2c(x)y^1(y^2)^2,$$
$$A_2 = \frac{\partial A}{\partial y^2} = 4a(x)(y^1 + y^2)^3 + 2b(x)(y^1 + y^2)y^1y^2 + b(x)(y^1 + y^2)^2y^1 + 2c(x)y^2(y^1)^2.$$

By some further computations

$$A_{11} = 12a(x)(y^1)^2 + (24a(x) + 6b(x))y^1y^2 + (12a(x) + 4b(x) + 2c(x))(y^2)^2,$$
$$A_{12} = A_{21} = (12a(x) + 3b(x))(y^1)^2 + (24a(x) + 8b(x) + 4c(x))y^1y^2 + (12a(x) + 3b(x))(y^2)^2,$$
$$A_{22} = (12a(x) + 4b(x) + 2c(x))(y^1)^2 + (24a(x) + 6b(x))y^1y^2 + 12a(x)(y^2)^2.$$

Introducing the functions

$$l(x) := a(x),\ m(x) := 4a(x) + b(x),\ n(x) := 6a(x) + 2b(x) + c(x),$$

we have that

$$A(x,y) = l(x)(y^1)^4 + m(x)(y^1)^3y^2 + n(x)(y^1)^2(y^2)^2 + m(x)y^1(y^2)^3 + l(y^2)^4, \tag{4.6}$$

$$\frac{\partial A}{\partial y^1} = 4l(x)(y^1)^3 + 3m(x)(y^1)^2y^2 + 2n(x)y^1(y^2)^2 + m(x)(y^2)^3, \tag{4.7}$$

$$\frac{\partial A}{\partial y^2} = m(x)(y^1)^3 + 2n(x)(y^1)^2y^2 + 3m(x)y^1(y^2)^2 + 4l(x)(y^2)^3. \tag{4.8}$$

Since

$$\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 6 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

is a regular linear transformation, the coefficients $a(x), b(x)$, $c(x)$ are uniquely determined by $l(x)$, $m(x)$, $n(x)$ and vice versa. We have

$$A_{11} = \begin{bmatrix} y^1 & y^2 \end{bmatrix} \begin{bmatrix} 12l & 3m \\ 3m & 2n \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}, \ A_{12} = A_{21} = \begin{bmatrix} y^1 & y^2 \end{bmatrix} \begin{bmatrix} 3m & 2n \\ 2n & 3m \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} y^1 & y^2 \end{bmatrix} \begin{bmatrix} 2n & 3m \\ 3m & 12l \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}.$$

Therefore $A_{ij} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is positive definite if and only if

$$\begin{bmatrix} 12\,l & 3\,m \\ 3\,m & 2\,n \end{bmatrix} \text{ and } \begin{bmatrix} 12\,l & 3\,m \\ 3\,m & 2\,n \end{bmatrix} \begin{bmatrix} 2\,n & 3\,m \\ 3\,m & 12\,l \end{bmatrix} - \begin{bmatrix} 3\,m & 2\,n \\ 2\,n & 3\,m \end{bmatrix}^2$$

are positive definite. Using some direct computations

$$\begin{bmatrix} 12\,l & 3\,m \\ 3\,m & 2\,n \end{bmatrix} \begin{bmatrix} 2\,n & 3\,m \\ 3\,m & 12\,l \end{bmatrix} - \begin{bmatrix} 3\,m & 2\,n \\ 2\,n & 3\,m \end{bmatrix}^2 = \begin{bmatrix} 24\,nl - 4\,n^2 & 72\,ml - 12\,nm \\ 0 & 24\,nl - 4\,n^2 \end{bmatrix}$$

and, consequently,

$$12l > 0, \quad 24ln - 9m^2 > 0 \text{ and } 24nl - 4n^2 > 0. \tag{4.9}$$

Especially, (4.9) is equivalent to

$$6l > n > 0 \text{ and } \frac{8}{3}nl > m^2,$$

for details see [53]. Let $\nabla$ be a linear connection on the base manifold $M$ equipped with a locally symmetric fourth root metric $F = \sqrt[4]{A}$ and suppose that the parallel transports preserve the Finslerian length of tangent vectors. The compatibility condition (2.7) can be written into the form

$$\frac{\partial A}{\partial x^i} - y^j \Gamma^k_{ij}(x) \frac{\partial A}{\partial y^k} = 0 \quad (i = 1, 2). \tag{4.10}$$

Substituting (4.6), (4.7) and (4.8) into (4.10), we get the following system of linear equations

$$\begin{bmatrix} 4l & 0 & m & 0 \\ 3m & 4l & 2n & m \\ 2n & 3m & 3m & 2n \\ m & 2n & 4l & 3m \\ 0 & m & 0 & 4l \end{bmatrix} \begin{bmatrix} \Gamma^1_{i1} \\ \Gamma^1_{i2} \\ \Gamma^2_{i1} \\ \Gamma^2_{i2} \end{bmatrix} = \begin{bmatrix} \partial l/\partial x^i \\ \partial m/\partial x^i \\ \partial n/\partial x^i \\ \partial m/\partial x^i \\ \partial l/\partial x^i \end{bmatrix} \quad (i = 1, 2).$$

**Theorem 4.11.** [53] *Let $M$ be a connected non-Riemannian Finsler surface with a locally symmetric fourth root metric $F = \sqrt[4]{A}$. It is a generalized Berwald surface if and only if the coefficient matrix*

$$B := \begin{bmatrix} 4l & 0 & m & 0 \\ 3m & 4l & 2n & m \\ 2n & 3m & 3m & 2n \\ m & 2n & 4l & 3m \\ 0 & m & 0 & 4l \end{bmatrix}$$

*is of constant rank* 4 *and*

$$\det \begin{bmatrix} 4l & 0 & m & 0 & \partial l/\partial x^i \\ 3m & 4l & 2n & m & \partial m/\partial x^i \\ 2n & 3m & 3m & 2n & \partial n/\partial x^i \\ m & 2n & 4l & 3m & \partial m/\partial x^i \\ 0 & m & 0 & 4l & \partial l/\partial x^i \end{bmatrix} = 0 \quad (i = 1, 2).$$

*The compatible linear connection is uniquely determined by the formula*

$$\begin{bmatrix} \Gamma^1_{i1} \\ \Gamma^1_{i2} \\ \Gamma^2_{i1} \\ \Gamma^2_{i2} \end{bmatrix} = \left(B^T B\right)^{-1} B^T \begin{bmatrix} \partial l/\partial x^i \\ \partial m/\partial x^i \\ \partial n/\partial x^i \\ \partial m/\partial x^i \\ \partial l/\partial x^i \end{bmatrix} \quad (i = 1, 2).$$

Examples and analogous computations in 3D can be found in [53].

### 4.5. A classical approach

In case of Finsler surfaces it is classical to introduce the vector field

$$V := \frac{\partial F}{\partial y^1}\frac{\partial}{\partial y^2} - \frac{\partial F}{\partial y^2}\frac{\partial}{\partial y^1}.$$

It is tangential to the indicatrix curve because of $VF = 0$. Since three vertical vector fields $\partial/\partial y^1, \partial/\partial y^2$ and $C$ (Liouville vector field) must be linearly dependent in 2D,

$$0 = \det \begin{pmatrix} g_{11} & g_{12} & \partial E/\partial y^1 \\ g_{12} & g_{22} & \partial E/\partial y^2 \\ \partial E/\partial y^1 & \partial E/\partial y^2 & 2E \end{pmatrix} = F^2 \left(\det g_{ij} - g(V,V)\right).$$

This means that $0 \neq \det g_{ij} = g(V,V)$ and, consequently,

$$V_0 := \frac{1}{\sqrt{g(V,V)}}V, \quad C_0 := \frac{1}{F}C, \quad V_0^h := V_0^i X_i^h = V_0^i\left(\frac{\partial}{\partial x^i} - G_i^l \frac{\partial}{\partial y^l}\right), \quad S_0 := \frac{1}{F}S = \frac{y^i}{F}X_i^h$$

form a local frame on the complement of the zero section in $\pi^{-1}(U)$. Such a collection of vector fields is called a *Berwald frame.*

**Definition 4.12.** *The main scalar of a Finsler surface is defined as* $\lambda := V_0^j V_0^k V_0^l \mathcal{C}_{jkl}$, *where* $V_0 = V/\sqrt{g(V,V)}$ *is the unit tangential vector field to the indicatrix curve.*

The vanishing of the main scalar implies that the surface is Riemannian and vice versa. The zero homogeneous version $I := F\lambda$ is also frequently used in the literature [8], [9], [18] and [22]. We also have that

$$\lambda := V_0^j V_0^k V_0^l \mathcal{C}_{jkl} = V_0^j \mathcal{C}_j = V_0\left(\ln\sqrt{\det g_{rs}}\right).$$

Let the indicatrix curve in $T_pM$ be parameterized as the integral curve of $V_0$:

$$V_0 \circ c_p(\theta) = c_p'(\theta) \quad \Rightarrow \quad \lambda \circ c_p(\theta) = \left(\ln\sqrt{\det g_{rs}} \circ c_p\right)'(\theta).$$

It is called the *central affine arcwise parametrization of the indicatrix curve.* The parameter $\theta$ is "the central affine length of the arc of the indicatrix" and the main scalar can be interpreted as its "central affine curvature" [60]. Let $\nabla$ be a linear connection on the base manifold $M$ and suppose that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition):

$$\frac{\partial E}{\partial x^i} - y^m\Gamma_{im}^l \circ \pi \frac{\partial E}{\partial y^l} = 0 \quad (i = 1,2), \quad \text{where} \quad E = \frac{1}{2}F^2$$

is the Finslerian energy. The classical approach is based on the comparison of $\nabla$ with the canonical horizontal distribution of the Finsler manifold [54]. Using the canonical horizontal sections we can write that

$$y^m\Gamma_{im}^l \circ \pi \frac{\partial E}{\partial y^l} - G_i^l \frac{\partial E}{\partial y^l} = 0.$$

Since the vertical vector fields are linear combinations of $V$ and $C$, it follows that

$$y^m\Gamma_{im}^l \circ \pi \frac{\partial}{\partial y^l} - G_i^l \frac{\partial}{\partial y^l} = f_i V + g_i C \quad (i = 1,2);$$

the coefficients $f_1$, $f_2$ are positively homogeneous of degree one, $g_1$ and $g_2$ are positively homogeneous of degree zero. Taking into account that $VE = 0$ and $CE = 2E$, we have that $g_1 = g_2 = 0$ and, consequently,

$$y^m\Gamma_{im}^l \circ \pi \frac{\partial}{\partial y^l} - G_i^l \frac{\partial}{\partial y^l} = f_i V \quad \Rightarrow \quad y^m\Gamma_{im}^k \circ \pi \frac{\partial}{\partial y^k} = G_i^k \frac{\partial}{\partial y^k} + f_i V \quad (i = 1,2).$$

To provide the linearity of the right hand side we should take the Lie brackets with the vertical coordinate vector fields two times:

$$0=\left[\left[y^m\Gamma^l_{im}\circ\pi\frac{\partial}{\partial y^l},\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]=\left[\left[G^l_i\frac{\partial}{\partial y^l},\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]+\left[\left[f_iV,\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]=$$

$$G^l_{ijk}\frac{\partial}{\partial y^l}+f_i\left[\left[V,\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]-\frac{\partial f_i}{\partial y^j}\left[V,\frac{\partial}{\partial y^k}\right]-\frac{\partial f_i}{\partial y^k}\left[V,\frac{\partial}{\partial y^j}\right]+\frac{\partial^2 f_i}{\partial y^j\partial y^k}V=:W_{ijk},$$

where

$$\left[V,\frac{\partial}{\partial y^j}\right]=\frac{\partial^2 F}{\partial y^j\partial y^2}\frac{\partial}{\partial y^1}-\frac{\partial^2 F}{\partial y^j\partial y^1}\frac{\partial}{\partial y^2},\quad \left[\left[V,\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]=-\frac{\partial^3 F}{\partial y^j\partial y^k\partial y^2}\frac{\partial}{\partial y^1}+\frac{\partial^3 F}{\partial y^j\partial y^k\partial y^1}\frac{\partial}{\partial y^2}.$$

Since $y^jW_{ijk}=y^kW_{ijk}=0$ it is enough to investigate the quantity $W_i=V^jV^kW_{ijk}$. By some direct computations

$$W_i=V^jV^kG^l_{ijk}\frac{\partial}{\partial y^l}-\frac{2V(f_i)}{F}\left(g\left(V,\frac{\partial}{\partial y^2}\right)\frac{\partial}{\partial y^1}-g\left(V,\frac{\partial}{\partial y^1}\right)\frac{\partial}{\partial y^2}\right)-$$

$$\frac{f_i}{F}\left(\left(2V^jV^k\mathcal{C}_{jk2}-\frac{1}{F}g(V,V)\frac{\partial F}{\partial y^2}\right)\frac{\partial}{\partial y^1}-\left(2V^jV^k\mathcal{C}_{jk1}-\frac{1}{F}g(V,V)\frac{\partial F}{\partial y^1}\right)\frac{\partial}{\partial y^2}\right)+$$

$$V^jV^k\frac{\partial^2 f_i}{\partial y^j\partial y^k}V.$$

The vanishing of $W_i$ is equivalent to

$$g(W_i,V_0)=0\ \text{ and }\ g(W_i,C_0)=0\quad (i=1,2), \tag{4.11}$$

where $V_0=V/\sqrt{g(V,V)}$ and $C_0=C/F$ are the normalized vector fields of the vertical Berwald frame. The vanishing of the orthogonal term to the indicatrix in (4.11) implies that

$$0=g(W_i,C)=W_iE=FV^jV^kG^l_{ijk}\frac{\partial F}{\partial y^l}-2V(f_i)g(V,V)-2f_iV^jV^kV^l\mathcal{C}_{jkl}.$$

Therefore

$$\frac{\alpha_i}{\sqrt{g(V,V)}}=\lambda f_i+(V_0f_i)\quad (i=1,2),$$

where $V_0=V/\sqrt{g(V,V)}$ is the unit tangential vector field to the indicatrix curve, $\lambda$ is the main scalar and

$$\alpha_i=\frac{1}{2}FV_0^jV_0^kG^l_{ijk}\frac{\partial F}{\partial y^l}\overset{(2.6)}{=}V_0^jV_0^kP_{ijk}.$$

Using that $\det g_{ij}=g(V,V)$, formula (4.5) says that

$$\alpha_i=V_0\left(f_i\sqrt{g(V,V)}\right)\quad (i=1,2).$$

Let the indicatrix curve $c_p$ in $T_pM$ be parameterized as the integral curve of $V_0$. Evaluating along $c_p$ we have

$$\alpha_i\circ c_p(\theta)=\left(f_i\circ c_p\sqrt{g(V,V)}\circ c_p\right)'(\theta)\quad (i=1,2)$$

for any $p\in U$. Therefore

$$\beta_i\circ c_p(t)=f_i\circ c_p(t)\sqrt{g(V,V)}\circ c_p(t)-f_i\circ c_p(0)\sqrt{g(V,V)}\circ c_p(0),$$

where $\beta_i\colon\pi^{-1}(U)\to\mathbb{R}$ $(i=1,2)$ are the 1-homogeneous extensions of the functions defined by

$$\beta_i\circ c_p(t)=\int_0^t\alpha_i\circ c_p(\theta)\,d\theta\quad (i=1,2)$$

along the central affine arcwise parametrization of the indicatrix curve. We can write that

$$f_i\circ c_p(t)=\frac{1}{\sqrt{g(V,V)\circ c_p(t)}}\left(\beta_i\circ c_p(t)+k_i(p)\right)\quad (i=1,2)$$

for some constants $k_i(p)$ $(i=1,2)$ depending only on the position. The vanishing of the tangential term to the indicatrix in (4.11) allows us to determine the integration constant $k_1(p)$ and $k_2(p)$ as follows.

**Theorem 4.13.** [54] *The compatible linear connection of a non-Riemannian connected generalized Berwald surface must be of the local form*

$$\Gamma^1_{ij}\circ\pi = G^1_{ij} - \frac{\partial f_i}{\partial y^j}\frac{\partial F}{\partial y^2} - f_i\frac{\partial^2 F}{\partial y^j\partial y^2},\quad \Gamma^2_{ij}\circ\pi = G^2_{ij} + \frac{\partial f_i}{\partial y^j}\frac{\partial F}{\partial y^1} + f_i\frac{\partial^2 F}{\partial y^j\partial y^1}\quad (i,j=1,2),$$

*where the* 1 *- homogeneous functions* $f_1$, $f_2$ *are given by*

$$f_i\circ c_p(t) = \frac{1}{\sqrt{g(V,V)\circ c_p(t)}}\left(\int_0^t \alpha_i\circ c_p(\theta)\,d\theta + k_i(p)\right)\quad (i=1,2)$$

*and the integration constants satisfy equations*

$$\omega_i\circ c_p(t) + (\alpha_i\circ c_p)'(t) = \left(\int_0^t \alpha_i\circ c_p(\theta)\,d\theta + k_i(p)\right)(\lambda\circ c_p)'(t) + \lambda\circ c_p(t)\alpha_i\circ c_p(t)\quad (i=1,2)$$

*for any* $p\in\pi^{-1}(U)$, *where*

$$\alpha_i = V_0^j V_0^k P_{ijk}\quad \text{and}\quad \omega_i = V_0^j V_0^k V_0^m G^l_{ijk} g_{ml}\quad (i=1,2).$$

**Corollary 4.14.** [54] *The compatible linear connection of a non-Riemannian generalized Berwald surface is uniquely determined.*

**Corollary 4.15.** [54] *A connected generalized Berwald surface is a Landsberg surface if and only if it is a Berwald surface.*

## 5. Special compatible linear connections in three-dimensional Finsler manifolds

We are going to give explicit formulas for the linear connections with totally anti-symmetric torsion preserving the Finslerian length of tangent vectors in case of three-dimensional Finsler manifolds [58]. The results are based on averaging of (intrinsic) Finslerian quantities by integration over the indicatrix surfaces. We also have some consequences for the base manifold as a Riemannian space with respect to the averaged Riemannian metric. The possible cases are Riemannian space forms of constant zero curvature, constant positive curvature or Riemannian spaces admitting Killing vector fields of constant Riemannian length (Theorem 5.4 and Theorem 5.6). The Riemannian consequences are simple to prove but they have essential influence on the differential topology of the base manifold (Remark 5.5 and Remark 5.7). However, the results are dominated by Theorem 5.3 as the explicit expression of the only possible compatible linear connection with totally anti-symmetric torsion for a Finsler metric in 3D.

Suppose that $\nabla$ is a compatible linear connection of a three-dimensional generalized Berwald manifold. By Theorem 2.4, such a linear connection must be metrical with respect to the averaged Riemannian metric (2.1) given by integration of the Riemann-Finsler metric on the indicatrix hypersurfaces. Therefore $\nabla$ is uniquely determined by its torsion tensor. In what follows $\nabla^*$ denotes the Lévi-Civita connection of the averaged Riemannian metric $\gamma$.

**Definition 5.1.** *The torsion tensor is totally anti-symmetric if its lowered tensor*

$$T_\flat(X,Y,Z) := \gamma(T(X,Y),Z)$$

*belongs to* $\wedge^3 M$.

**Corollary 5.2.** *If* $\nabla$ *is a metric linear connection with totally anti-symmetric torsion then*

$$\nabla^*_X Y = \nabla_X Y - \frac{1}{2}T(X,Y)$$

*and the geodesics of* $\nabla^*$ *and* $\nabla$ *coincide.*

If $\dim M = 3$ then $\dim\wedge^3 M = 1$ and, consequently,

$$T_\flat\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j},\frac{\partial}{\partial u^k}\right) = f\gamma\left(\frac{\partial}{\partial u^i}\times_\gamma\frac{\partial}{\partial u^j},\frac{\partial}{\partial u^k}\right) = f\det\gamma_{ij}$$

for some local function $f\colon U\to\mathbb{R}$, where the (local) orientation is choosen such that the coordinate vector fields represent a positive basis. This means that

$$\nabla^*_X Y = \nabla_X Y - \frac{f}{2}X\times_\gamma Y. \tag{5.1}$$

Taking the Riemannian energy $E^*(v) := \gamma(v,v)/2$, the Riemann-Finsler metric and the cross product of vertical vector fields are defined by $g^*_{ij} = \gamma_{ij} \circ \pi$ and

$$g^*\left(\frac{\partial}{\partial y^i} \times_{g^*} \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = \det g^*_{ij} = \det \gamma_{ij} \circ \pi$$

with bilinear extension. Since the horizontal distributions induced by $\nabla^*$ and $\nabla$ are spanned by the vector fields

$$X_i^{h^*} = \frac{\partial}{\partial x^i} - y^j {\Gamma^*}^l_{ij} \circ \pi \frac{\partial}{\partial y^l} \quad \text{and} \quad X_i^{h^\nabla} = \frac{\partial}{\partial x^i} - y^j \Gamma^l_{ij} \circ \pi \frac{\partial}{\partial y^l},$$

respectively, we have, by formula (5.1), that

$$X_i^{h^*} = X_i^{h^\nabla} + f \circ \pi V_i, \quad \text{where} \quad V_i = \frac{1}{2}\frac{\partial}{\partial y^i} \times_{g^*} C \quad (i = 1,2,3). \tag{5.2}$$

Using the comparison formula (5.2), the compatibility condition (2.7) gives that

$$X_i^{h^*} E = f \circ \pi V_i E \quad (i = 1,2,3) \quad \Rightarrow \quad VE = f \circ \pi \sum_{i=1}^{3} (V_i E)^2,$$

where the vector field $V$ is defined by the formula $V := \sum_{i=1}^{3} (V_i E) X_i^{h^*}$.

**Theorem 5.3.** [58] *For a three-dimensional non-Riemannian Finsler manifold, the compatible linear connection with totally anti-symmetric torsion tensor must be of the form*

$$\nabla_X Y = \nabla^*_X Y + \frac{f}{2} X \times_\gamma Y,$$

*where $\nabla^*$ is the Lévi-Civita connection of the averaged Riemannian metric $\gamma$ and the function $f$ is given by*

$$f(p) = \frac{1}{\sigma(p)} \int_{\partial K_p} VE\, \mu,$$

*where*

$$V = \sum_{i=1}^{3} (V_i E) X_i^{h^*}, \quad \sigma(p) = \sum_{i=1}^{3} \int_{\partial K_p} (V_i E)^2\, \mu, \quad V_i = \frac{1}{2}\frac{\partial}{\partial y^i} \times_{g^*} C \quad (i = 1,2,3).$$

### *5.1. Curvature properties*

Let a point $p \in M$ be fixed and consider the subgroup $G$ of orthogonal transformations with respect to the averaged inner product leaving the indicatrix $\partial K_p$ invariant in $T_pM$. Such a group is obviously closed in $O(3)$ and, consequently, it is compact. If we have a generalized Berwald manifold then the group $G$ is essentially independent of the choice of $p$ because the parallel translations with respect to the compatible linear connection $\nabla$ makes them isomorphic provided that the manifold is connected. On the other hand $G$ must be finite or reducible unless the manifold is Riemannian, for a more general context of the problem see [50]. According to Theorem 2.4 it follows that Hol $\nabla \subset G$, i.e. the holonomy group of a compatible linear connection is finite or reducible in case of a non-Riemannian generalized Berwald manifold.

#### *5.1.1. The case of finite holonomy groups*

Suppose that $G$ is finite and, consequently, the holonomy group of the compatible linear connection is also finite, i.e. its curvature is zero.

**Theorem 5.4.** [58] *If $M$ is a connected three-dimensional non-Riemannian Finsler manifold admitting a compatible flat linear connection with totally anti-symmetric torsion tensor then the sectional curvature of the averaged Riemannian metric is constant and*

- *$M$ is a classical Berwald manifold provided that the curvature is zero, or*
- *$M$ is a proper generalized Berwald manifold provided that the curvature is positive.*

**Remark 5.5.** If $M$ is complete then, by the Killing-Hopf theorem of Riemannian geometry, it follows that the universal cover of $M$ (as a Riemannian space with respect to the averaged Riemannian metric) is isometric to $\mathbb{R}^3$ or the Euclidean unit sphere $S^3 \subset \mathbb{R}^4$. Otherwise the manifold (as a non-Riemannian Finsler space) does not admit a compatible flat linear connection with totally anti-symmetric torsion tensor.

#### 5.1.2. The case of non-finite reducible holonomy groups

**Theorem 5.6.** [58] *If $M$ is a connected three-dimensional non-Riemannian Finsler manifold admitting a compatible non-flat linear connection $\nabla$ with totally anti-symmetric torsion tensor then there exists a one-dimensional distribution $\mathcal{D}$ such that*

- *any local section of constant length is a covariant constant vector field with respect to $\nabla$,*
- *any local section of constant length is a Killing vector field of constant length with respect to the averaged Riemannian metric.*

**Remark 5.7.** Killing vector fields of constant length naturally appear in different geometry of $K$-contact and Sasakian manifolds [7], [11] and [12]. There are many restrictions to the existence of Killing vector fields of constant length on a Riemannian manifold; for a comprehensive survey see [10]: for example, if a compact Riemannian manifold admits such a vector field then its Euler characteristic must be zero in the sense of a theorem due to H. Hopf [10, Section 1].

## 6. Extremal compatible linear connections

The notion of generalized Berwald manifolds goes back to V. Wagner [60]. They are Finsler manifolds admitting linear connections such that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition). We are interested in the unicity of the compatible linear connection and its expression in terms of the canonical data of the Finsler manifold (intrinsic characterization). If the torsion is zero (classical Berwald manifolds), the intrinsic characterization is the vanishing of the mixed curvature tensor of the canonical horizontal distribution. The problem of the intrinsic characterization is solved in the more general case of Finsler manifolds admitting semi-symmetric compatible linear connections [43], see also [48]. We also have a unicity statement because the torsion tensor of the semi-symmetric compatible linear connection can be explicitly expressed in terms of metrics and differential forms given by averaging. Especially, the integration of the Riemann-Finsler metric on the indicatrix hypersurfaces (the so-called averaged Riemannian metric) provides a Riemannian environment for the investigations. The fundamental result of the theory [36] states that a linear connection satisfying the compatibility condition must be metrical with respect to the averaged Riemannian metric. Therefore the compatible linear connection is uniquely determined by its torsion tensor. Unfortunately, the unicity statement for the compatible linear connection of a generalized Berwald manifold is false in general. As an example consider an additional term in Formula (3.2) of the compatible linear connection on a Randers manifold [45]:

$$\nabla_X Y = \nabla^*_X Y + A(X,Y) + B(X,Y),$$

where

$$\alpha(B(X,Y),Z) = -\alpha(B(X,Z),Y) \text{ and } \alpha(B(X,Y),\beta^\sharp) = 0. \tag{6.1}$$

The first condition in (6.1) provides us a metric connection with respect to $\alpha$. The second condition in (6.1) says that $B(X,Y)$ is orthogonal to $\beta^\sharp$ for any vector field $X$ and $Y$ on the manifold $M$. This provides that $\beta$ is parallel with respect to $\nabla$. The example shows that the compatible linear connection is not uniquely determined in general[5]. To provide the unicity of the compatible linear connection we should prescribe restrictions for its torsion. To avoid the difficulties of different possible solutions, the idea is to look for the extremal solution in some sense: the extremal compatible linear connection of a generalized Berwald manifold keeps its torsion as close to the zero as possible [55]. It is a conditional extremum problem involving functions defined on a local neighbourhood of the tangent manifold. In case of a given point $p \in M$ we can evaluate them at a refence element $v \in T_pM$. Note that the unknown components of the torsion tensor depends only on the position. The solution of the conditional extremum problem with a reference element can be expressed in terms of the canonical data. Such a solution allows us to reformulate the original problem by adding new constrains. Therefore the original problem can be solved algorithmically in finitely many steps at each point of the manifold. The pointwise solutions should form a continuous section of the torsion tensor bundle. The continuity of the components of the torsion tensor implies the continuity of the connection parameters. Using parallel translations with respect to such a connection we can conclude that the Finsler metric is monochromatic. By the fundamental result of the theory [5] it is sufficient and necessary for a Finsler metric to be a generalized Berwald metric. Therefore we have an intrinsic algorithm to check the existence of compatible linear connections on a Finsler manifold because it is equivalent to the existence of the extremal compatible linear connection [55].

[5] As an explicit example consider the case of dimension three: $B(X,Y) := \eta(X)\beta^\sharp \times Y$, where $\eta \in \wedge^1 M$.

Let us substitute the connection parameters with the components of the torsion tensor in equations of the compatibility condition (2.7). Using the standard Christoffel process

$$\Gamma^r_{ij} = \Gamma^{*r}_{ij} - \frac{1}{2}\left(T^l_{jk}\gamma^{kr}\gamma_{il} + T^l_{ik}\gamma^{kr}\gamma_{jl} - T^r_{ij}\right),$$

where the symbol $*$ refers to the quantities associated with the Lévi-Civita connection of the averaged Riemannian metric. Therefore the compatibility condition (2.7) can be written into the form

$$X^{h^*}_i F + \frac{1}{2} y^j \left(T^l_{jk}\gamma^{kr}\gamma_{il} + T^l_{ik}\gamma^{kr}\gamma_{jl} - T^r_{ij}\right) \circ \pi \frac{\partial F}{\partial y^r} = 0 \quad (i = 1, \ldots, n).$$

The reformulation is based on the one-to-one correspondence $\nabla \rightleftharpoons T$ between metric linear connections and their torsion tensors. Such an identification allows us to work in a tensor bundle over the base manifold. Moreover

$$\lambda\nabla_1 + (1-\lambda)\nabla_2 \rightleftharpoons \lambda T_1 + (1-\lambda)T_2$$

for any real number $\lambda \in \mathbb{R}$, i. e. the correspondence $\nabla \rightleftharpoons T$ preserves the affine combinations of the linear connections. It is also clear that if $\nabla_1$ and $\nabla_2$ satisfy the compatibility condition (2.7) then so does

$$\lambda\nabla_1 + (1-\lambda)\nabla_2.$$

This means that the set containing the restrictions of the torsion tensors of the compatible linear connections to the Cartesian product $T_pM \times T_pM$ is an affine subspace in the linear space $\wedge^2 T^*_pM \otimes T_pM$ for any $p \in M$. As the point is varying we have an affine distribution of the torsion tensor bundle $\wedge^2 T^*M \otimes TM$. Since the torsion tensor bundle over a Riemannian manifold can be equipped with a Riemannian metric in a natural way, we are going to look for the closest point of an affine subspace to the origin fiber by fiber.

**Definition 6.1.** [55] *Suppose that the coordinate vector fields $\partial/\partial u^1, \ldots, \partial/\partial u^n$ form an orthonormal basis at a point $p \in M$ with respect to the averaged Riemannian metric. We introduce a Riemannian metric on $\wedge^2 T^*_pM \otimes T_pM$ in the following way: if $T = \sum_{i<j,k} T^k_{ij} du^i \wedge du^j \otimes \frac{\partial}{\partial u^k}$, then*

$$\langle T_p, S_p\rangle := \sum_{i<j,k} T^k_{ij}(p) S^k_{ij}(p). \tag{6.2}$$

*The extremal compatible linear connection on a generalized Berwald space is the uniquely determined compatible linear connection whose torsion minimizes the norm arising from* (6.2).

Measuring the length of the torsion tensor point by point we can formulate an extremum problem for the compatible linear connection keeping its torsion as close to the origin as possible [55]: let a point $p \in M$ be given and consider the conditional extremum problem

$$\min \frac{1}{2}\|T_p\|^2 \quad \text{subject to} \quad T_p \in A_p, \tag{6.3}$$

where the affine subspace $A_p \subset \wedge^2 T^*_pM \otimes T_pM$ is defined by

$$X^{h^*}_i F + \frac{1}{2} y^j \left(T^l_{jk}\gamma^{kr}\gamma_{il} + T^l_{ik}\gamma^{kr}\gamma_{jl} - T^r_{ij}\right)(p) \frac{\partial F}{\partial y^r} = 0 \quad (i = 1, \ldots, n).$$

The solution of the conditional extremum problem (6.3) can be constructed algorithmically at each point of the manifold. The algorithm needs at most $\binom{n}{2}n$ steps by choosing so-called reference elements $v_1, v_2, \ldots$ in the corresponding tangent space [55].

**Remark 6.2.** Let $H_p \subset \wedge^2 T^*_pM \otimes T_pM$ be the directional space of the affine subspace $A_p$. By some direct computations, it can be proved [59], see also [55], that

$$T_q(v, w) := \varphi \circ T_p(\varphi^{-1}(v), \varphi^{-1}(w))$$

belongs to the directional space $H_q$ for any $T_p \in H_p$, where $\varphi$ is a linear isometry with respect to the Finslerian metric between the tangent spaces at the corresponding points $p$ and $q$. Such an isometry can be given (for example) by parallel transports with respect to a compatible linear connection $\nabla$. Therefore, in case of a connected generalized Berwald manifold, the mapping $p \in M \mapsto A_p \subset \wedge^2 T^*_pM \otimes T_pM$ is a smooth affine distribution of constant rank on the torsion tensor bundle.

Special metrics admit special methods to determine the extremal compatible linear connections. Detailed computations are in [56] to find all compatible linear connections of a generalized Berwald Randers manifold in terms of the torsion tensor components. Especially, we can find the extremal one among them: computing the components of the torsion tensor of an arbitrary compatible linear connection, we get the extremal connection by choosing the free parameters in such a way that the quadratic sum of all the components is minimal. According to the number of the tensor components that can be chosen arbitrarily, we have that

$$\dim A_p = n\binom{n-1}{2}$$

for a generalized Berwald Randers manifold of dimension $n \geq 3$. Especially, it is of dimension zero (it is a singleton) in case of dimension two [56].

**Theorem 6.3.** [56] *Let $F = \alpha + \beta$ be a non-Riemannian generalized Berwald Randers metric on a connected manifold $M$. If $p \in M$ is a given point together with a local coordinate system on the base manifold such that $\partial/\partial u^1(p), \ldots, \partial/\partial u^n(p)$ is an orthonormal basis of $T_pM$ with respect to $\alpha$, $\beta_1(p) = \cdots = \beta_{n-1}(p) = 0$ and $\beta_n(p) \neq 0$, then the torsion components $T^c_{ab}$ of the extremal compatible linear connection are given as follows:*

$$T^c_{ab} = 0 \quad (a < b < n, c < n),$$

$$T^n_{ab}(p) = \frac{1}{\beta_n}(x)\left(\frac{\partial \beta_b}{\partial x^a}(x) - \frac{\partial \beta_a}{\partial x^b}(x)\right) \quad (a < b < n),$$

$$T^a_{an}(p) = \Gamma^{*n}_{aa}(p) - \frac{1}{\beta_n(x)}\frac{\partial \beta_a}{\partial x^a}(x) \quad (a = 1, \ldots, n-1),$$

$$T^n_{an}(p) = \Gamma^{*n}_{an}(p) - \frac{1}{\beta_n(x)}\frac{\partial \beta_a}{\partial x^n}(x) \quad (a = 1, \ldots, n-1),$$

$$T^c_{an}(p) = \Gamma^{*n}_{ac}(p) - \frac{1}{2\beta_n(x)}\left(\frac{\partial \beta_c}{\partial x^a}(x) + \frac{\partial \beta_a}{\partial x^c}(x)\right) \quad (a, c = 1, \ldots, n-1, a \neq c).$$

**Remark 6.4.** A linear connection can always be introduced on a Randers manifold by the formulas for the components of the torsion tensor in Theorem 6.3. However, it is not compatible to the Randers metric in general. The necessary and sufficient condition for such a linear connection to be compatible to the Randers metric is that the dual vector field of $\beta$ has constant length with respect to the Riemannian metric $\alpha$ (cf. Theorem 3.2), see also [45] and [56].

## References


[1] T. Aikou, Averaged Riemannian metrics and connections with application to locally conformal Berwald manifolds, Publ. Math. Debrecen, 81(1-2) (2012), 179-198.

[2] V. Balan and N. Brinzei, Einstein equations for (h, v)-Bervald-Moór relativistic models, Balcan J. Geom. Appl. 11 (2006), 20-27.

[3] V. Balan, Spectra of symmetric tensors and m-root Finsler models, Linear Algebra and its Applications, 436(1) (2012), 152-162

[4] D. Bao, On two curvature-driven problems in Riemann-Finsler geometry, Advanced Studies in Pure Mathematics, 48 (2007), 19-71.

[5] N. Bartelmeß and V. Matveev, Monochromatic metrics are generalized Berwald, J. Diff. Geom. Appl. 58 (2018), 264-271.

[6] S. Bácsó, M. Hashiguchi and M. Matsumoto, Generalized Berwald spaces and Wagner spaces, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N. S.) 43 (1997), 307-321.

[7] F. Belgun, A. Moroianu, and U. Semmelmann, Symmetries of contact metric manifolds, Geom. Dedic. 101(1) (2003), 203-216.

[8] L. Berwald, Über zweidimensionale allgemeine metrische Räume, J. reine angew. Math. 156 (1927), 191-210 and 211-222.

[9] L. Berwald, On Finsler and Cartan Geometries III, Two-dimensional Finsler spaces with rectilinear extremals, Ann. of Math. 42 (1941), 84-112.

[10] V. N. Berestovskii and Yu. G. Nikonorov, Killing vector fields of constant length on Riemannian manifolds, Siberian Mathematical Journal, 49(3) (2008), 395-407.

[11] D. Blair, Contact manifolds in Riemannian geometry, Springer Lectures Notes in Math., V. 509, Springer Verlag, Berlin and New York, 1976.

[12] C. Boyer and K. Galicki, On Sasakian-Einstein geometry, Internat. J. Math. 11(7) (2000), 873-909.

[13] N. Brinzei, Projective relations for m-th root metric spaces, arXiv:0711.4781v2 (2008).

[14] L. S. Charlap, Bieberbach groups and flat manifolds, Springer 1986.

[15] M. Crampin, On the inverse problem for sprays, Publ. Math. Debrecen 70(3-4) 2007, 319-335.

[16] M. Crampin, On the construction of Riemannian metrics for Berwald spaces by averaging, Houston J. Math. 40(3) (2014), 737-750.

[17] M. Hashiguchi, On Wagner's generalized Berwald space, J. Korean Math. Soc. 12(1) (1975), 51-61.

[18] M. Hashiguchi, On conformal transformations of Finsler metrics, J. Math. Kyoto Univ. 16 (1976), 25-50.

[19] M. Hashiguchi and Y. Ichijyō, On conformal transformations of Wagner spaces, Rep. Fac. Sci. Kagoshima Univ. (Math., Phys., Chem.) 10 (1977), 19-25.

[20] S. Kikuchi, On the condition that a Finsler space be conformally flat, Tensor (N. S.) 55 (1994), 97-100.

[21] M. Matsumoto and K. Okubo, Theory of Finsler spaces with m-th root metric, Tensor (N. S.) 56 (1995), 9-104

[22] M. Matsumoto, Foundations of Finsler Geometry and Special Finsler spaces, Kaisheisa Press, Otsu (1986).

[23] M. Matsumoto, Conformally Berwald and conformally flat Finsler spaces, Publ. Math. Debrecen, 58(1-2) (2001), 275-285.

[24] V. S. Matveev, H-B. Rademacher, M. Troyanov and A. Zeghib, Finsler Conformal Lichnerovitz-Obata Conjecture, Ann. Inst. Fourier, Grenoble 59(3) (2009), 937-949.

[25] V. S. Matveev and M. Troyanov, The Binet-Legendre metric in Finsler geometry, Geometry and Topology 16 (2012), 2135-2170.

[26] V. S. Matveev and M. Troyanov, Completeness and incompleteness of the Binet-Legendre metric, European Journal of Mathematics, 1(3) (2015), 483-502.

[27] H. Shimada, On Finsler spaces with the metric $L = \sqrt[m]{a_{i_1 \dots i_m} y^{i_1} \cdot \dots \cdot y^{i_m}}$, Tensor (N.S.) 33 (1979), 365-372.

[28] Z. I. Szabó, Positive definite Berwald spaces. Structure theorems on Berwald spaces, Tensor (N. S.) 35(1) (1981), 25-39.

[29] L. Tamássy, Finsler Spaces with Polynomial Metric, Hypercomplex Numbers in Geometry and Physics, 2(6) Vol. 3 (2006), 85-92.

[30] A. Tayebi and B. Najafi, On $m$-th root metrics, J. Geom. Phys. 61 (2011),1479-1484.

[31] A. Tayebi and M. Barzegari, Generalized Berwald manifolds with $(\alpha, \beta)$-metrics, Indagationes Mathematicae, 27(3) (2016), 670-683.

[32] R. G. Torromé, A Riemannian structure associated with a Finsler structure, manuscript, 2005. arXiv:math/0501058.

[33] Cs. Vincze, An intrinsic version of Hashiguchi-Ichijyō's theorems for Wagner manifolds, SUT J. Math. 35(2) (1999), 263-270.

[34] Cs. Vincze, On Wagner connections and Wagner manifolds, Acta Math. Hung. 89(1-2) (2000), 111-133.

[35] Cs. Vincze, On conformal equivalence of Berwald manifolds all of whose indicatrices have positive curvature, SUT J. Math. 39 (1) (2003), 15-40.

[36] Cs. Vincze, A new proof of Szabó's theorem on the Riemann-metrizability of Berwald manifolds, Acta Math. Acad. Paedagog. Nyházi 21 (2005), 199-204.

[37] Cs. Vincze, On a scale function for testing the conformality of Finsler manifolds to a Berwald manifold, Journal of Geometry and Physics, 54 (2005), 454-475.

[38] Cs. Vincze, On geometric vector fields of Minkowski spaces and their applications, J. Diff. Geom. and Its Appl. 24 (2006), 1-20.

[39] Cs. Vincze, On an existence theorem of Wagner manifolds, Indagationes Mathematicae, 17(1) (2006), 129-145.

[40] Cs. Vincze, On Berwald and Wagner manifolds, Acta Math. Acad. Paedagog. Nyházi, 24 (2008), 169-178.

[41] Cs. Vincze and Á. Nagy, An introduction to the theory of generalized conics and their applications, Journal of Geom. and Phys. 61 (2011), 815-828.

[42] Cs. Vincze, On generalized conics' theory and averaged Riemannian metrics in Finsler geometry, In: Proceeding of the 47th Symposium on Finsler Geometry, Kagoshima (2012), 62-70.

[43] Cs. Vincze, Generalized Berwald manifolds with semi-symmetric compatible linear connections, Publ. Math. Debrecen, 83 (4) (2013) 741-755.

[44] Cs. Vincze, On generalized conics' theory and averaged Riemannian metrics in Finsler geometry, Tensor (N. S.) 74 (1) (2013), 101-116.

[45] Cs. Vincze, On Randers manifolds with semi-symmetric compatible linear connections, Indagationes Mathematicae, 26(2), 2014,363-379.

[46] Cs. Vincze, Average methods and their applications in differential geometry I, Journal of Geom. and Physics, 92 (2015), 194-209, arXiv:1309.0827.

[47] Cs. Vincze, A short review on averaging processes in Finsler geometry, Acta Math. Acad. Paedagog. Nyházi 31(1) (2015), 171-185, www.emis.de/journals.

[48] Cs. Vincze, On a special type of generalized Berwald manifolds: semi-symmetric linear connections preserving the Finslerian length of tangent vectors, Finsler geometry, new methods and perspectives, European J. of Math. 3(4), 1098-1171.

[49] Cs. Vincze, On Asanov's Finsleroid-Finsler metrics as the solutions of a conformal rigidity problem, J. of Diff. Geom. and Its Appl. 53 (2017), 148-168. arXiv:1601.08177.

[50] Cs. Vincze, Lazy orbits: an optimization problem on the sphere, J. of Geom. and Phys. 124 (2018), 180-198. arXiv:1709.06410.

[51] Cs. Vincze, Analytic properties and the asymptotic behavior of the area function of a Funk metric, Houston J. of Math. 44(2) (2018), 495-520. arXive:1602.06565.

[52] Cs. Vincze, An observation on Asanov's Unicorn metrics, Publ. Math. Debrecen, 90(1-2) (2017), 251-268. arXiv:1605.04407.

[53] Cs. Vincze, T. Khoshdani and M. Oláh, On generalized Berwald surfaces with locally symmetric fourth root metrics, Balkan Journal of Geometry and Its Appl., Vol. 24(2) (2019), 63-78. arXiv:1808.10855.

[54] Cs. Vincze, T. Khoshdani, S. Mehdi Zadeh and M. Oláh, On compatible linear connections of two-dimensional generalized Berwald manifolds: a classical approach, Communications in Math., Vol. 27(1) (2019), pp. 51-68. arXiv version: On compatible linear connections of two-dimensional generalized Berwald manifolds, arXiv:1808.02644.

[55] Cs. Vincze, On the extremal compatible linear connection of a generalized Berwald manifold, submitted to Aequat. Math. arXiv:1909.03096.

[56] Cs. Vincze and M. Oláh, On the extremal compatible linear connection of a Randers space, J. Geom. 111(19) (2020). https://doi.org/10.1007/s00022-020-00532-9, arXiv:2001.04389.

[57] Cs. Vincze, M. Oláh and L. Muhsin, On the divergence representation of the Gauss curvature of Riemannian surfaces and its applications, Rend. Circ. Mat. Palermo, II. Ser 69 (2020), 1-13. https://doi.org/10.1007/s12215-018-0382-6.

[58] Cs. Vincze, *On compatible linear connections with totally anti-symmetric torsion tensor of three-dimensional generalized Berwald manifolds*, Contributions to Algebra and Geometry 61(1) (2020), 117-128. arXiv:1903.06665.

[59] Cs. Vincze an M. Oláh, On generalized Berwald manifolds of dimension three, accepted for publication in Publ. Math. Debrecen.

[60] V. Wagner, On generalized Berwald spaces, CR Dokl. Acad. Sci. USSR (N.S.) 39 (1943), 3-5.

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